

**BENJAMIN CROWELL**

**LIGHT AND MATTER**





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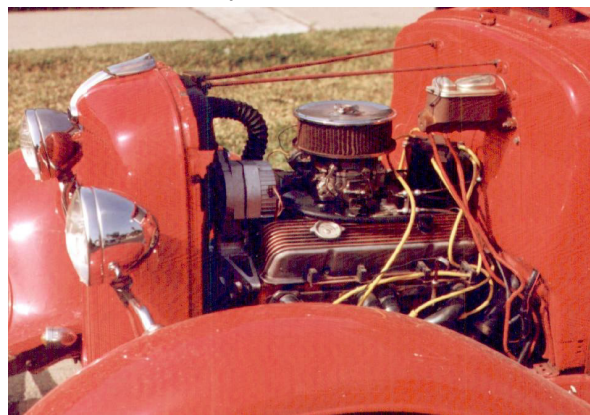
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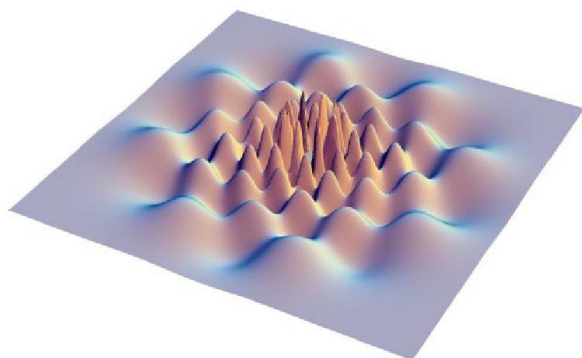


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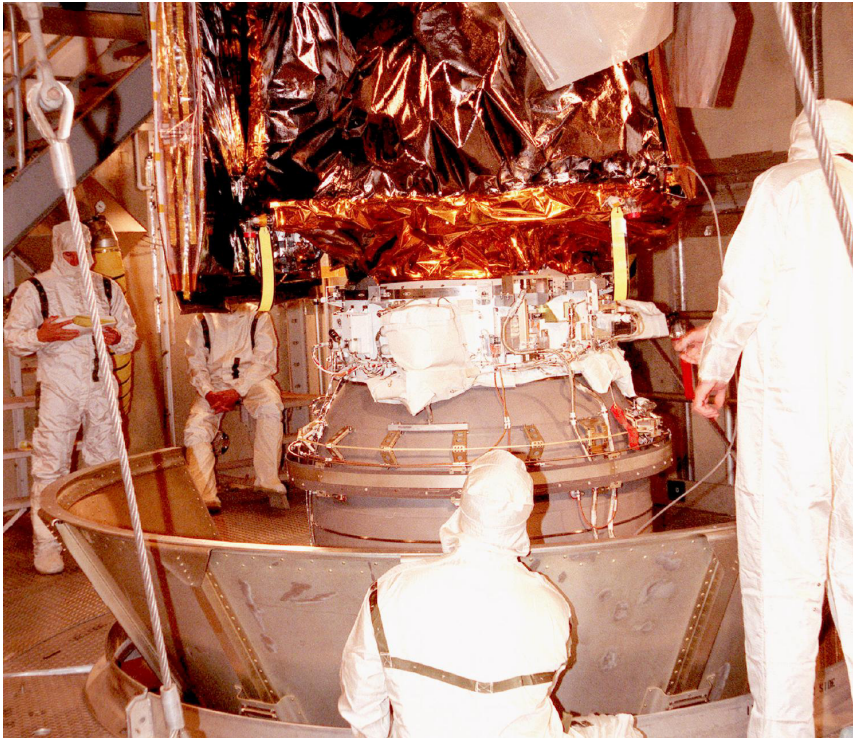
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The Mars Climate Orbiter is prepared for its mission. The laws of physics are the same everywhere, even on Mars, so the probe could be designed based on the laws of physics as discovered on earth. There is unfortunately another reason why this spacecraft is relevant to the topics of this chapter: it was destroyed attempting to enter Mars' atmosphere because engineers at Lockheed Martin forgot to convert data on engine thrusts from pounds into the metric unit of force (newtons) before giving the information to NASA. Conversions are important!

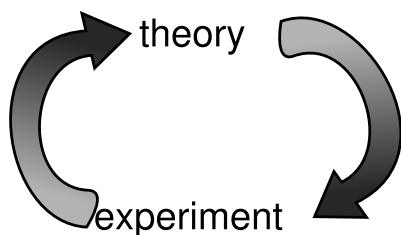
## Chapter 0

# Introduction and review

If you drop your shoe and a coin side by side, they hit the ground at the same time. Why doesn't the shoe get there first, since gravity is pulling harder on it? How does the lens of your eye work, and why do your eye's muscles need to squash its lens into different shapes in order to focus on objects nearby or far away? These are the kinds of questions that physics tries to answer about the behavior of light and matter, the two things that the universe is made of.

### 0.1 The scientific method

Until very recently in history, no progress was made in answering questions like these. Worse than that, the *wrong* answers written by thinkers like the ancient Greek physicist Aristotle were accepted without question for thousands of years. Why is it that scientific knowledge has progressed more since the Renaissance than it had in all the preceding millennia since the beginning of recorded history? Undoubtedly the industrial revolution is part of the answer. Building its centerpiece, the steam engine, required improved tech-



a / Science is a cycle of theory and experiment.



b / A satirical drawing of an alchemist's laboratory. H. Cock, after a drawing by Peter Brueghel the Elder (16th century).

niques for precise construction and measurement. (Early on, it was considered a major advance when English machine shops learned to build pistons and cylinders that fit together with a gap narrower than the thickness of a penny.) But even before the industrial revolution, the pace of discovery had picked up, mainly because of the introduction of the modern scientific method. Although it evolved over time, most scientists today would agree on something like the following list of the basic principles of the scientific method:

(1) *Science is a cycle of theory and experiment.* Scientific theories<sup>1</sup> are created to explain the results of experiments that were created under certain conditions. A successful theory will also make new predictions about new experiments under new conditions. Eventually, though, it always seems to happen that a new experiment comes along, showing that under certain conditions the theory is not a good approximation or is not valid at all. The ball is then back in the theorists' court. If an experiment disagrees with the current theory, the theory has to be changed, not the experiment.

(2) *Theories should both predict and explain.* The requirement of predictive power means that a theory is only meaningful if it predicts something that can be checked against experimental measurements that the theorist did not already have at hand. That is, a theory should be testable. Explanatory value means that many phenomena should be accounted for with few basic principles. If you answer every "why" question with "because that's the way it is," then your theory has no explanatory value. Collecting lots of data without being able to find any basic underlying principles is not science.

(3) *Experiments should be reproducible.* An experiment should be treated with suspicion if it only works for one person, or only in one part of the world. Anyone with the necessary skills and equipment should be able to get the same results from the same experiment. This implies that science transcends national and ethnic boundaries; you can be sure that nobody is doing actual science who claims that their work is "Aryan, not Jewish," "Marxist, not bourgeois," or "Christian, not atheistic." An experiment cannot be reproduced if it is secret, so science is necessarily a public enterprise.

As an example of the cycle of theory and experiment, a vital step toward modern chemistry was the experimental observation that the chemical elements could not be transformed into each other, e.g., lead could not be turned into gold. This led to the theory that chemical reactions consisted of rearrangements of the elements in

<sup>1</sup>The term "theory" in science does not just mean "what someone thinks," or even "what a lot of scientists think." It means an interrelated set of statements that have predictive value, and that have survived a broad set of empirical tests. Thus, both Newton's law of gravity and Darwinian evolution are scientific theories. A "hypothesis," in contrast to a theory, is any statement of interest that can be empirically tested. That the moon is made of cheese is a hypothesis, which was empirically tested, for example, by the Apollo astronauts.



different combinations, without any change in the identities of the elements themselves. The theory worked for hundreds of years, and was confirmed experimentally over a wide range of pressures and temperatures and with many combinations of elements. Only in the twentieth century did we learn that one element could be transformed into one another under the conditions of extremely high pressure and temperature existing in a nuclear bomb or inside a star. That observation didn't completely invalidate the original theory of the immutability of the elements, but it showed that it was only an approximation, valid at ordinary temperatures and pressures.

*self-check A*

A psychic conducts seances in which the spirits of the dead speak to the participants. He says he has special psychic powers not possessed by other people, which allow him to "channel" the communications with the spirits. What part of the scientific method is being violated here?

▷ Answer, p. 530

The scientific method as described here is an idealization, and should not be understood as a set procedure for doing science. Scientists have as many weaknesses and character flaws as any other group, and it is very common for scientists to try to discredit other people's experiments when the results run contrary to their own favored point of view. Successful science also has more to do with luck, intuition, and creativity than most people realize, and the restrictions of the scientific method do not stifle individuality and self-expression any more than the fugue and sonata forms stifled Bach and Haydn. There is a recent tendency among social scientists to go even further and to deny that the scientific method even exists, claiming that science is no more than an arbitrary social system that determines what ideas to accept based on an in-group's criteria. I think that's going too far. If science is an arbitrary social ritual, it would seem difficult to explain its effectiveness in building such useful items as airplanes, CD players, and sewers. If alchemy and astrology were no less scientific in their methods than chemistry and astronomy, what was it that kept them from producing anything useful?

**Discussion questions**

Consider whether or not the scientific method is being applied in the following examples. If the scientific method is not being applied, are the people whose actions are being described performing a useful human activity, albeit an unscientific one?

**A** Acupuncture is a traditional medical technique of Asian origin in which small needles are inserted in the patient's body to relieve pain. Many doctors trained in the west consider acupuncture unworthy of experimental study because if it had therapeutic effects, such effects could not be explained by their theories of the nervous system. Who is being more scientific, the western or eastern practitioners?

**B** Goethe, a German poet, is less well known for his theory of color. He published a book on the subject, in which he argued that scientific apparatus for measuring and quantifying color, such as prisms, lenses and colored filters, could not give us full insight into the ultimate meaning of color, for instance the cold feeling evoked by blue and green or the heroic sentiments inspired by red. Was his work scientific?

**C** A child asks why things fall down, and an adult answers “because of gravity.” The ancient Greek philosopher Aristotle explained that rocks fell because it was their nature to seek out their natural place, in contact with the earth. Are these explanations scientific?

**D** Buddhism is partly a psychological explanation of human suffering, and psychology is of course a science. The Buddha could be said to have engaged in a cycle of theory and experiment, since he worked by trial and error, and even late in his life he asked his followers to challenge his ideas. Buddhism could also be considered reproducible, since the Buddha told his followers they could find enlightenment for themselves if they followed a certain course of study and discipline. Is Buddhism a scientific pursuit?

## 0.2 What is physics?

Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the things which compose it...nothing would be uncertain, and the future as the past would be laid out before its eyes.

*Pierre Simon de Laplace*

Physics is the use of the scientific method to find out the basic principles governing light and matter, and to discover the implications of those laws. Part of what distinguishes the modern outlook from the ancient mind-set is the assumption that there are rules by which the universe functions, and that those laws can be at least partially understood by humans. From the Age of Reason through the nineteenth century, many scientists began to be convinced that the laws of nature not only could be known but, as claimed by Laplace, those laws could in principle be used to predict everything about the universe’s future if complete information was available about the present state of all light and matter. In subsequent sections, I’ll describe two general types of limitations on prediction using the laws of physics, which were only recognized in the twentieth century.

Matter can be defined as anything that is affected by gravity, i.e., that has weight or would have weight if it was near the Earth or another star or planet massive enough to produce measurable gravity. Light can be defined as anything that can travel from one place to another through empty space and can influence matter, but has no weight. For example, sunlight can influence your body by heating it or by damaging your DNA and giving you skin cancer. The physicist’s definition of light includes a variety of phenomena

that are not visible to the eye, including radio waves, microwaves, x-rays, and gamma rays. These are the “colors” of light that do not happen to fall within the narrow violet-to-red range of the rainbow that we can see.

### *self-check B*

At the turn of the 20th century, a strange new phenomenon was discovered in vacuum tubes: mysterious rays of unknown origin and nature. These rays are the same as the ones that shoot from the back of your TV's picture tube and hit the front to make the picture. Physicists in 1895 didn't have the faintest idea what the rays were, so they simply named them “cathode rays,” after the name for the electrical contact from which they sprang. A fierce debate raged, complete with nationalistic overtones, over whether the rays were a form of light or of matter. What would they have had to do in order to settle the issue? ▷

Answer, p. 530

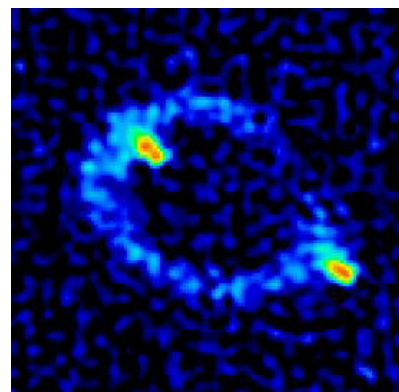
Many physical phenomena are not themselves light or matter, but are properties of light or matter or interactions between light and matter. For instance, motion is a property of all light and some matter, but it is not itself light or matter. The pressure that keeps a bicycle tire blown up is an interaction between the air and the tire. Pressure is not a form of matter in and of itself. It is as much a property of the tire as of the air. Analogously, sisterhood and employment are relationships among people but are not people themselves.

Some things that appear weightless actually do have weight, and so qualify as matter. Air has weight, and is thus a form of matter even though a cubic inch of air weighs less than a grain of sand. A helium balloon has weight, but is kept from falling by the force of the surrounding more dense air, which pushes up on it. Astronauts in orbit around the Earth have weight, and are falling along a curved arc, but they are moving so fast that the curved arc of their fall is broad enough to carry them all the way around the Earth in a circle. They perceive themselves as being weightless because their space capsule is falling along with them, and the floor therefore does not push up on their feet.

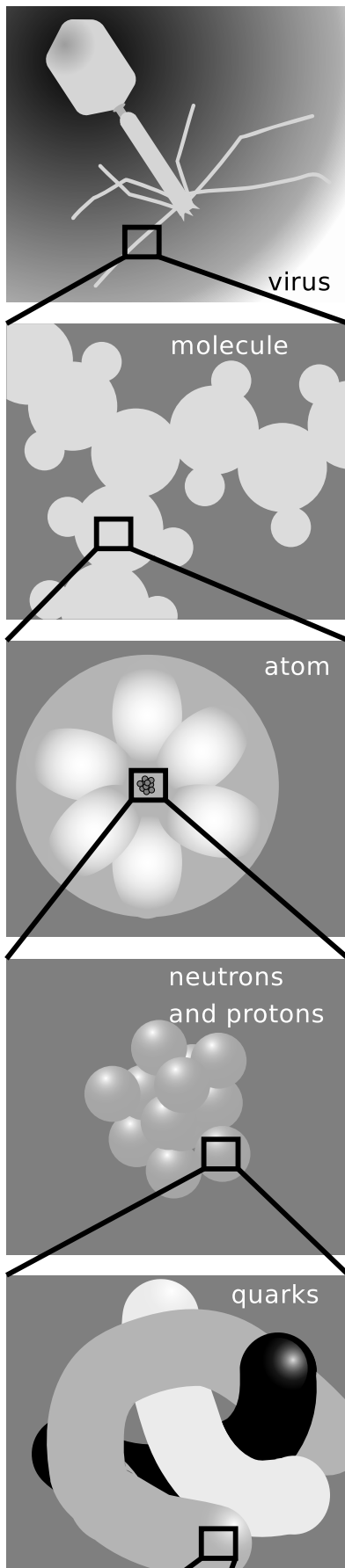
### **Optional Topic: Modern Changes in the Definition of Light and Matter**

Einstein predicted as a consequence of his theory of relativity that light would after all be affected by gravity, although the effect would be extremely weak under normal conditions. His prediction was borne out by observations of the bending of light rays from stars as they passed close to the sun on their way to the Earth. Einstein's theory also implied the existence of black holes, stars so massive and compact that their intense gravity would not even allow light to escape. (These days there is strong evidence that black holes exist.)

Einstein's interpretation was that light doesn't really have mass, but that energy is affected by gravity just like mass is. The energy in a light



c / This telescope picture shows two images of the same distant object, an exotic, very luminous object called a quasar. This is interpreted as evidence that a massive, dark object, possibly a black hole, happens to be between us and it. Light rays that would otherwise have missed the earth on either side have been bent by the dark object's gravity so that they reach us. The actual direction to the quasar is presumably in the center of the image, but the light along that central line doesn't get to us because it is absorbed by the dark object. The quasar is known by its catalog number, MG1131+0456, or more informally as Einstein's Ring.



beam is equivalent to a certain amount of mass, given by the famous equation  $E = mc^2$ , where  $c$  is the speed of light. Because the speed of light is such a big number, a large amount of energy is equivalent to only a very small amount of mass, so the gravitational force on a light ray can be ignored for most practical purposes.

There is however a more satisfactory and fundamental distinction between light and matter, which should be understandable to you if you have had a chemistry course. In chemistry, one learns that electrons obey the Pauli exclusion principle, which forbids more than one electron from occupying the same orbital if they have the same spin. The Pauli exclusion principle is obeyed by the subatomic particles of which matter is composed, but disobeyed by the particles, called photons, of which a beam of light is made.

Einstein's theory of relativity is discussed more fully in book 6 of this series.

The boundary between physics and the other sciences is not always clear. For instance, chemists study atoms and molecules, which are what matter is built from, and there are some scientists who would be equally willing to call themselves physical chemists or chemical physicists. It might seem that the distinction between physics and biology would be clearer, since physics seems to deal with inanimate objects. In fact, almost all physicists would agree that the basic laws of physics that apply to molecules in a test tube work equally well for the combination of molecules that constitutes a bacterium. (Some might believe that something more happens in the minds of humans, or even those of cats and dogs.) What differentiates physics from biology is that many of the scientific theories that describe living things, while ultimately resulting from the fundamental laws of physics, cannot be rigorously derived from physical principles.

### Isolated systems and reductionism

To avoid having to study everything at once, scientists isolate the things they are trying to study. For instance, a physicist who wants to study the motion of a rotating gyroscope would probably prefer that it be isolated from vibrations and air currents. Even in biology, where field work is indispensable for understanding how living things relate to their entire environment, it is interesting to note the vital historical role played by Darwin's study of the Galápagos Islands, which were conveniently isolated from the rest of the world. Any part of the universe that is considered apart from the rest can be called a "system."

Physics has had some of its greatest successes by carrying this process of isolation to extremes, subdividing the universe into smaller and smaller parts. Matter can be divided into atoms, and the behavior of individual atoms can be studied. Atoms can be split apart



into their constituent neutrons, protons and electrons. Protons and neutrons appear to be made out of even smaller particles called quarks, and there have even been some claims of experimental evidence that quarks have smaller parts inside them. This method of splitting things into smaller and smaller parts and studying how those parts influence each other is called reductionism. The hope is that the seemingly complex rules governing the larger units can be better understood in terms of simpler rules governing the smaller units. To appreciate what reductionism has done for science, it is only necessary to examine a 19th-century chemistry textbook. At that time, the existence of atoms was still doubted by some, electrons were not even suspected to exist, and almost nothing was understood of what basic rules governed the way atoms interacted with each other in chemical reactions. Students had to memorize long lists of chemicals and their reactions, and there was no way to understand any of it systematically. Today, the student only needs to remember a small set of rules about how atoms interact, for instance that atoms of one element cannot be converted into another via chemical reactions, or that atoms from the right side of the periodic table tend to form strong bonds with atoms from the left side.

### Discussion questions

**A** I've suggested replacing the ordinary dictionary definition of light with a more technical, more precise one that involves weightlessness. It's still possible, though, that the stuff a lightbulb makes, ordinarily called "light," does have some small amount of weight. Suggest an experiment to attempt to measure whether it does.

**B** Heat is weightless (i.e., an object becomes no heavier when heated), and can travel across an empty room from the fireplace to your skin, where it influences you by heating you. Should heat therefore be considered a form of light by our definition? Why or why not?

**C** Similarly, should sound be considered a form of light?

## 0.3 How to learn physics

For as knowledges are now delivered, there is a kind of contract of error between the deliverer and the receiver; for he that delivereth knowledge desireth to deliver it in such a form as may be best believed, and not as may be best examined; and he that receiveth knowledge desireth rather present satisfaction than expectant inquiry.

*Francis Bacon*

Many students approach a science course with the idea that they can succeed by memorizing the formulas, so that when a problem

is assigned on the homework or an exam, they will be able to plug numbers in to the formula and get a numerical result on their calculator. Wrong! That's not what learning science is about! There is a big difference between memorizing formulas and understanding concepts. To start with, different formulas may apply in different situations. One equation might represent a definition, which is always true. Another might be a very specific equation for the speed of an object sliding down an inclined plane, which would not be true if the object was a rock drifting down to the bottom of the ocean. If you don't work to understand physics on a conceptual level, you won't know which formulas can be used when.

Most students taking college science courses for the first time also have very little experience with interpreting the meaning of an equation. Consider the equation  $w = A/h$  relating the width of a rectangle to its height and area. A student who has not developed skill at interpretation might view this as yet another equation to memorize and plug in to when needed. A slightly more savvy student might realize that it is simply the familiar formula  $A = wh$  in a different form. When asked whether a rectangle would have a greater or smaller width than another with the same area but a smaller height, the unsophisticated student might be at a loss, not having any numbers to plug in on a calculator. The more experienced student would know how to reason about an equation involving division — if  $h$  is smaller, and  $A$  stays the same, then  $w$  must be bigger. Often, students fail to recognize a sequence of equations as a derivation leading to a final result, so they think all the intermediate steps are equally important formulas that they should memorize.

When learning any subject at all, it is important to become as actively involved as possible, rather than trying to read through all the information quickly without thinking about it. It is a good idea to read and think about the questions posed at the end of each section of these notes as you encounter them, so that you know you have understood what you were reading.

Many students' difficulties in physics boil down mainly to difficulties with math. Suppose you feel confident that you have enough mathematical preparation to succeed in this course, but you are having trouble with a few specific things. In some areas, the brief review given in this chapter may be sufficient, but in other areas it probably will not. Once you identify the areas of math in which you are having problems, get help in those areas. Don't limp along through the whole course with a vague feeling of dread about something like scientific notation. The problem will not go away if you ignore it. The same applies to essential mathematical skills that you are learning in this course for the first time, such as vector addition.

Sometimes students tell me they keep trying to understand a

certain topic in the book, and it just doesn't make sense. The worst thing you can possibly do in that situation is to keep on staring at the same page. Every textbook explains certain things badly — even mine! — so the best thing to do in this situation is to look at a different book. Instead of college textbooks aimed at the same mathematical level as the course you're taking, you may in some cases find that high school books or books at a lower math level give clearer explanations.

Finally, when reviewing for an exam, don't simply read back over the text and your lecture notes. Instead, try to use an active method of reviewing, for instance by discussing some of the discussion questions with another student, or doing homework problems you hadn't done the first time.

## 0.4 Self-evaluation

The introductory part of a book like this is hard to write, because every student arrives at this starting point with a different preparation. One student may have grown up outside the U.S. and so may be completely comfortable with the metric system, but may have had an algebra course in which the instructor passed too quickly over scientific notation. Another student may have already taken calculus, but may have never learned the metric system. The following self-evaluation is a checklist to help you figure out what you need to study to be prepared for the rest of the course.

<b>If you disagree with this statement...</b>	<b>you should study this section:</b>
I am familiar with the basic metric units of meters, kilograms, and seconds, and the most common metric prefixes: milli- (m), kilo- (k), and centi- (c).	section 0.5 Basic of the Metric System
I know about the newton, a unit of force	section 0.6 The newton, the Metric Unit of Force
I am familiar with these less common metric prefixes: mega- (M), micro- ( $\mu$ ), and nano- (n).	section 0.7 Less Common Metric Prefixes
I am comfortable with scientific notation.	section 0.8 Scientific Notation
I can confidently do metric conversions.	section 0.9 Conversions
I understand the purpose and use of significant figures.	section 0.10 Significant Figures

It wouldn't hurt you to skim the sections you think you already know about, and to do the self-checks in those sections.

## 0.5 Basics of the metric system

### The metric system

Units were not standardized until fairly recently in history, so when the physicist Isaac Newton gave the result of an experiment with a pendulum, he had to specify not just that the string was  $37\frac{7}{8}$  inches long but that it was “ $37\frac{7}{8}$  London inches long.” The inch as defined in Yorkshire would have been different. Even after the British Empire standardized its units, it was still very inconvenient to do calculations involving money, volume, distance, time, or weight, because of all the odd conversion factors, like 16 ounces in a pound, and 5280 feet in a mile. Through the nineteenth century, schoolchildren squandered most of their mathematical education in preparing to do calculations such as making change when a customer in a shop offered a one-crown note for a book costing two pounds, thirteen shillings and tuppence. The dollar has always been decimal, and British money went decimal decades ago, but the United States is still saddled with the antiquated system of feet, inches, pounds, ounces and so on.

Every country in the world besides the U.S. uses a system of units known in English as the “metric system.”<sup>2</sup> This system is entirely decimal, thanks to the same eminently logical people who brought about the French Revolution. In deference to France, the system’s official name is the *Système International*, or SI, meaning International System. (The phrase “SI system” is therefore redundant.)

The wonderful thing about the SI is that people who live in countries more modern than ours do not need to memorize how many ounces there are in a pound, how many cups in a pint, how many feet in a mile, etc. The whole system works with a single, consistent set of Greek and Latin prefixes that modify the basic units. Each prefix stands for a power of ten, and has an abbreviation that can be combined with the symbol for the unit. For instance, the meter is a unit of distance. The prefix kilo- stands for  $10^3$ , so a kilometer, 1 km, is a thousand meters.

The basic units of the metric system are the meter for distance, the second for time, and the gram for mass.

The following are the most common metric prefixes. You should memorize them.

prefix		meaning		example
kilo-	k	$10^3$	60 kg	= a person’s mass
centi-	c	$10^{-2}$	28 cm	= height of a piece of paper
milli-	m	$10^{-3}$	1 ms	= time for one vibration of a guitar string playing the note D

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<sup>2</sup>Liberia and Myanmar have not legally adopted metric units, but use them in everyday life.



The prefix centi-, meaning  $10^{-2}$ , is only used in the centimeter; a hundredth of a gram would not be written as 1 cg but as 10 mg. The centi- prefix can be easily remembered because a cent is  $10^{-2}$  dollars. The official SI abbreviation for seconds is “s” (not “sec”) and grams are “g” (not “gm”).

## The second

When I stated briefly above that the second was a unit of time, it may not have occurred to you that this was not much of a definition. We can make a dictionary-style definition of a term like “time,” or give a general description like Isaac Newton’s: “Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external. . .” Newton’s characterization sounds impressive, but physicists today would consider it useless as a definition of time. Today, the physical sciences are based on operational definitions, which means definitions that spell out the actual steps (operations) required to measure something numerically.

In an era when our toasters, pens, and coffee pots tell us the time, it is far from obvious to most people what is the fundamental operational definition of time. Until recently, the hour, minute, and second were defined operationally in terms of the time required for the earth to rotate about its axis. Unfortunately, the Earth’s rotation is slowing down slightly, and by 1967 this was becoming an issue in scientific experiments requiring precise time measurements. The second was therefore redefined as the time required for a certain number of vibrations of the light waves emitted by a cesium atoms in a lamp constructed like a familiar neon sign but with the neon replaced by cesium. The new definition not only promises to stay constant indefinitely, but for scientists is a more convenient way of calibrating a clock than having to carry out astronomical measurements.

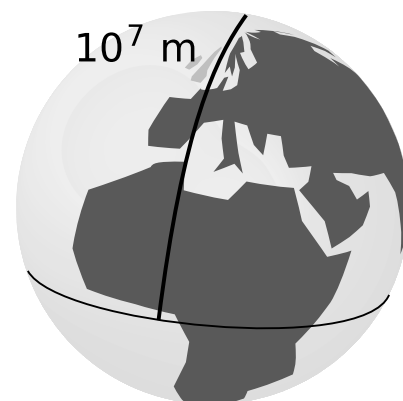
### *self-check C*

What is a possible operational definition of how strong a person is? ▷

Answer, p. 530

## The meter

The French originally defined the meter as  $10^{-7}$  times the distance from the equator to the north pole, as measured through Paris (of course). Even if the definition was operational, the operation of traveling to the north pole and laying a surveying chain behind you was not one that most working scientists wanted to carry out. Fairly soon, a standard was created in the form of a metal bar with two scratches on it. This was replaced by an atomic standard in 1960, and finally in 1983 by the current definition, which is that the meter is the distance traveled by light in a vacuum over a period of  $(1/299792458)$  seconds.



e / The original definition of the meter.

## The kilogram

The third base unit of the SI is the kilogram, a unit of mass. Mass is intended to be a measure of the amount of a substance, but that is not an operational definition. Bathroom scales work by measuring our planet's gravitational attraction for the object being weighed, but using that type of scale to define mass operationally would be undesirable because gravity varies in strength from place to place on the earth.



f / A duplicate of the Paris kilogram, maintained at the Danish National Metrology Institute.

There's a surprising amount of disagreement among physics textbooks about how mass should be defined, but here's how it's actually handled by the few working physicists who specialize in ultra-high-precision measurements. They maintain a physical object in Paris, which is the standard kilogram, a cylinder made of platinum-iridium alloy. Duplicates are checked against this mother of all kilograms by putting the original and the copy on the two opposite pans of a balance. Although this method of comparison depends on gravity, the problems associated with differences in gravity in different geographical locations are bypassed, because the two objects are being compared in the same place. The duplicates can then be removed from the Parisian kilogram shrine and transported elsewhere in the world. It would be desirable to replace this at some point with a universally accessible atomic standard rather than one based on a specific artifact, but as of 2010 the technology for automated counting of large numbers of atoms has not gotten good enough to make that work with the desired precision.

## Combinations of metric units

Just about anything you want to measure can be measured with some combination of meters, kilograms, and seconds. Speed can be measured in  $\text{m/s}$ , volume in  $\text{m}^3$ , and density in  $\text{kg/m}^3$ . Part of what makes the SI great is this basic simplicity. No more funny units like a cord of wood, a bolt of cloth, or a jigger of whiskey. No more liquid and dry measure. Just a simple, consistent set of units. The SI measures put together from meters, kilograms, and seconds make up the mks system. For example, the mks unit of speed is  $\text{m/s}$ , not  $\text{km/hr}$ .

## Checking units

A useful technique for finding mistakes in one's algebra is to analyze the units associated with the variables.

### Checking units

### example 1

▷ Jae starts from the formula  $V = \frac{1}{3}Ah$  for the volume of a cone, where  $A$  is the area of its base, and  $h$  is its height. He wants to find an equation that will tell him how tall a conical tent has to be in order to have a certain volume, given its radius. His algebra goes like this:

$$\begin{array}{ll} [1] & V = \frac{1}{3}Ah \\ [2] & A = \pi r^2 \\ [3] & V = \frac{1}{3}\pi r^2 h \\ [4] & h = \frac{\pi r^2}{3V} \end{array}$$

Is his algebra correct? If not, find the mistake.

▷ Line 4 is supposed to be an equation for the height, so the units of the expression on the right-hand side had better equal meters. The pi and the 3 are unitless, so we can ignore them. In terms of units, line 4 becomes

$$m = \frac{m^2}{m^3} = \frac{1}{m} \quad .$$

This is false, so there must be a mistake in the algebra. The units of lines 1, 2, and 3 check out, so the mistake must be in the step from line 3 to line 4. In fact the result should have been

$$h = \frac{3V}{\pi r^2} \quad .$$

Now the units check:  $m = m^3/m^2$ .

### Discussion question

**A** Isaac Newton wrote, "... the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time. . . It may be that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated or retarded. . ." Newton was right. Even the modern definition of the second in terms of light emitted by cesium atoms is subject to variation. For instance, magnetic fields could cause the cesium atoms to emit light with a slightly different rate of vibration. What makes us think, though, that a pendulum clock is more accurate than a sundial, or that a cesium atom is a more accurate timekeeper than a pendulum clock? That is, how can one test experimentally how the accuracies of different time standards compare?

## 0.6 The Newton, the metric unit of force

A force is a push or a pull, or more generally anything that can change an object's speed or direction of motion. A force is required to start a car moving, to slow down a baseball player sliding in to home base, or to make an airplane turn. (Forces may fail to change an object's motion if they are canceled by other forces, e.g., the force of gravity pulling you down right now is being canceled by the force of the chair pushing up on you.) The metric unit of force is the Newton, defined as the force which, if applied for one second, will cause a 1-kilogram object starting from rest to reach a speed of 1 m/s. Later chapters will discuss the force concept in more detail. In fact, this entire book is about the relationship between force and motion.

In section 0.5, I gave a gravitational definition of mass, but by defining a numerical scale of force, we can also turn around and define a scale of mass without reference to gravity. For instance, if a force of two Newtons is required to accelerate a certain object from rest to 1 m/s in 1 s, then that object must have a mass of 2 kg. From this point of view, mass characterizes an object's resistance to a change in its motion, which we call inertia or inertial mass. Although there is no fundamental reason why an object's resistance to a change in its motion must be related to how strongly gravity affects it, careful and precise experiments have shown that the inertial definition and the gravitational definition of mass are highly consistent for a variety of objects. It therefore doesn't really matter for any practical purpose which definition one adopts.

### Discussion question

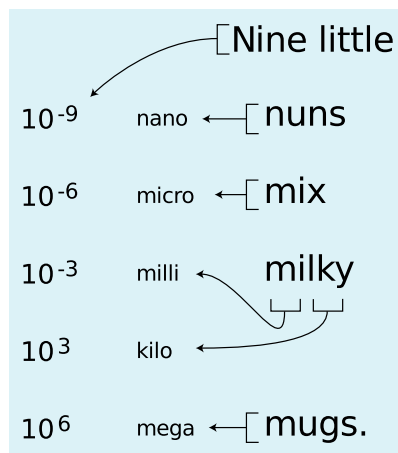
**A** Spending a long time in weightlessness is unhealthy. One of the most important negative effects experienced by astronauts is a loss of muscle and bone mass. Since an ordinary scale won't work for an astronaut in orbit, what is a possible way of monitoring this change in mass? (Measuring the astronaut's waist or biceps with a measuring tape is not good enough, because it doesn't tell anything about bone mass, or about the replacement of muscle with fat.)

## 0.7 Less common metric prefixes

The following are three metric prefixes which, while less common than the ones discussed previously, are well worth memorizing.

prefix	meaning	example
mega-	M $10^6$	6.4 Mm = radius of the earth
micro-	$\mu$ $10^{-6}$	10 $\mu$ m = size of a white blood cell
nano-	n $10^{-9}$	0.154 nm = distance between carbon nuclei in an ethane molecule

Note that the abbreviation for micro is the Greek letter mu,  $\mu$  — a common mistake is to confuse it with m (milli) or M (mega).



**g** / This is a mnemonic to help you remember the most important metric prefixes. The word "little" is to remind you that the list starts with the prefixes used for small quantities and builds upward. The exponent changes by 3, except that of course that we do not need a special prefix for  $10^0$ , which equals one.



There are other prefixes even less common, used for extremely large and small quantities. For instance, 1 femtometer =  $10^{-15}$  m is a convenient unit of distance in nuclear physics, and 1 gigabyte =  $10^9$  bytes is used for computers' hard disks. The international committee that makes decisions about the SI has recently even added some new prefixes that sound like jokes, e.g., 1 yoctogram =  $10^{-24}$  g is about half the mass of a proton. In the immediate future, however, you're unlikely to see prefixes like "yocto-" and "zepto-" used except perhaps in trivia contests at science-fiction conventions or other geekfests.

*self-check D*

Suppose you could slow down time so that according to your perception, a beam of light would move across a room at the speed of a slow walk. If you perceived a nanosecond as if it was a second, how would you perceive a microsecond?

▷ Answer, p. 530

## 0.8 Scientific notation

Most of the interesting phenomena in our universe are not on the human scale. It would take about 1,000,000,000,000,000,000 bacteria to equal the mass of a human body. When the physicist Thomas Young discovered that light was a wave, it was back in the bad old days before scientific notation, and he was obliged to write that the time required for one vibration of the wave was  $1/500$  of a millionth of a millionth of a second. Scientific notation is a less awkward way to write very large and very small numbers such as these. Here's a quick review.

Scientific notation means writing a number in terms of a product of something from 1 to 10 and something else that is a power of ten. For instance,

$$\begin{aligned} 32 &= 3.2 \times 10^1 \\ 320 &= 3.2 \times 10^2 \\ 3200 &= 3.2 \times 10^3 \quad \dots \end{aligned}$$

Each number is ten times bigger than the previous one.

Since  $10^1$  is ten times smaller than  $10^2$ , it makes sense to use the notation  $10^0$  to stand for one, the number that is in turn ten times smaller than  $10^1$ . Continuing on, we can write  $10^{-1}$  to stand for 0.1, the number ten times smaller than  $10^0$ . Negative exponents are used for small numbers:

$$\begin{aligned} 3.2 &= 3.2 \times 10^0 \\ 0.32 &= 3.2 \times 10^{-1} \\ 0.032 &= 3.2 \times 10^{-2} \quad \dots \end{aligned}$$

A common source of confusion is the notation used on the displays of many calculators. Examples:

$3.2 \times 10^6$	(written notation)
3.2E+6	(notation on some calculators)
$3.2^6$	(notation on some other calculators)

The last example is particularly unfortunate, because  $3.2^6$  really stands for the number  $3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2 = 1074$ , a totally different number from  $3.2 \times 10^6 = 3200000$ . The calculator notation should never be used in writing. It's just a way for the manufacturer to save money by making a simpler display.

*self-check E*

A student learns that  $10^4$  bacteria, standing in line to register for classes at Paramecium Community College, would form a queue of this size:

\_\_\_\_\_

The student concludes that  $10^2$  bacteria would form a line of this length:

\_\_\_\_\_

Why is the student incorrect?

▷ Answer, p. 530

## 0.9 Conversions

Conversions are one of the three essential mathematical skills, summarized on pp.514-515, that you need for success in this course.

I suggest you avoid memorizing lots of conversion factors between SI units and U.S. units, but two that do come in handy are:

$$1 \text{ inch} = 2.54 \text{ cm}$$

An object with a weight on Earth of 2.2 pounds-force has a mass of 1 kg.

The first one is the present definition of the inch, so it's exact. The second one is not exact, but is good enough for most purposes. (U.S. units of force and mass are confusing, so it's a good thing they're not used in science. In U.S. units, the unit of force is the pound-force, and the best unit to use for mass is the slug, which is about 14.6 kg.)

More important than memorizing conversion factors is understanding the right method for doing conversions. Even within the SI, you may need to convert, say, from grams to kilograms. Different people have different ways of thinking about conversions, but the method I'll describe here is systematic and easy to understand. The idea is that if 1 kg and 1000 g represent the same mass, then

we can consider a fraction like

$$\frac{10^3 \text{ g}}{1 \text{ kg}}$$

to be a way of expressing the number one. This may bother you. For instance, if you type 1000/1 into your calculator, you will get 1000, not one. Again, different people have different ways of thinking about it, but the justification is that it helps us to do conversions, and it works! Now if we want to convert 0.7 kg to units of grams, we can multiply kg by the number one:

$$0.7 \text{ kg} \times \frac{10^3 \text{ g}}{1 \text{ kg}}$$

If you're willing to treat symbols such as "kg" as if they were variables as used in algebra (which they're really not), you can then cancel the kg on top with the kg on the bottom, resulting in

$$0.7 \cancel{\text{kg}} \times \frac{10^3 \text{ g}}{1 \cancel{\text{kg}}} = 700 \text{ g} \quad .$$

To convert grams to kilograms, you would simply flip the fraction upside down.

One advantage of this method is that it can easily be applied to a series of conversions. For instance, to convert one year to units of seconds,

$$1 \text{ year} \times \frac{365 \cancel{\text{ days}}}{1 \cancel{\text{ year}}} \times \frac{24 \cancel{\text{ hours}}}{1 \cancel{\text{ day}}} \times \frac{60 \cancel{\text{ min}}}{1 \cancel{\text{ hour}}} \times \frac{60 \text{ s}}{1 \cancel{\text{ min}}} = 3.15 \times 10^7 \text{ s} \quad .$$

### Should that exponent be positive, or negative?

A common mistake is to write the conversion fraction incorrectly. For instance the fraction

$$\frac{10^3 \text{ kg}}{1 \text{ g}} \quad (\text{incorrect})$$

does not equal one, because  $10^3 \text{ kg}$  is the mass of a car, and  $1 \text{ g}$  is the mass of a raisin. One correct way of setting up the conversion factor would be

$$\frac{10^{-3} \text{ kg}}{1 \text{ g}} \quad (\text{correct}) \quad .$$

You can usually detect such a mistake if you take the time to check your answer and see if it is reasonable.

If common sense doesn't rule out either a positive or a negative exponent, here's another way to make sure you get it right. There are big prefixes and small prefixes:

big prefixes:    k    M  
 small prefixes: m     $\mu$     n

(It's not hard to keep straight which are which, since "mega" and "micro" are evocative, and it's easy to remember that a kilometer is bigger than a meter and a millimeter is smaller.) In the example above, we want the top of the fraction to be the same as the bottom. Since  $k$  is a big prefix, we need to *compensate* by putting a small number like  $10^{-3}$  in front of it, not a big number like  $10^3$ .

▷ *Solved problem: a simple conversion*                      page 37, problem 6

▷ *Solved problem: the geometric mean*                      page 38, problem 8

### Discussion question

**A** Each of the following conversions contains an error. In each case, explain what the error is.

(a)  $1000 \text{ kg} \times \frac{1 \text{ kg}}{1000 \text{ g}} = 1 \text{ g}$

(b)  $50 \text{ m} \times \frac{1 \text{ cm}}{100 \text{ m}} = 0.5 \text{ cm}$

(c) "Nano" is  $10^{-9}$ , so there are  $10^{-9}$  nm in a meter.

(d) "Micro" is  $10^{-6}$ , so 1 kg is  $10^6$   $\mu\text{g}$ .

## 0.10 Significant figures

An engineer is designing a car engine, and has been told that the diameter of the pistons (which are being designed by someone else) is 5 cm. He knows that 0.02 cm of clearance is required for a piston of this size, so he designs the cylinder to have an inside diameter of 5.04 cm. Luckily, his supervisor catches his mistake before the car goes into production. She explains his error to him, and mentally puts him in the "do not promote" category.

What was his mistake? The person who told him the pistons were 5 cm in diameter was wise to the ways of significant figures, as was his boss, who explained to him that he needed to go back and get a more accurate number for the diameter of the pistons. That person said "5 cm" rather than "5.00 cm" specifically to avoid creating the impression that the number was extremely accurate. In reality, the pistons' diameter was 5.13 cm. They would never have fit in the 5.04-cm cylinders.

The number of digits of accuracy in a number is referred to as the number of significant figures, or "sig figs" for short. As in the example above, sig figs provide a way of showing the accuracy of a number. In most cases, the result of a calculation involving several pieces of data can be no more accurate than the least accurate piece of data. In other words, "garbage in, garbage out." Since the 5 cm diameter of the pistons was not very accurate, the result of the engineer's calculation, 5.04 cm, was really not as accurate as he



thought. In general, your result should not have more than the number of sig figs in the least accurate piece of data you started with. The calculation above should have been done as follows:

$$\begin{array}{rcl} 5 \text{ cm} & (1 \text{ sig fig}) & \\ +0.04 \text{ cm} & (1 \text{ sig fig}) & \\ \hline =5 \text{ cm} & (\text{rounded off to 1 sig fig}) & \end{array}$$

The fact that the final result only has one significant figure then alerts you to the fact that the result is not very accurate, and would not be appropriate for use in designing the engine.

Note that the leading zeroes in the number 0.04 do not count as significant figures, because they are only placeholders. On the other hand, a number such as 50 cm is ambiguous — the zero could be intended as a significant figure, or it might just be there as a placeholder. The ambiguity involving trailing zeroes can be avoided by using scientific notation, in which  $5 \times 10^1$  cm would imply one sig fig of accuracy, while  $5.0 \times 10^1$  cm would imply two sig figs.

*self-check F*

The following quote is taken from an editorial by Norimitsu Onishi in the New York Times, August 18, 2002.

Consider Nigeria. Everyone agrees it is Africa's most populous nation. But what is its population? The United Nations says 114 million; the State Department, 120 million. The World Bank says 126.9 million, while the Central Intelligence Agency puts it at 126,635,626.

What should bother you about this?

▷ Answer, p. 530

Dealing correctly with significant figures can save you time! Often, students copy down numbers from their calculators with eight significant figures of precision, then type them back in for a later calculation. That's a waste of time, unless your original data had that kind of incredible precision.

The rules about significant figures are only rules of thumb, and are not a substitute for careful thinking. For instance,  $\$20.00 + \$0.05$  is  $\$20.05$ . It need not and should not be rounded off to  $\$20$ . In general, the sig fig rules work best for multiplication and division, and we also apply them when doing a complicated calculation that involves many types of operations. For simple addition and subtraction, it makes more sense to maintain a fixed number of digits after the decimal point.

When in doubt, don't use the sig fig rules at all. Instead, intentionally change one piece of your initial data by the maximum amount by which you think it could have been off, and recalculate the final result. The digits on the end that are completely reshuffled are the ones that are meaningless, and should be omitted.

*self-check G*

How many significant figures are there in each of the following measurements?

(1) 9.937 m

(2) 4.0 s

(3) 0.000000000000000037 kg

▷ Answer, p. 530

## Summary

### Selected vocabulary

matter . . . . .	Anything that is affected by gravity.
light . . . . .	Anything that can travel from one place to another through empty space and can influence matter, but is not affected by gravity.
operational definition . . . . .	A definition that states what operations should be carried out to measure the thing being defined.
Système International . . . . .	A fancy name for the metric system.
mks system . . .	The use of metric units based on the meter, kilogram, and second. Example: meters per second is the mks unit of speed, not cm/s or km/hr.
mass . . . . .	A numerical measure of how difficult it is to change an object's motion.
significant figures	Digits that contribute to the accuracy of a measurement.

### Notation

m . . . . .	meter, the metric distance unit
kg . . . . .	kilogram, the metric unit of mass
s . . . . .	second, the metric unit of time
M- . . . . .	the metric prefix mega-, $10^6$
k- . . . . .	the metric prefix kilo-, $10^3$
m- . . . . .	the metric prefix milli-, $10^{-3}$
$\mu$ - . . . . .	the metric prefix micro-, $10^{-6}$
n- . . . . .	the metric prefix nano-, $10^{-9}$

## Summary

Physics is the use of the scientific method to study the behavior of light and matter. The scientific method requires a cycle of theory and experiment, theories with both predictive and explanatory value, and reproducible experiments.

The metric system is a simple, consistent framework for measurement built out of the meter, the kilogram, and the second plus a set of prefixes denoting powers of ten. The most systematic method for doing conversions is shown in the following example:

$$370 \text{ ms} \times \frac{10^{-3} \text{ s}}{1 \text{ ms}} = 0.37 \text{ s}$$

Mass is a measure of the amount of a substance. Mass can be defined gravitationally, by comparing an object to a standard mass on a double-pan balance, or in terms of inertia, by comparing the effect of a force on an object to the effect of the same force on a standard mass. The two definitions are found experimentally to be proportional to each other to a high degree of precision, so we

usually refer simply to “mass,” without bothering to specify which type.

A force is that which can change the motion of an object. The metric unit of force is the Newton, defined as the force required to accelerate a standard 1-kg mass from rest to a speed of 1 m/s in 1 s.

Scientific notation means, for example, writing  $3.2 \times 10^5$  rather than 320000.

Writing numbers with the correct number of significant figures correctly communicates how accurate they are. As a rule of thumb, the final result of a calculation is no more accurate than, and should have no more significant figures than, the least accurate piece of data.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** Correct use of a calculator: (a) Calculate  $\frac{74658}{53222+97554}$  on a calculator. [Self-check: The most common mistake results in 97555.40.]

✓

(b) Which would be more like the price of a TV, and which would be more like the price of a house,  $\$3.5 \times 10^5$  or  $\$3.5^5$ ?

**2** Compute the following things. If they don't make sense because of units, say so.

(a) 3 cm + 5 cm

(b) 1.11 m + 22 cm

(c) 120 miles + 2.0 hours

(d) 120 miles / 2.0 hours

**3** Your backyard has brick walls on both ends. You measure a distance of 23.4 m from the inside of one wall to the inside of the other. Each wall is 29.4 cm thick. How far is it from the outside of one wall to the outside of the other? Pay attention to significant figures.

**4** The speed of light is  $3.0 \times 10^8$  m/s. Convert this to furlongs per fortnight. A furlong is 220 yards, and a fortnight is 14 days. An inch is 2.54 cm.

✓

**5** Express each of the following quantities in micrograms:

(a) 10 mg, (b)  $10^4$  g, (c) 10 kg, (d)  $100 \times 10^3$  g, (e) 1000 ng.

✓

**6** Convert 134 mg to units of kg, writing your answer in scientific notation.

▷ Solution, p. 516

**7** In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implanted steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas?

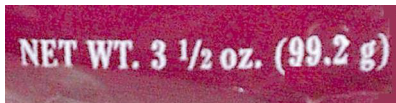
✓



**8** The usual definition of the mean (average) of two numbers  $a$  and  $b$  is  $(a+b)/2$ . This is called the arithmetic mean. The geometric mean, however, is defined as  $(ab)^{1/2}$  (i.e., the square root of  $ab$ ). For the sake of definiteness, let's say both numbers have units of mass. (a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If  $a$  and  $b$  both have units of grams, what should we call the units of  $ab$ ? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as  $(ab)^{1/3}$ . Is this reasonable? ▷ Solution, p. 516

**9** In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

**10** The photo shows the corner of a bag of pretzels. What's wrong here?



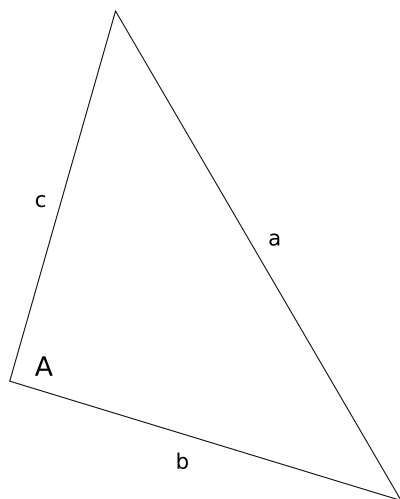
Problem 10.

**11** The distance to the horizon is given by the expression  $\sqrt{2rh}$ , where  $r$  is the radius of the Earth, and  $h$  is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. (See example 1 on p. 27 for an example of how to do this.) Don't try to prove the result, just check its units.

**12** (a) Based on the definitions of the sine, cosine, and tangent, what units must they have? (b) A cute formula from trigonometry lets you find any angle of a triangle if you know the lengths of its sides. Using the notation shown in the figure, and letting  $s = (a + b + c)/2$  be half the perimeter, we have

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Show that the units of this equation make sense. In other words, check that the units of the right-hand side are the same as your answer to part a of the question. ▷ Solution, p. 516



Problem 12.

## Exercise 0: Models and idealization

Equipment:

- coffee filters
- ramps (one per group)
- balls of various sizes
- sticky tape
- vacuum pump and “guinea and feather” apparatus (one)

The motion of falling objects has been recognized since ancient times as an important piece of physics, but the motion is inconveniently fast, so in our everyday experience it can be hard to tell exactly what objects are doing when they fall. In this exercise you will use several techniques to get around this problem and study the motion. Your goal is to construct a scientific *model* of falling. A model means an explanation that makes testable predictions. Often models contain simplifications or idealizations that make them easier to work with, even though they are not strictly realistic.

1. One method of making falling easier to observe is to use objects like feathers that we know from everyday experience will not fall as fast. You will use coffee filters, in stacks of various sizes, to test the following two hypotheses and see which one is true, or whether neither is true:

Hypothesis 1A: When an object is dropped, it rapidly speeds up to a certain natural falling speed, and then continues to fall at that speed. The falling speed is *proportional* to the object’s weight. (A proportionality is not just a statement that if one thing gets bigger, the other does too. It says that if one becomes three times bigger, the other also gets three times bigger, etc.)

Hypothesis 1B: Different objects fall the same way, regardless of weight.

Test these hypotheses and discuss your results with your instructor.

2. A second way to slow down the action is to let a ball roll down a ramp. The steeper the ramp, the closer to free fall. Based on your experience in part 1, write a hypothesis about what will happen when you race a heavier ball against a lighter ball down the same ramp, starting them both from rest.

Hypothesis:\_\_\_\_\_

Show your hypothesis to your instructor, and then test it.

You have probably found that falling was more complicated than you thought! Is there more than one factor that affects the motion of a falling object? Can you imagine certain idealized situations that are simpler? Try to agree verbally with your group on an informal model of falling that can make predictions about the experiments described in parts 3 and 4.

3. You have three balls: a standard “comparison ball” of medium weight, a light ball, and a heavy ball. Suppose you stand on a chair and (a) drop the light ball side by side with the comparison ball, then (b) drop the heavy ball side by side with the comparison ball, then (c) join the light and heavy balls together with sticky tape and drop them side by side with the comparison ball.

Use your model to make a prediction:\_\_\_\_\_

Test your prediction.

4. Your instructor will pump nearly all the air out of a chamber containing a feather and a heavier object, then let them fall side by side in the chamber.

Use your model to make a prediction:\_\_\_\_\_



Life would be very different if you were the size of an insect.

# Chapter 1

## Scaling and estimation

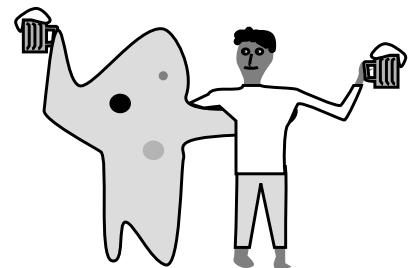
### 1.1 Introduction

Why can't an insect be the size of a dog? Some skinny stretched-out cells in your spinal cord are a meter tall — why does nature display no single cells that are not just a meter tall, but a meter wide, and a meter thick as well? Believe it or not, these are questions that can be answered fairly easily without knowing much more about physics than you already do. The only mathematical technique you really need is the humble conversion, applied to area and volume.

#### Area and volume

Area can be defined by saying that we can copy the shape of interest onto graph paper with  $1\text{ cm} \times 1\text{ cm}$  squares and count the number of squares inside. Fractions of squares can be estimated by eye. We then say the area equals the number of squares, in units of square cm. Although this might seem less “pure” than computing areas using formulae like  $A = \pi r^2$  for a circle or  $A = wh/2$  for a triangle, those formulae are not useful as definitions of area because they cannot be applied to irregularly shaped areas.

Units of square cm are more commonly written as  $\text{cm}^2$  in science. Of course, the unit of measurement symbolized by “cm” is not an



a / Amoebas this size are seldom encountered.

algebra symbol standing for a number that can be literally multiplied by itself. But it is advantageous to write the units of area that way and treat the units as if they were algebra symbols. For instance, if you have a rectangle with an area of  $6\text{m}^2$  and a width of  $2\text{ m}$ , then calculating its length as  $(6\text{ m}^2)/(2\text{ m}) = 3\text{ m}$  gives a result that makes sense both numerically and in terms of units. This algebra-style treatment of the units also ensures that our methods of converting units work out correctly. For instance, if we accept the fraction

$$\frac{100\text{ cm}}{1\text{ m}}$$

as a valid way of writing the number one, then one times one equals one, so we should also say that one can be represented by

$$\frac{100\text{ cm}}{1\text{ m}} \times \frac{100\text{ cm}}{1\text{ m}} \quad ,$$

which is the same as

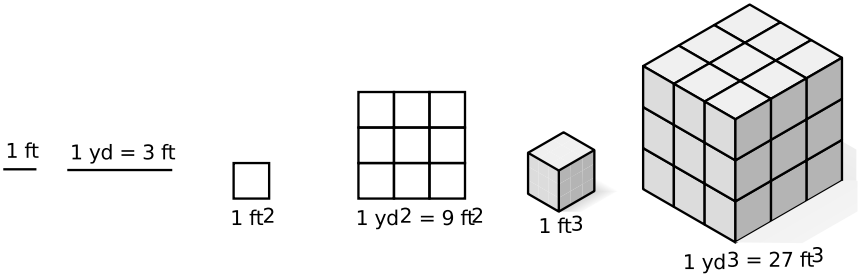
$$\frac{10000\text{ cm}^2}{1\text{ m}^2} \quad .$$

That means the conversion factor from square meters to square centimeters is a factor of  $10^4$ , i.e., a square meter has  $10^4$  square centimeters in it.

All of the above can be easily applied to volume as well, using one-cubic-centimeter blocks instead of squares on graph paper.

To many people, it seems hard to believe that a square meter equals 10000 square centimeters, or that a cubic meter equals a million cubic centimeters — they think it would make more sense if there were  $100\text{ cm}^2$  in  $1\text{ m}^2$ , and  $100\text{ cm}^3$  in  $1\text{ m}^3$ , but that would be incorrect. The examples shown in figure b aim to make the correct answer more believable, using the traditional U.S. units of feet and yards. (One foot is 12 inches, and one yard is three feet.)

b / Visualizing conversions of area and volume using traditional U.S. units.



*self-check A*

Based on figure b, convince yourself that there are  $9\text{ ft}^2$  in a square yard, and  $27\text{ ft}^3$  in a cubic yard, then demonstrate the same thing symbolically (i.e., with the method using fractions that equal one). ▷ Answer, p. 530



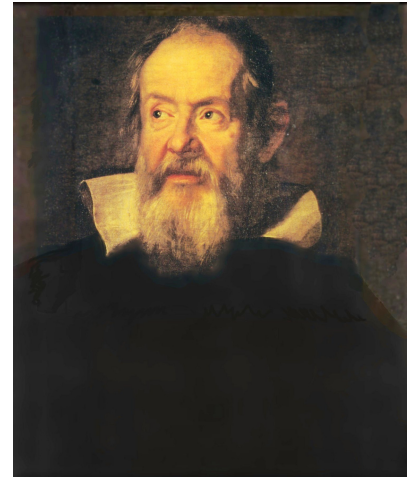
▷ Solved problem: converting  $\text{mm}^2$  to  $\text{cm}^2$       page 59, problem 10

▷ Solved problem: scaling a liter      page 60, problem 19

### Discussion question

**A** How many square centimeters are there in a square inch? (1 inch = 2.54 cm) First find an approximate answer by making a drawing, then derive the conversion factor more accurately using the symbolic method.

c / Galileo Galilei (1564-1642) was a Renaissance Italian who brought the scientific method to bear on physics, creating the modern version of the science. Coming from a noble but very poor family, Galileo had to drop out of medical school at the University of Pisa when he ran out of money. Eventually becoming a lecturer in mathematics at the same school, he began a career as a notorious troublemaker by writing a burlesque ridiculing the university's regulations — he was forced to resign, but found a new teaching position at Padua. He invented the pendulum clock, investigated the motion of falling bodies, and discovered the moons of Jupiter. The thrust of his life's work was to discredit Aristotle's physics by confronting it with contradictory experiments, a program that paved the way for Newton's discovery of the relationship between force and motion. In chapter 3 we'll come to the story of Galileo's ultimate fate at the hands of the Church.



## 1.2 Scaling of area and volume

Great fleas have lesser fleas  
Upon their backs to bite 'em.  
And lesser fleas have lesser still,  
And so ad infinitum.

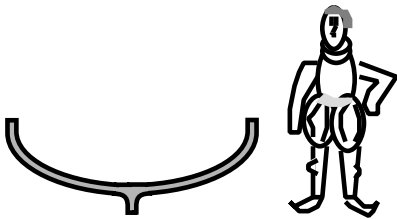
*Jonathan Swift*

Now how do these conversions of area and volume relate to the questions I posed about sizes of living things? Well, imagine that you are shrunk like Alice in Wonderland to the size of an insect. One way of thinking about the change of scale is that what used to look like a centimeter now looks like perhaps a meter to you, because you're so much smaller. If area and volume scaled according to most people's intuitive, incorrect expectations, with  $1 \text{ m}^2$  being the same as  $100 \text{ cm}^2$ , then there would be no particular reason why nature should behave any differently on your new, reduced scale. But nature does behave differently now that you're small. For instance, you will find that you can walk on water, and jump to many times your own height. The physicist Galileo Galilei had the basic insight that the scaling of area and volume determines how natural phenomena behave differently on different scales. He first reasoned about mechanical structures, but later extended his insights to living things, taking the then-radical point of view that at the fundamental level, a living organism should follow the same laws

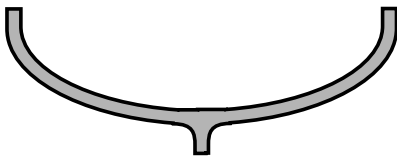
of nature as a machine. We will follow his lead by first discussing machines and then living things.

### Galileo on the behavior of nature on large and small scales

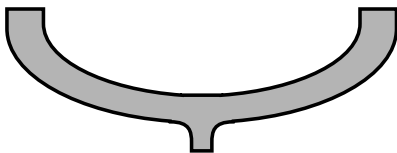
One of the world's most famous pieces of scientific writing is Galileo's *Dialogues Concerning the Two New Sciences*. Galileo was an entertaining writer who wanted to explain things clearly to laypeople, and he livened up his work by casting it in the form of a dialogue among three people. Salviati is really Galileo's alter ego. Simplicio is the stupid character, and one of the reasons Galileo got in trouble with the Church was that there were rumors that Simplicio represented the Pope. Sagredo is the earnest and intelligent student, with whom the reader is supposed to identify. (The following excerpts are from the 1914 translation by Crew and de Salvio.)



d / The small boat holds up just fine.



e / A larger boat built with the same proportions as the small one will collapse under its own weight.



f / A boat this large needs to have timbers that are thicker compared to its size.

SAGREDO: Yes, that is what I mean; and I refer especially to his last assertion which I have always regarded as false. . . ; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale. Now, since mechanics has its foundations in geometry, where mere size [is unimportant], I do not see that the properties of circles, triangles, cylinders, cones and other solid figures will change with their size. If, therefore, a large machine be constructed in such a way that its parts bear to one another the same ratio as in a smaller one, and if the smaller is sufficiently strong for the purpose for which it is designed, I do not see why the larger should not be able to withstand any severe and destructive tests to which it may be subjected.

Salviati contradicts Sagredo:

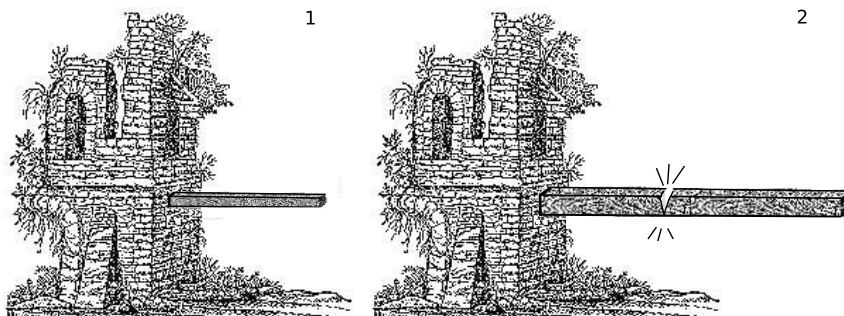
SALVIATI: . . . Please observe, gentlemen, how facts which at first seem improbable will, even on scant explanation, drop the cloak which has hidden them and stand forth in naked and simple beauty. Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.

The point Galileo is making here is that small things are sturdier in proportion to their size. There are a lot of objections that could be raised, however. After all, what does it really mean for something to be “strong”, to be “strong in proportion to its size,” or to be strong “out of proportion to its size?” Galileo hasn't given operational definitions of things like “strength,” i.e., definitions that spell out how to measure them numerically.

Also, a cat is shaped differently from a horse — an enlarged photograph of a cat would not be mistaken for a horse, even if the photo-doctoring experts at the National Inquirer made it look like a person was riding on its back. A grasshopper is not even a mammal, and it has an exoskeleton instead of an internal skeleton. The whole argument would be a lot more convincing if we could do some isolation of variables, a scientific term that means to change only one thing at a time, isolating it from the other variables that might have an effect. If size is the variable whose effect we're interested in seeing, then we don't really want to compare things that are different in size but also different in other ways.

**SALVIATI:** ...we asked the reason why [shipbuilders] employed stocks, scaffolding, and bracing of larger dimensions for launching a big vessel than they do for a small one; and [an old man] answered that they did this in order to avoid the danger of the ship parting under its own heavy weight, a danger to which small boats are not subject?

After this entertaining but not scientifically rigorous beginning, Galileo starts to do something worthwhile by modern standards. He simplifies everything by considering the strength of a wooden plank. The variables involved can then be narrowed down to the type of wood, the width, the thickness, and the length. He also gives an operational definition of what it means for the plank to have a certain strength “in proportion to its size,” by introducing the concept of a plank that is the longest one that would not snap under its own weight if supported at one end. If you increased its length by the slightest amount, without increasing its width or thickness, it would break. He says that if one plank is the same shape as another but a different size, appearing like a reduced or enlarged photograph of the other, then the planks would be strong “in proportion to their sizes” if both were just barely able to support their own weight.



g / Galileo discusses planks made of wood, but the concept may be easier to imagine with clay. All three clay rods in the figure were originally the same shape. The medium-sized one was twice the height, twice the length, and twice the width of the small one, and similarly the large one was twice as big as the medium one in all its linear dimensions. The big one has four times the linear dimensions of the small one, 16 times the cross-sectional area when cut perpendicular to the page, and 64 times the volume. That means that the big one has 64 times the weight to support, but only 16 times the strength compared to the smallest one.

h / 1. This plank is as long as it can be without collapsing under its own weight. If it was a hundredth of an inch longer, it would collapse. 2. This plank is made out of the same kind of wood. It is twice as thick, twice as long, and twice as wide. It will collapse under its own weight.

Also, Galileo is doing something that would be frowned on in modern science: he is mixing experiments whose results he has actually observed (building boats of different sizes), with experiments that he could not possibly have done (dropping an ant from the height of the moon). He now relates how he has done actual experiments with such planks, and found that, according to this operational definition, they are not strong in proportion to their sizes. The larger one breaks. He makes sure to tell the reader how important the result is, via Sagredo's astonished response:

**SAGREDO:** My brain already reels. My mind, like a cloud momentarily illuminated by a lightning flash, is for an instant filled with an unusual light, which now beckons to me and which now suddenly mingles and obscures strange, crude ideas. From what you have said it appears to me impossible to build two similar structures of the same material, but of different sizes and have them proportionately strong.

In other words, this specific experiment, using things like wooden planks that have no intrinsic scientific interest, has very wide implications because it points out a general principle, that nature acts differently on different scales.

To finish the discussion, Galileo gives an explanation. He says that the strength of a plank (defined as, say, the weight of the heaviest boulder you could put on the end without breaking it) is proportional to its cross-sectional area, that is, the surface area of the fresh wood that would be exposed if you sawed through it in the middle. Its weight, however, is proportional to its volume.<sup>1</sup>

How do the volume and cross-sectional area of the longer plank compare with those of the shorter plank? We have already seen, while discussing conversions of the units of area and volume, that these quantities don't act the way most people naively expect. You might think that the volume and area of the longer plank would both be doubled compared to the shorter plank, so they would increase in proportion to each other, and the longer plank would be equally able to support its weight. You would be wrong, but Galileo knows that this is a common misconception, so he has Salviati address the point specifically:

**SALVIATI:** ... Take, for example, a cube two inches on a side so that each face has an area of four square inches and the total area, i.e., the sum of the six faces, amounts to twenty-four square inches; now imagine this cube to be sawed through three times [with cuts in three perpendicular planes] so as to divide it into eight smaller cubes, each one inch on the side, each face one inch square, and the total

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<sup>1</sup>Galileo makes a slightly more complicated argument, taking into account the effect of leverage (torque). The result I'm referring to comes out the same regardless of this effect.

surface of each cube six square inches instead of twenty-four in the case of the larger cube. It is evident therefore, that the surface of the little cube is only one-fourth that of the larger, namely, the ratio of six to twenty-four; but the volume of the solid cube itself is only one-eighth; the volume, and hence also the weight, diminishes therefore much more rapidly than the surface. . . You see, therefore, Simplicio, that I was not mistaken when . . . I said that the surface of a small solid is comparatively greater than that of a large one.

The same reasoning applies to the planks. Even though they are not cubes, the large one could be sawed into eight small ones, each with half the length, half the thickness, and half the width. The small plank, therefore, has more surface area in proportion to its weight, and is therefore able to support its own weight while the large one breaks.

### Scaling of area and volume for irregularly shaped objects

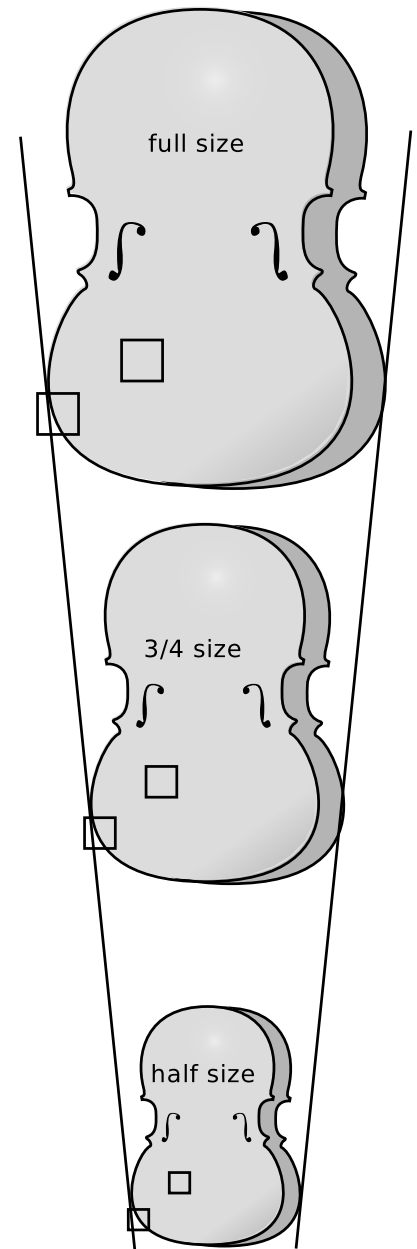
You probably are not going to believe Galileo's claim that this has deep implications for all of nature unless you can be convinced that the same is true for any shape. Every drawing you've seen so far has been of squares, rectangles, and rectangular solids. Clearly the reasoning about sawing things up into smaller pieces would not prove anything about, say, an egg, which cannot be cut up into eight smaller egg-shaped objects with half the length.

Is it always true that something half the size has one quarter the surface area and one eighth the volume, even if it has an irregular shape? Take the example of a child's violin. Violins are made for small children in smaller size to accomodate their small bodies. Figure i shows a full-size violin, along with two violins made with half and 3/4 of the normal length.<sup>2</sup> Let's study the surface area of the front panels of the three violins.

Consider the square in the interior of the panel of the full-size violin. In the 3/4-size violin, its height and width are both smaller by a factor of 3/4, so the area of the corresponding, smaller square becomes  $3/4 \times 3/4 = 9/16$  of the original area, not 3/4 of the original area. Similarly, the corresponding square on the smallest violin has half the height and half the width of the original one, so its area is 1/4 the original area, not half.

The same reasoning works for parts of the panel near the edge, such as the part that only partially fills in the other square. The entire square scales down the same as a square in the interior, and in each violin the same fraction (about 70%) of the square is full, so the contribution of this part to the total area scales down just the same.

<sup>2</sup>The customary terms "half-size" and "3/4-size" actually don't describe the sizes in any accurate way. They're really just standard, arbitrary marketing labels.



i / The area of a shape is proportional to the square of its linear dimensions, even if the shape is irregular.

Since any small square region or any small region covering part of a square scales down like a square object, the entire surface area of an irregularly shaped object changes in the same manner as the surface area of a square: scaling it down by  $3/4$  reduces the area by a factor of  $9/16$ , and so on.

In general, we can see that any time there are two objects with the same shape, but different linear dimensions (i.e., one looks like a reduced photo of the other), the ratio of their areas equals the ratio of the squares of their linear dimensions:

$$\frac{A_1}{A_2} = \left( \frac{L_1}{L_2} \right)^2 .$$

Note that it doesn't matter where we choose to measure the linear size,  $L$ , of an object. In the case of the violins, for instance, it could have been measured vertically, horizontally, diagonally, or even from the bottom of the left f-hole to the middle of the right f-hole. We just have to measure it in a consistent way on each violin. Since all the parts are assumed to shrink or expand in the same manner, the ratio  $L_1/L_2$  is independent of the choice of measurement.

It is also important to realize that it is completely unnecessary to have a formula for the area of a violin. It is only possible to derive simple formulas for the areas of certain shapes like circles, rectangles, triangles and so on, but that is no impediment to the type of reasoning we are using.

Sometimes it is inconvenient to write all the equations in terms of ratios, especially when more than two objects are being compared. A more compact way of rewriting the previous equation is

$$A \propto L^2 .$$

The symbol “ $\propto$ ” means “is proportional to.” Scientists and engineers often speak about such relationships verbally using the phrases “scales like” or “goes like,” for instance “area goes like length squared.”

All of the above reasoning works just as well in the case of volume. Volume goes like length cubed:

$$V \propto L^3 .$$

#### *self-check B*

When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost  $C$  per kilometer of travel is related to the weight per axle  $w$  by  $C \propto w^4$ . Translate this into a statement about ratios. ▷ Answer, p. 530

If different objects are made of the same material with the same density,  $\rho = m/V$ , then their masses,  $m = \rho V$ , are proportional to  $L^3$ . (The symbol for density is  $\rho$ , the lower-case Greek letter “rho.”)



j/ The muffin comes out of the oven too hot to eat. Breaking it up into four pieces increases its surface area while keeping the total volume the same. It cools faster because of the greater surface-to-volume ratio. In general, smaller things have greater surface-to-volume ratios, but in this example there is no easy way to compute the effect exactly, because the small pieces aren't the same shape as the original muffin.



An important point is that all of the above reasoning about scaling only applies to objects that are the same shape. For instance, a piece of paper is larger than a pencil, but has a much greater surface-to-volume ratio.

#### Scaling of the area of a triangle

#### example 1

▷ In figure k, the larger triangle has sides twice as long. How many times greater is its area?

Correct solution #1: Area scales in proportion to the square of the linear dimensions, so the larger triangle has four times more area ( $2^2 = 4$ ).

Correct solution #2: You could cut the larger triangle into four of the smaller size, as shown in fig. (b), so its area is four times greater. (This solution is correct, but it would not work for a shape like a circle, which can't be cut up into smaller circles.)

Correct solution #3: The area of a triangle is given by

$A = bh/2$ , where  $b$  is the base and  $h$  is the height. The areas of the triangles are

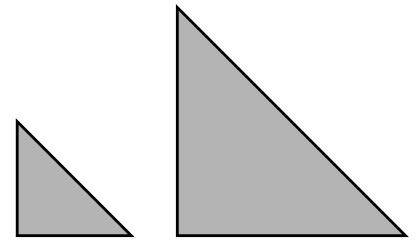
$$\begin{aligned} A_1 &= b_1 h_1 / 2 \\ A_2 &= b_2 h_2 / 2 \\ &= (2b_1)(2h_1) / 2 \\ &= 2b_1 h_1 \\ A_2 / A_1 &= (2b_1 h_1) / (b_1 h_1 / 2) \\ &= 4 \end{aligned}$$

(Although this solution is correct, it is a lot more work than solution #1, and it can only be used in this case because a triangle is a simple geometric shape, and we happen to know a formula for its area.)

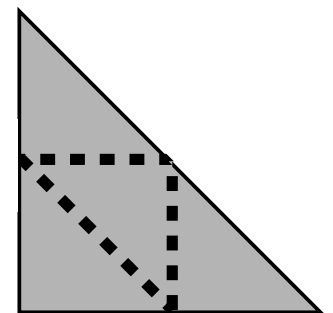
Correct solution #4: The area of a triangle is  $A = bh/2$ . The comparison of the areas will come out the same as long as the ratios of the linear sizes of the triangles is as specified, so let's just say  $b_1 = 1.00$  m and  $b_2 = 2.00$  m. The heights are then also  $h_1 = 1.00$  m and  $h_2 = 2.00$  m, giving areas  $A_1 = 0.50$  m<sup>2</sup> and  $A_2 = 2.00$  m<sup>2</sup>, so  $A_2/A_1 = 4.00$ .

(The solution is correct, but it wouldn't work with a shape for whose area we don't have a formula. Also, the numerical calculation might make the answer of 4.00 appear inexact, whereas solution #1 makes it clear that it is exactly 4.)

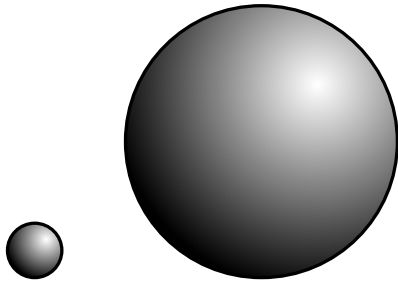
Incorrect solution: The area of a triangle is  $A = bh/2$ , and if you plug in  $b = 2.00$  m and  $h = 2.00$  m, you get  $A = 2.00$  m<sup>2</sup>, so the bigger triangle has 2.00 times more area. (This solution is incorrect because no comparison has been made with the smaller triangle.)



k / Example 1. The big triangle has four times more area than the little one.



l / A tricky way of solving example 1, explained in solution #2.



m / Example 2. The big sphere has 125 times more volume than the little one.

### Scaling of the volume of a sphere

example 2

▷ In figure m, the larger sphere has a radius that is five times greater. How many times greater is its volume?

Correct solution #1: Volume scales like the third power of the linear size, so the larger sphere has a volume that is 125 times greater ( $5^3 = 125$ ).

Correct solution #2: The volume of a sphere is  $V = (4/3)\pi r^3$ , so

$$\begin{aligned} V_1 &= \frac{4}{3}\pi r_1^3 \\ V_2 &= \frac{4}{3}\pi r_2^3 \\ &= \frac{4}{3}\pi(5r_1)^3 \\ &= \frac{500}{3}\pi r_1^3 \\ V_2/V_1 &= \left(\frac{500}{3}\pi r_1^3\right) / \left(\frac{4}{3}\pi r_1^3\right) = 125 \end{aligned}$$

Incorrect solution: The volume of a sphere is  $V = (4/3)\pi r^3$ , so

$$\begin{aligned} V_1 &= \frac{4}{3}\pi r_1^3 \\ V_2 &= \frac{4}{3}\pi r_2^3 \\ &= \frac{4}{3}\pi \cdot 5r_1^3 \\ &= \frac{20}{3}\pi r_1^3 \\ V_2/V_1 &= \left(\frac{20}{3}\pi r_1^3\right) / \left(\frac{4}{3}\pi r_1^3\right) = 5 \end{aligned}$$

(The solution is incorrect because  $(5r_1)^3$  is not the same as  $5r_1^3$ .)



n / Example 3. The 48-point “S” has 1.78 times more area than the 36-point “S.”

### Scaling of a more complex shape

example 3

▷ The first letter “S” in figure n is in a 36-point font, the second in 48-point. How many times more ink is required to make the larger “S”? (Points are a unit of length used in typography.)

Correct solution: The amount of ink depends on the area to be covered with ink, and area is proportional to the square of the linear dimensions, so the amount of ink required for the second “S” is greater by a factor of  $(48/36)^2 = 1.78$ .

Incorrect solution: The length of the curve of the second “S” is longer by a factor of  $48/36 = 1.33$ , so 1.33 times more ink is required.

(The solution is wrong because it assumes incorrectly that the width of the curve is the same in both cases. Actually both the

width and the length of the curve are greater by a factor of 48/36, so the area is greater by a factor of  $(48/36)^2 = 1.78$ .)

Reasoning about ratios and proportionalities is one of the three essential mathematical skills, summarized on pp.514-515, that you need for success in this course.

▷ *Solved problem: a telescope gathers light*      page 59, problem 11

▷ *Solved problem: distance from an earthquake*      page 59, problem 12

### Discussion questions

**A** A toy fire engine is 1/30 the size of the real one, but is constructed from the same metal with the same proportions. How many times smaller is its weight? How many times less red paint would be needed to paint it?

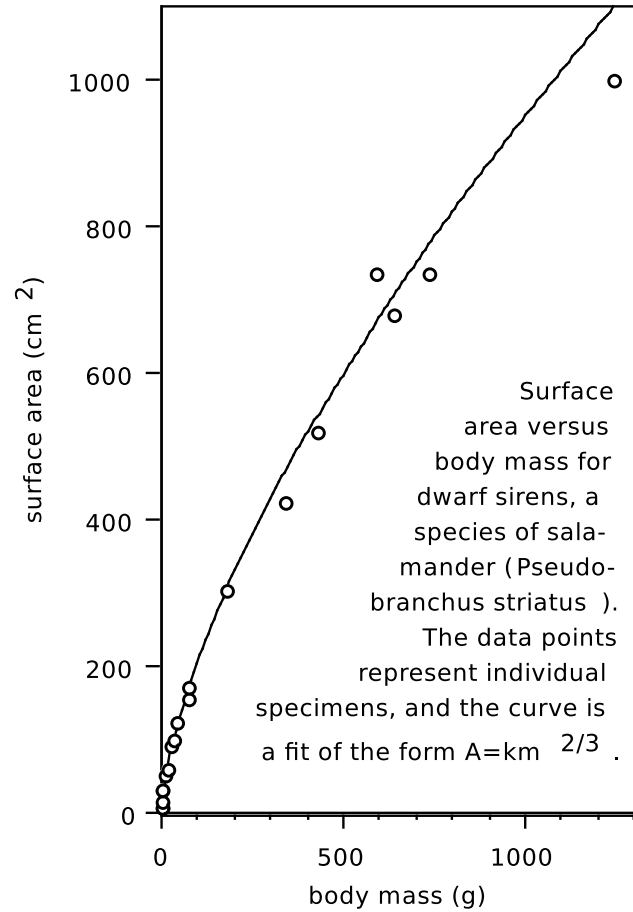
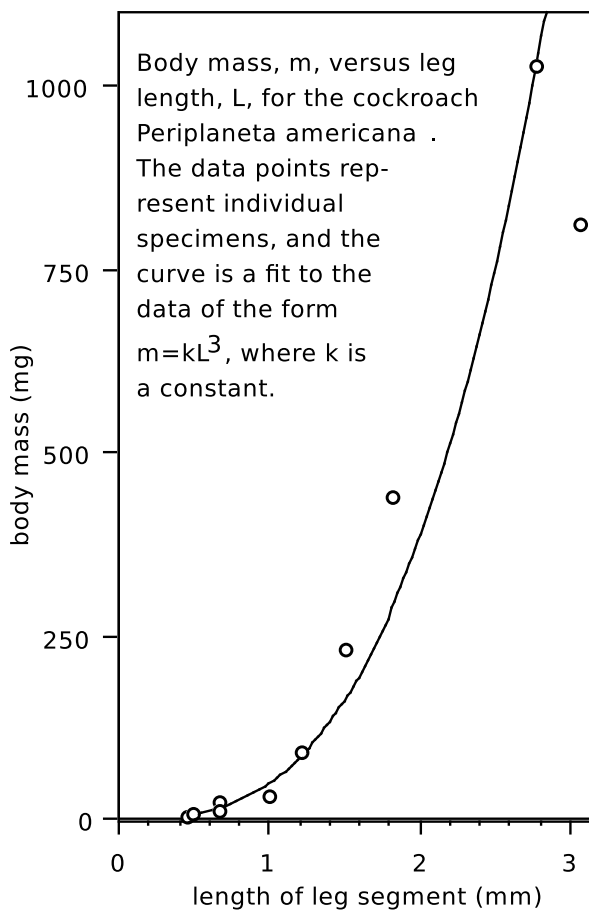
**B** Galileo spends a lot of time in his dialog discussing what really happens when things break. He discusses everything in terms of Aristotle's now-discredited explanation that things are hard to break, because if something breaks, there has to be a gap between the two halves with nothing in between, at least initially. Nature, according to Aristotle, "abhors a vacuum," i.e., nature doesn't "like" empty space to exist. Of course, air will rush into the gap immediately, but at the very moment of breaking, Aristotle imagined a vacuum in the gap. Is Aristotle's explanation of why it is hard to break things an experimentally testable statement? If so, how could it be tested experimentally?

## 1.3 ★ Scaling applied to biology

### Organisms of different sizes with the same shape

The left-hand panel in figure o shows the approximate validity of the proportionality  $m \propto L^3$  for cockroaches (redrawn from McMahon and Bonner). The scatter of the points around the curve indicates that some cockroaches are proportioned slightly differently from others, but in general the data seem well described by  $m \propto L^3$ . That means that the largest cockroaches the experimenter could raise (is there a 4-H prize?) had roughly the same shape as the smallest ones.

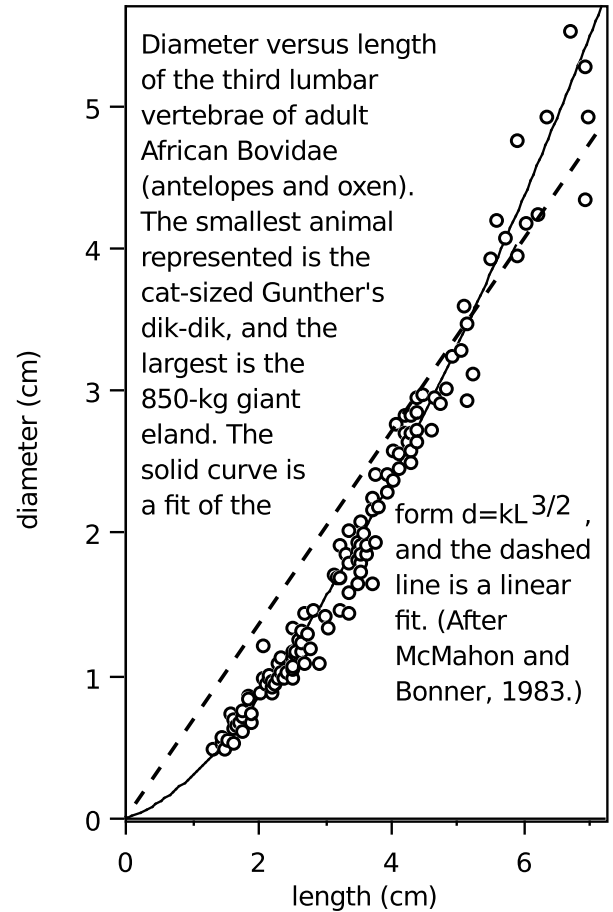
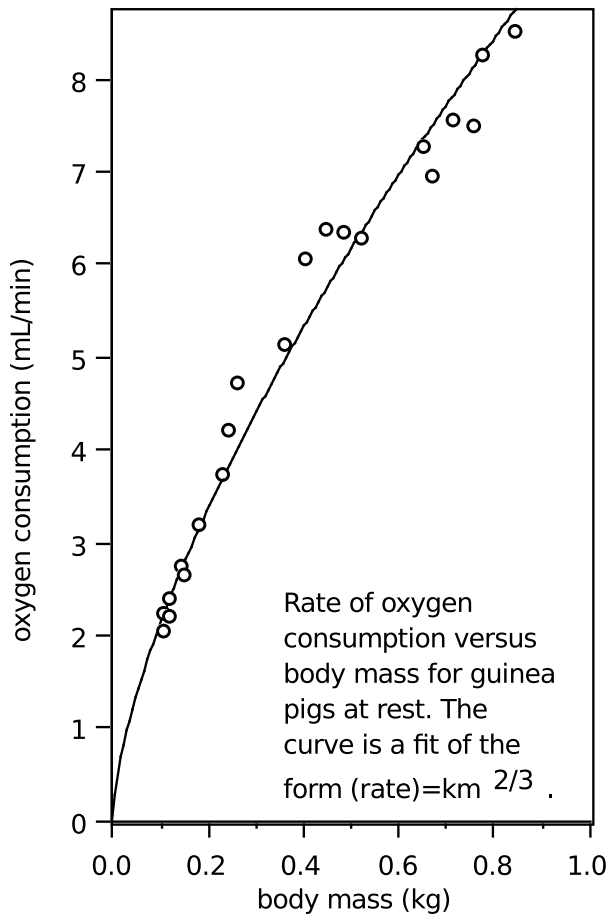
Another relationship that should exist for animals of different sizes shaped in the same way is that between surface area and body mass. If all the animals have the same average density, then body mass should be proportional to the cube of the animal's linear size,  $m \propto L^3$ , while surface area should vary proportionately to  $L^2$ . Therefore, the animals' surface areas should be proportional to  $m^{2/3}$ . As shown in the right-hand panel of figure o, this relationship appears to hold quite well for the dwarf siren, a type of salamander. Notice how the curve bends over, meaning that the surface area does not increase as quickly as body mass, e.g., a salamander with eight



o / Geometrical scaling of animals.

times more body mass will have only four times more surface area.

This behavior of the ratio of surface area to mass (or, equivalently, the ratio of surface area to volume) has important consequences for mammals, which must maintain a constant body temperature. It would make sense for the rate of heat loss through the animal's skin to be proportional to its surface area, so we should expect small animals, having large ratios of surface area to volume, to need to produce a great deal of heat in comparison to their size to avoid dying from low body temperature. This expectation is borne out by the data of the left-hand panel of figure p, showing the rate of oxygen consumption of guinea pigs as a function of their body mass. Neither an animal's heat production nor its surface area is convenient to measure, but in order to produce heat, the animal must metabolize oxygen, so oxygen consumption is a good indicator of the rate of heat production. Since surface area is proportional to  $m^{2/3}$ , the proportionality of the rate of oxygen consumption to  $m^{2/3}$  is consistent with the idea that the animal needs to produce heat at a rate in proportion to its surface area. Although the smaller animals

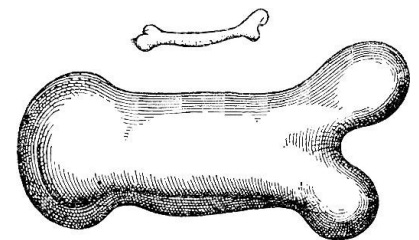


p / Scaling of animals' bodies related to metabolic rate and skeletal strength.

metabolize less oxygen and produce less heat in absolute terms, the amount of food and oxygen they must consume is greater in proportion to their own mass. The Etruscan pigmy shrew, weighing in at 2 grams as an adult, is at about the lower size limit for mammals. It must eat continually, consuming many times its body weight each day to survive.

### Changes in shape to accommodate changes in size

Large mammals, such as elephants, have a small ratio of surface area to volume, and have problems getting rid of their heat fast enough. An elephant cannot simply eat small enough amounts to keep from producing excessive heat, because cells need to have a certain minimum metabolic rate to run their internal machinery. Hence the elephant's large ears, which add to its surface area and help it to cool itself. Previously, we have seen several examples of data within a given species that were consistent with a fixed shape, scaled up and down in the cases of individual specimens. The elephant's ears are an example of a change in shape necessitated by a change in scale.



q / Galileo's original drawing, showing how larger animals' bones must be greater in diameter compared to their lengths.

Large animals also must be able to support their own weight. Returning to the example of the strengths of planks of different sizes, we can see that if the strength of the plank depends on area while its weight depends on volume, then the ratio of strength to weight goes as follows:

$$\text{strength/weight} \propto A/V \propto 1/L \quad .$$

Thus, the ability of objects to support their own weights decreases inversely in proportion to their linear dimensions. If an object is to be just barely able to support its own weight, then a larger version will have to be proportioned differently, with a different shape.

Since the data on the cockroaches seemed to be consistent with roughly similar shapes within the species, it appears that the ability to support its own weight was not the tightest design constraint that Nature was working under when she designed them. For large animals, structural strength is important. Galileo was the first to quantify this reasoning and to explain why, for instance, a large animal must have bones that are thicker in proportion to their length. Consider a roughly cylindrical bone such as a leg bone or a vertebra. The length of the bone,  $L$ , is dictated by the overall linear size of the animal, since the animal's skeleton must reach the animal's whole length. We expect the animal's mass to scale as  $L^3$ , so the strength of the bone must also scale as  $L^3$ . Strength is proportional to cross-sectional area, as with the wooden planks, so if the diameter of the bone is  $d$ , then

$$d^2 \propto L^3$$

or

$$d \propto L^{3/2} \quad .$$

If the shape stayed the same regardless of size, then all linear dimensions, including  $d$  and  $L$ , would be proportional to one another. If our reasoning holds, then the fact that  $d$  is proportional to  $L^{3/2}$ , not  $L$ , implies a change in proportions of the bone. As shown in the right-hand panel of figure p, the vertebrae of African Bovidae follow the rule  $d \propto L^{3/2}$  fairly well. The vertebrae of the giant eland are as chunky as a coffee mug, while those of a Gunther's dik-dik are as slender as the cap of a pen.

### Discussion questions

**A** Single-celled animals must passively absorb nutrients and oxygen from their surroundings, unlike humans who have lungs to pump air in and out and a heart to distribute the oxygenated blood throughout their bodies. Even the cells composing the bodies of multicellular animals must absorb oxygen from a nearby capillary through their surfaces. Based on these facts, explain why cells are always microscopic in size.

**B** The reasoning of the previous question would seem to be contradicted by the fact that human nerve cells in the spinal cord can be as much as a meter long, although their widths are still very small. Why is this possible?



## 1.4 Order-of-magnitude estimates

It is the mark of an instructed mind to rest satisfied with the degree of precision that the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible.

*Aristotle*

It is a common misconception that science must be exact. For instance, in the Star Trek TV series, it would often happen that Captain Kirk would ask Mr. Spock, “Spock, we’re in a pretty bad situation. What do you think are our chances of getting out of here?” The scientific Mr. Spock would answer with something like, “Captain, I estimate the odds as 237.345 to one.” In reality, he could not have estimated the odds with six significant figures of accuracy, but nevertheless one of the hallmarks of a person with a good education in science is the ability to make estimates that are likely to be at least somewhere in the right ballpark. In many such situations, it is often only necessary to get an answer that is off by no more than a factor of ten in either direction. Since things that differ by a factor of ten are said to differ by one order of magnitude, such an estimate is called an order-of-magnitude estimate. The tilde,  $\sim$ , is used to indicate that things are only of the same order of magnitude, but not exactly equal, as in

odds of survival  $\sim 100$  to one .

The tilde can also be used in front of an individual number to emphasize that the number is only of the right order of magnitude.

Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it’s a mode of reasoning that is completely unfamiliar to most college students. Some of the typical mental steps can be illustrated in the following example.

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*Cost of transporting tomatoes (incorrect solution)      example 4*

▷ Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?

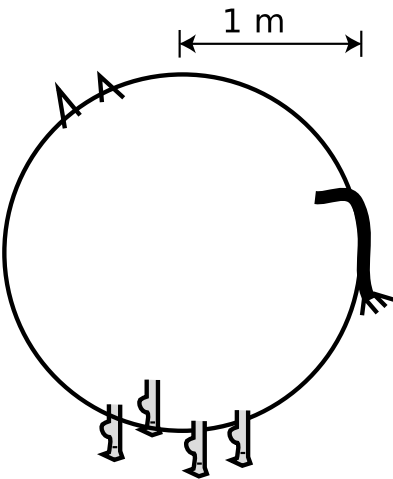
▷ The following incorrect solution illustrates one of the main ways you can go wrong in order-of-magnitude estimates.

Incorrect solution: Let’s say the trucker needs to make a \$400 profit on the trip. Taking into account her benefits, the cost of gas, and maintenance and payments on the truck, let’s say the total cost is more like \$2000. I’d guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself. Transportation really adds a lot to the cost of produce, I guess.

The problem is that the human brain is not very good at estimating area or volume, so it turns out the estimate of 5000 tomatoes



r / Can you guess how many jelly beans are in the jar? If you try to guess directly, you will almost certainly underestimate. The right way to do it is to estimate the linear dimensions, then get the volume indirectly. See problem 26, p. 62.



s / Consider a spherical cow.

fitting in the truck is way off. That's why people have a hard time at those contests where you are supposed to estimate the number of jellybeans in a big jar. Another example is that most people think their families use about 10 gallons of water per day, but in reality the average is about 300 gallons per day. When estimating area or volume, you are much better off estimating linear dimensions, and computing volume from the linear dimensions. Here's a better solution to the problem about the tomato truck:

*Cost of transporting tomatoes (correct solution)* *example 5*

As in the previous solution, say the cost of the trip is \$2000. The dimensions of the bin are probably  $4\text{ m} \times 2\text{ m} \times 1\text{ m}$ , for a volume of  $8\text{ m}^3$ . Since the whole thing is just an order-of-magnitude estimate, let's round that off to the nearest power of ten,  $10\text{ m}^3$ . The shape of a tomato is complicated, and I don't know any formula for the volume of a tomato shape, but since this is just an estimate, let's pretend that a tomato is a cube,  $0.05\text{ m} \times 0.05\text{ m} \times 0.05\text{ m}$ , for a volume of  $1.25 \times 10^{-4}\text{ m}^3$ . Since this is just a rough estimate, let's round that to  $10^{-4}\text{ m}^3$ . We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato:  $10\text{ m}^3 / 10^{-4}\text{ m}^3 = 10^5$  tomatoes. The transportation cost per tomato is  $\$2000 / 10^5 \text{ tomatoes} = \$0.02/\text{tomato}$ . That means that transportation really doesn't contribute very much to the cost of a tomato.

Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates. A similar situation would occur if you were trying to estimate how many  $\text{m}^2$  of leather could be produced from a herd of ten thousand cattle. There is no point in trying to take into account the shape of the cows' bodies. A reasonable plan of attack might be to consider a spherical cow. Probably a cow has roughly the same surface area as a sphere with a radius of about 1 m, which would be  $4\pi(1\text{ m})^2$ . Using the well-known facts that pi equals three, and four times three equals about ten, we can guess that a cow has a surface area of about  $10\text{ m}^2$ , so the herd as a whole might yield  $10^5\text{ m}^2$  of leather.

*Estimating mass indirectly* *example 6*

Usually the best way to estimate mass is to estimate linear dimensions, then use those to infer volume, and then get the mass based on the volume. For example, *Amphicoelias*, shown in the figure, may have been the largest land animal ever to live. Fossils tell us the linear dimensions of an animal, but we can only indirectly guess its mass. Given the length scale in the figure, let's estimate the mass of an *Amphicoelias*.

Its torso looks like it can be approximated by a rectangular box with dimensions  $10\text{ m} \times 5\text{ m} \times 3\text{ m}$ , giving about  $2 \times 10^2\text{ m}^3$ . Living things are mostly made of water, so we assume the animal to have the density of water,  $1\text{ g/cm}^3$ , which converts to  $10^3\text{ kg/m}^3$ .

This gives a mass of about  $2 \times 10^5$  kg, or 200 metric tons.



The following list summarizes the strategies for getting a good order-of-magnitude estimate.

1. Don't even attempt more than one significant figure of precision.
2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them.
3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.
4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield  $0.01 \text{ m}^2$  of leather, then you have probably made a mistake with conversion factors somewhere.

# Summary

## Notation

- $\propto$  . . . . . is proportional to
- $\sim$  . . . . . on the order of, is on the order of

## Summary

Nature behaves differently on large and small scales. Galileo showed that this results fundamentally from the way area and volume scale. Area scales as the second power of length,  $A \propto L^2$ , while volume scales as length to the third power,  $V \propto L^3$ .

An order of magnitude estimate is one in which we do not attempt or expect an exact answer. The main reason why the uninitiated have trouble with order-of-magnitude estimates is that the human brain does not intuitively make accurate estimates of area and volume. Estimates of area and volume should be approached by first estimating linear dimensions, which one's brain has a feel for.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** How many cubic inches are there in a cubic foot? The answer is not 12. ✓

**2** Assume a dog's brain is twice as great in diameter as a cat's, but each animal's brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

**3** The population density of Los Angeles is about 4000 people/km<sup>2</sup>. That of San Francisco is about 6000 people/km<sup>2</sup>. How many times farther away is the average person's nearest neighbor in LA than in San Francisco? The answer is not 1.5. ✓

**4** A hunting dog's nose has about 10 square inches of active surface. How is this possible, since the dog's nose is only about 1 in × 1 in × 1 in = 1 in<sup>3</sup>? After all, 10 is greater than 1, so how can it fit?

**5** Estimate the number of blades of grass on a football field.

**6** In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

**7** Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world's population to live.

**8** A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

**9** Estimate the volume of a human body, in cm<sup>3</sup>.

**10** How many cm<sup>2</sup> is 1 mm<sup>2</sup>? ▷ Solution, p. 516

**11** Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope. ▷ Solution, p. 516

**12** One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out

from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters. ▷ Solution, p. 517

**13** In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box? ✓

**14** Estimate the mass of one of the hairs in Albert Einstein's moustache, in units of kg.

**15** According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar's last words. Is this true? If so, how many?

**16** The Earth's surface is about 70% water. Mars's diameter is about half the Earth's, but it has no surface water. Compare the land areas of the two planets. ✓

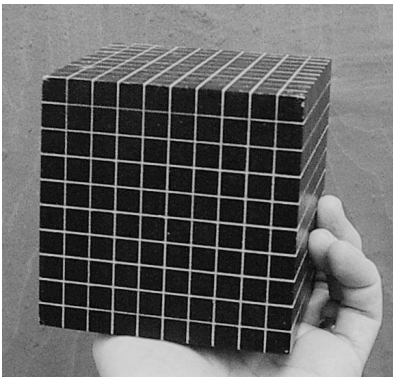
**17** The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth? ▷ Solution, p. 517

**18** The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD's area is therefore lost? ✓

**19** The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes? ▷ Solution, p. 517



Albert Einstein, and his moustache, problem 14.



Problem 19.

**20** [This problem is now problem 0-12 on p. 38.]

**21** Estimate the number of man-hours required for building the Great Wall of China.  
▷ Solution, p. 517

**22** (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is  $1\text{ }\mu\text{m}$ . Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of  $10^{13}$ . (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?



Problem 22.

**23** A taxon (plural taxa) is a group of living things. For example, *Homo sapiens* and *Homo neanderthalensis* are both taxa — specifically, they are two different species within the genus *Homo*. Surveys by botanists show that the number of plant taxa native to a given contiguous land area  $A$  is usually approximately proportional to  $A^{1/3}$ . (a) There are 70 different species of lupine native to Southern California, which has an area of about  $200,000\text{ km}^2$ . The San Gabriel Mountains cover about  $1,600\text{ km}^2$ . Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with?  
▷ Answer, p. 535 ✓  
(b) What is the interpretation of the fact that the exponent,  $1/3$ , is less than one?



**24** X-ray images aren't only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other. ✓



Problem 25.

**25** Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I've indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don't have similar photos of our own Milky Way galaxy, because we can't see it from the outside.) So far we haven't received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010? ✓

**26** Estimate the number of jellybeans in figure r on p. 56.

▷ Solution, p. 517



Problem 27.

**27** At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let's call this fraction  $f$ . Mathematicians have proved that the best possible result is  $f \approx 0.7405$ , which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has  $f \approx 0.64$ . Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box? ✓

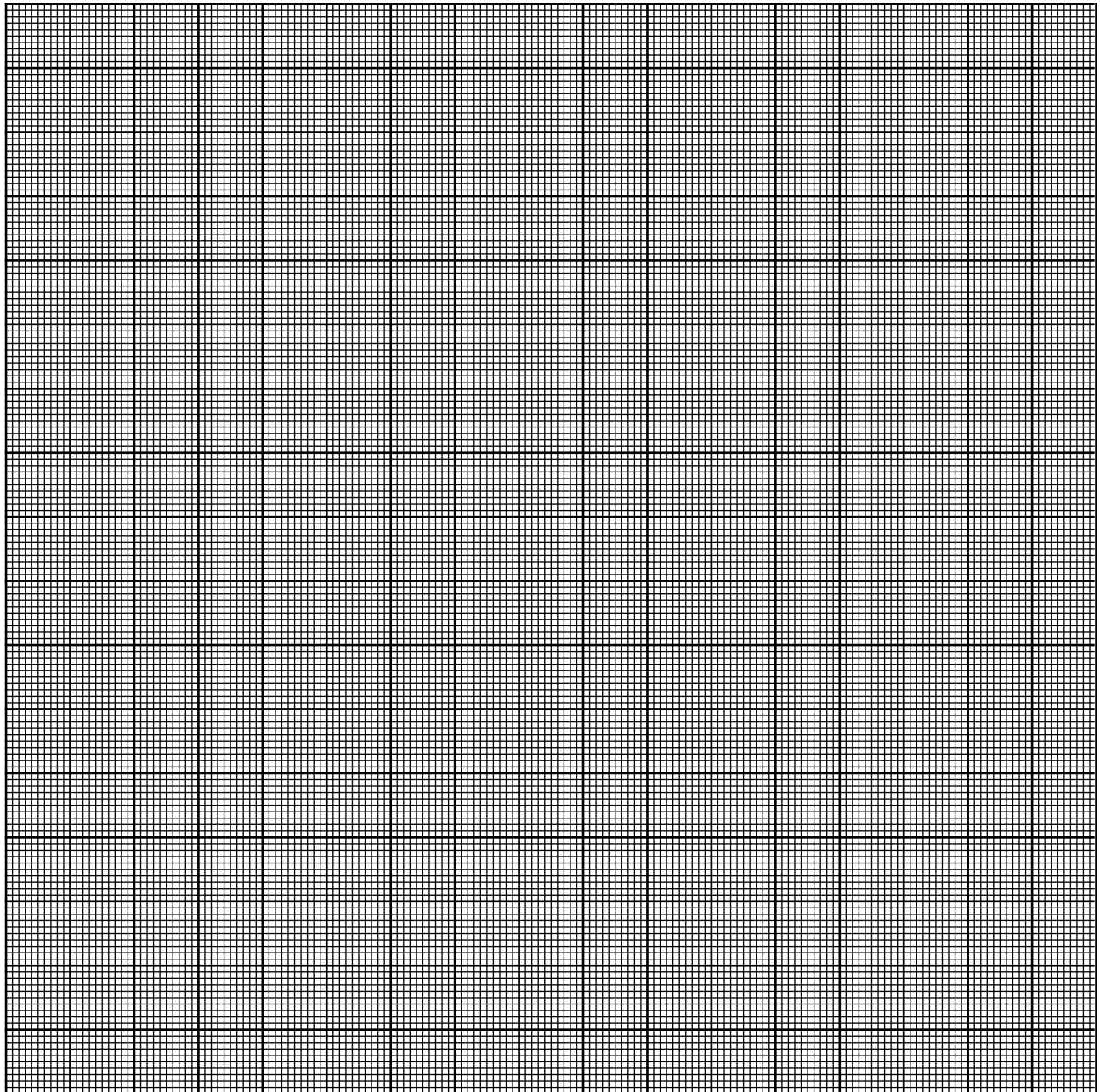
**28** Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world's waste stockpiles have about  $10^3$  metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than  $2 \times 10^{-13}$  g/cm<sup>3</sup> of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world's inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world's oceans, thereby becoming mixed uniformly into the world's water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.

## Exercise 1: Scaling applied to leaves

Equipment:

leaves of three sizes, having roughly similar proportions of length, width, and thickness  
balance

Each group will have one leaf, and should measure its surface area and volume, and determine its surface-to-volume ratio. For consistency, every group should use units of  $\text{cm}^2$  and  $\text{cm}^3$ , and should only find the area of one side of the leaf. The area can be found by tracing the area of the leaf on graph paper and counting squares. The volume can be found by weighing the leaf and assuming that its density is  $1 \text{ g/cm}^3$  (the density of water). What implications do your results have for the plants' abilities to survive in different environments?



# **Motion in one dimension**



# Chapter 2

## Velocity and relative motion

### 2.1 Types of motion

If you had to think consciously in order to move your body, you would be severely disabled. Even walking, which we consider to be no great feat, requires an intricate series of motions that your cerebrum would be utterly incapable of coordinating. The task of putting one foot in front of the other is controlled by the more primitive parts of your brain, the ones that have not changed much since the mammals and reptiles went their separate evolutionary ways. The thinking part of your brain limits itself to general directives such as “walk faster,” or “don’t step on her toes,” rather than micromanaging every contraction and relaxation of the hundred or so muscles of your hips, legs, and feet.

Physics is all about the conscious understanding of motion, but we’re obviously not immediately prepared to understand the most complicated types of motion. Instead, we’ll use the divide-and-conquer technique. We’ll first classify the various types of motion, and then begin our campaign with an attack on the simplest cases. To make it clear what we are and are not ready to consider, we need to examine and define carefully what types of motion can exist.

#### **Rigid-body motion distinguished from motion that changes an object’s shape**

Nobody, with the possible exception of Fred Astaire, can simply glide forward without bending their joints. Walking is thus an example in which there is both a general motion of the whole object and a change in the shape of the object. Another example is the motion of a jiggling water balloon as it flies through the air. We are not presently attempting a mathematical description of the way in which the shape of an object changes. Motion without a change in shape is called rigid-body motion. (The word “body” is often used in physics as a synonym for “object.”)

#### **Center-of-mass motion as opposed to rotation**

A ballerina leaps into the air and spins around once before landing. We feel intuitively that her rigid-body motion while her feet are off the ground consists of two kinds of motion going on simul-



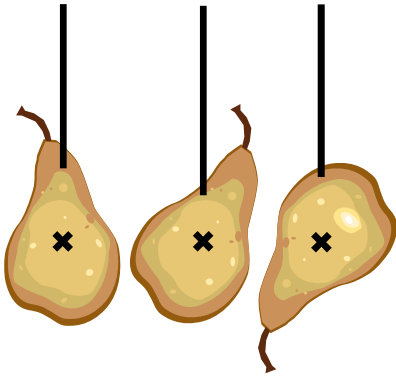
a / Rotation.



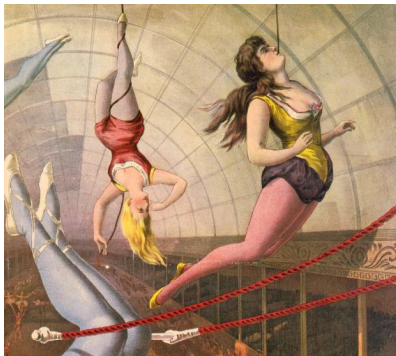
b / Simultaneous rotation and motion through space.



c / One person might say that the tipping chair was only rotating in a circle about its point of contact with the floor, but another could describe it as having both rotation and motion through space.

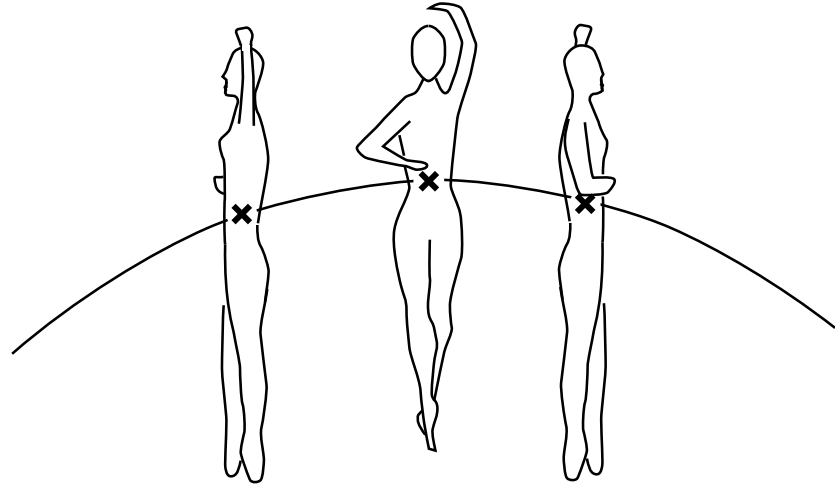


e / No matter what point you hang the pear from, the string lines up with the pear's center of mass. The center of mass can therefore be defined as the intersection of all the lines made by hanging the pear in this way. Note that the X in the figure should not be interpreted as implying that the center of mass is on the surface — it is actually inside the pear.



f / The circus performers hang with the ropes passing through their centers of mass.

taneously: a rotation and a motion of her body as a whole through space, along an arc. It is not immediately obvious, however, what is the most useful way to define the distinction between rotation and motion through space. Imagine that you attempt to balance a chair and it falls over. One person might say that the only motion was a rotation about the chair's point of contact with the floor, but another might say that there was both rotation and motion down and to the side.



— X — center of mass

d / The leaping dancer's motion is complicated, but the motion of her center of mass is simple.

It turns out that there is one particularly natural and useful way to make a clear definition, but it requires a brief digression. Every object has a balance point, referred to in physics as the *center of mass*. For a two-dimensional object such as a cardboard cutout, the center of mass is the point at which you could hang the object from a string and make it balance. In the case of the ballerina (who is likely to be three-dimensional unless her diet is particularly severe), it might be a point either inside or outside her body, depending on how she holds her arms. Even if it is not practical to attach a string to the balance point itself, the center of mass can be defined as shown in figure e.

Why is the center of mass concept relevant to the question of classifying rotational motion as opposed to motion through space? As illustrated in figures d and g, it turns out that the motion of an object's center of mass is nearly always far simpler than the motion of any other part of the object. The ballerina's body is a large object with a complex shape. We might expect that her motion would be much more complicated than the motion of a small, simply-shaped

object, say a marble, thrown up at the same angle as the angle at which she leapt. But it turns out that the motion of the ballerina's center of mass is exactly the same as the motion of the marble. That is, the motion of the center of mass is the same as the motion the ballerina would have if all her mass was concentrated at a point. By restricting our attention to the motion of the center of mass, we can therefore simplify things greatly.



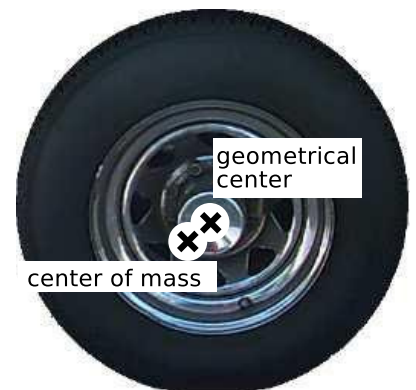
g / The same leaping dancer, viewed from above. Her center of mass traces a straight line, but a point away from her center of mass, such as her elbow, traces the much more complicated path shown by the dots.

We can now replace the ambiguous idea of “motion as a whole through space” with the more useful and better defined concept of “center-of-mass motion.” The motion of any rigid body can be cleanly split into rotation and center-of-mass motion. By this definition, the tipping chair does have both rotational and center-of-mass motion. Concentrating on the center of mass motion allows us to make a simplified model of the motion, as if a complicated object like a human body was just a marble or a point-like particle. Science really never deals with reality; it deals with models of reality.

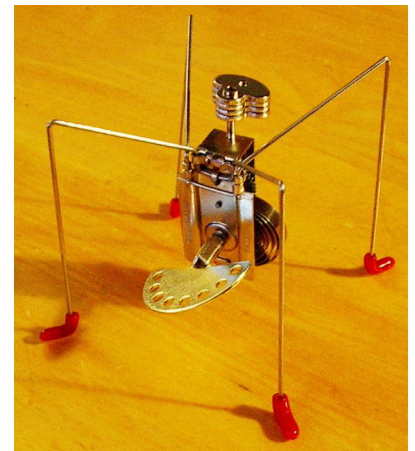
Note that the word “center” in “center of mass” is not meant to imply that the center of mass must lie at the geometrical center of an object. A car wheel that has not been balanced properly has a center of mass that does not coincide with its geometrical center. An object such as the human body does not even have an obvious geometrical center.

It can be helpful to think of the center of mass as the average location of all the mass in the object. With this interpretation, we can see for example that raising your arms above your head raises your center of mass, since the higher position of the arms' mass raises the average. We won't be concerned right now with calculating centers of mass mathematically; the relevant equations are in ch. 14.

Ballerinas and professional basketball players can create an illusion of flying horizontally through the air because our brains intuitively expect them to have rigid-body motion, but the body does not stay rigid while executing a grand jete or a slam dunk. The legs are low at the beginning and end of the jump, but come up higher at

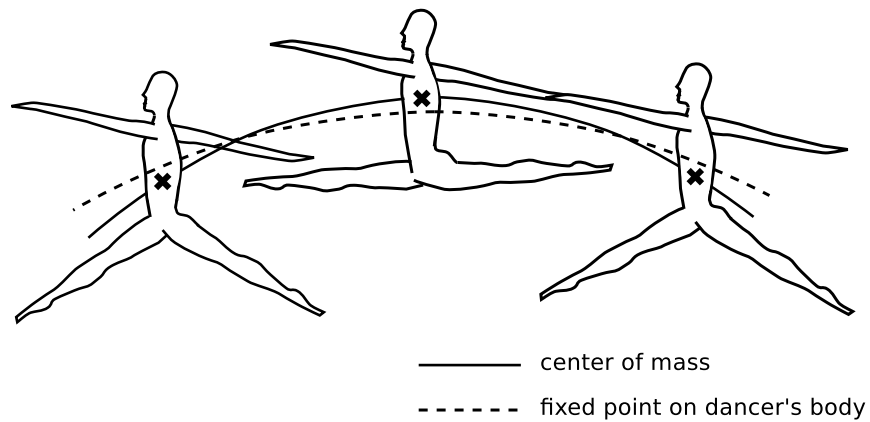


h / An improperly balanced wheel has a center of mass that is not at its geometrical center. When you get a new tire, the mechanic clamps little weights to the rim to balance the wheel.



i / This toy was intentionally designed so that the mushroom-shaped piece of metal on top would throw off the center of mass. When you wind it up, the mushroom spins, but the center of mass doesn't want to move, so the rest of the toy tends to counter the mushroom's motion, causing the whole thing to jump around.

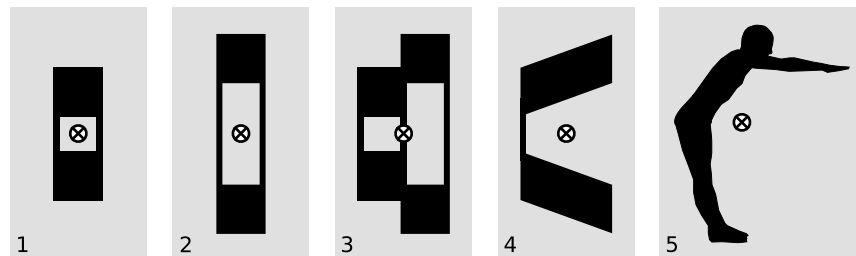




j / A fixed point on the dancer's body follows a trajectory that is flatter than what we expect, creating an illusion of flight.

the middle. Regardless of what the limbs do, the center of mass will follow the same arc, but the low position of the legs at the beginning and end means that the torso is higher compared to the center of mass, while in the middle of the jump it is lower compared to the center of mass. Our eye follows the motion of the torso and tries to interpret it as the center-of-mass motion of a rigid body. But since the torso follows a path that is flatter than we expect, this attempted interpretation fails, and we experience an illusion that the person is flying horizontally.

k / Example 1.



*The center of mass as an average* example 1

▷ Explain how we know that the center of mass of each object is at the location shown in figure k.

▷ The center of mass is a sort of average, so the height of the centers of mass in 1 and 2 has to be midway between the two squares, because that height is the average of the heights of the two squares. Example 3 is a combination of examples 1 and 2, so we can find its center of mass by averaging the horizontal positions of their centers of mass. In example 4, each square has been skewed a little, but just as much mass has been moved up as down, so the average vertical position of the mass hasn't changed. Example 5 is clearly not all that different from example 4, the main difference being a slight clockwise rotation, so just as

in example 4, the center of mass must be hanging in empty space, where there isn't actually any mass. Horizontally, the center of mass must be between the heels and toes, or else it wouldn't be possible to stand without tipping over.

Another interesting example from the sports world is the high jump, in which the jumper's curved body passes over the bar, but the center of mass passes under the bar! Here the jumper lowers his legs and upper body at the peak of the jump in order to bring his waist higher compared to the center of mass.

Later in this course, we'll find that there are more fundamental reasons (based on Newton's laws of motion) why the center of mass behaves in such a simple way compared to the other parts of an object. We're also postponing any discussion of numerical methods for finding an object's center of mass. Until later in the course, we will only deal with the motion of objects' centers of mass.

### Center-of-mass motion in one dimension

In addition to restricting our study of motion to center-of-mass motion, we will begin by considering only cases in which the center of mass moves along a straight line. This will include cases such as objects falling straight down, or a car that speeds up and slows down but does not turn.

Note that even though we are not explicitly studying the more complex aspects of motion, we can still analyze the center-of-mass motion while ignoring other types of motion that might be occurring simultaneously. For instance, if a cat is falling out of a tree and is initially upside-down, it goes through a series of contortions that bring its feet under it. This is definitely not an example of rigid-body motion, but we can still analyze the motion of the cat's center of mass just as we would for a dropping rock.

#### self-check A

Consider a person running, a person pedaling on a bicycle, a person coasting on a bicycle, and a person coasting on ice skates. In which cases is the center-of-mass motion one-dimensional? Which cases are examples of rigid-body motion?

▷ Answer, p. 530

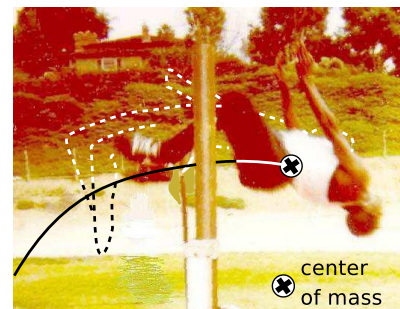
#### self-check B

The figure shows a gymnast holding onto the inside of a big wheel. From inside the wheel, how could he make it roll one way or the other?

▷ Answer, p. 531

## 2.2 Describing distance and time

Center-of-mass motion in one dimension is particularly easy to deal with because all the information about it can be encapsulated in two variables:  $x$ , the position of the center of mass relative to the origin, and  $t$ , which measures a point in time. For instance, if someone



1 / The high-jumper's body passes over the bar, but his center of mass passes under it.



m / Self-check B.

supplied you with a sufficiently detailed table of  $x$  and  $t$  values, you would know pretty much all there was to know about the motion of the object's center of mass.

### A point in time as opposed to duration

In ordinary speech, we use the word “time” in two different senses, which are to be distinguished in physics. It can be used, as in “a short time” or “our time here on earth,” to mean a length or duration of time, or it can be used to indicate a clock reading, as in “I didn’t know what time it was,” or “now’s the time.” In symbols,  $t$  is ordinarily used to mean a point in time, while  $\Delta t$  signifies an interval or duration in time. The capital Greek letter delta,  $\Delta$ , means “the change in...,” i.e. a duration in time is the change or difference between one clock reading and another. The notation  $\Delta t$  does not signify the product of two numbers,  $\Delta$  and  $t$ , but rather one single number,  $\Delta t$ . If a matinee begins at a point in time  $t = 1$  o’clock and ends at  $t = 3$  o’clock, the duration of the movie was the change in  $t$ ,

$$\Delta t = 3 \text{ hours} - 1 \text{ hour} = 2 \text{ hours} \quad .$$

To avoid the use of negative numbers for  $\Delta t$ , we write the clock reading “after” to the left of the minus sign, and the clock reading “before” to the right of the minus sign. A more specific definition of the delta notation is therefore that delta stands for “after minus before.”

Even though our definition of the delta notation guarantees that  $\Delta t$  is positive, there is no reason why  $t$  can’t be negative. If  $t$  could not be negative, what would have happened one second before  $t = 0$ ? That doesn’t mean that time “goes backward” in the sense that adults can shrink into infants and retreat into the womb. It just means that we have to pick a reference point and call it  $t = 0$ , and then times before that are represented by negative values of  $t$ . An example is that a year like 2007 A.D. can be thought of as a positive  $t$  value, while one like 370 B.C. is negative. Similarly, when you hear a countdown for a rocket launch, the phrase “t minus ten seconds” is a way of saying  $t = -10$  s, where  $t = 0$  is the time of blastoff, and  $t > 0$  refers to times after launch.

Although a point in time can be thought of as a clock reading, it is usually a good idea to avoid doing computations with expressions such as “2:35” that are combinations of hours and minutes. Times can instead be expressed entirely in terms of a single unit, such as hours. Fractions of an hour can be represented by decimals rather than minutes, and similarly if a problem is being worked in terms of minutes, decimals can be used instead of seconds.

#### *self-check C*

Of the following phrases, which refer to points in time, which refer to time intervals, and which refer to time in the abstract rather than as a measurable number?

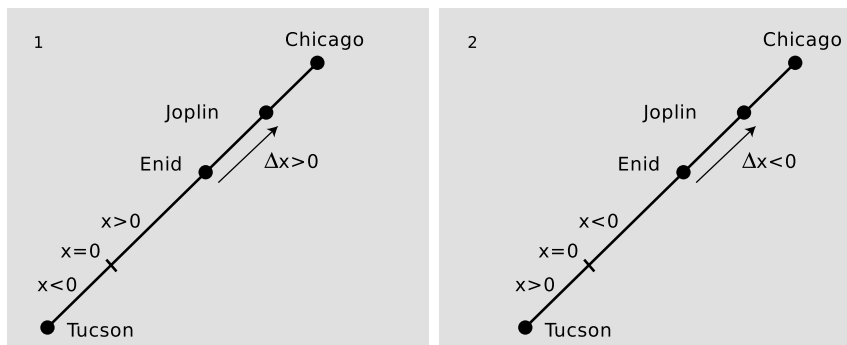
- (1) "The time has come."
- (2) "Time waits for no man."
- (3) "The whole time, he had spit on his chin." ▷ Answer, p. 531

### Position as opposed to change in position

As with time, a distinction should be made between a point in space, symbolized as a coordinate  $x$ , and a change in position, symbolized as  $\Delta x$ .

As with  $t$ ,  $x$  can be negative. If a train is moving down the tracks, not only do you have the freedom to choose any point along the tracks and call it  $x = 0$ , but it's also up to you to decide which side of the  $x = 0$  point is positive  $x$  and which side is negative  $x$ .

Since we've defined the delta notation to mean "after minus before," it is possible that  $\Delta x$  will be negative, unlike  $\Delta t$  which is guaranteed to be positive. Suppose we are describing the motion of a train on tracks linking Tucson and Chicago. As shown in the figure, it is entirely up to you to decide which way is positive.



n / Two equally valid ways of describing the motion of a train from Tucson to Chicago. In example 1, the train has a positive  $\Delta x$  as it goes from Enid to Joplin. In 2, the same train going forward in the same direction has a negative  $\Delta x$ .

Note that in addition to  $x$  and  $\Delta x$ , there is a third quantity we could define, which would be like an odometer reading, or actual distance traveled. If you drive 10 miles, make a U-turn, and drive back 10 miles, then your  $\Delta x$  is zero, but your car's odometer reading has increased by 20 miles. However important the odometer reading is to car owners and used car dealers, it is not very important in physics, and there is not even a standard name or notation for it. The change in position,  $\Delta x$ , is more useful because it is so much easier to calculate: to compute  $\Delta x$ , we only need to know the beginning and ending positions of the object, not all the information about how it got from one position to the other.

### self-check D

A ball falls vertically, hits the floor, bounces to a height of one meter, falls, and hits the floor again. Is the  $\Delta x$  between the two impacts equal to zero, one, or two meters? ▷ Answer, p. 531

## Frames of reference

The example above shows that there are two arbitrary choices you have to make in order to define a position variable,  $x$ . You have to decide where to put  $x = 0$ , and also which direction will be positive. This is referred to as choosing a coordinate system or choosing a frame of reference. (The two terms are nearly synonymous, but the first focuses more on the actual  $x$  variable, while the second is more of a general way of referring to one's point of view.) As long as you are consistent, any frame is equally valid. You just don't want to change coordinate systems in the middle of a calculation.

Have you ever been sitting in a train in a station when suddenly you notice that the station is moving backward? Most people would describe the situation by saying that you just failed to notice that the train was moving — it only seemed like the station was moving. But this shows that there is yet a third arbitrary choice that goes into choosing a coordinate system: valid frames of reference can differ from each other by moving relative to one another. It might seem strange that anyone would bother with a coordinate system that was moving relative to the earth, but for instance the frame of reference moving along with a train might be far more convenient for describing things happening inside the train.

## 2.3 Graphs of motion; velocity

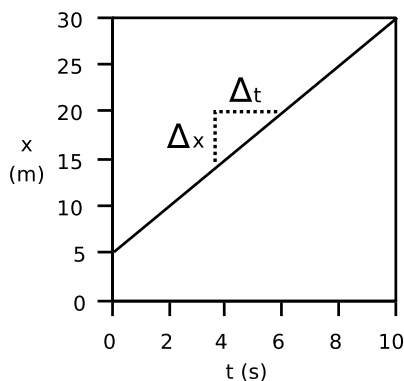
### Motion with constant velocity

In example o, an object is moving at constant speed in one direction. We can tell this because every two seconds, its position changes by five meters.

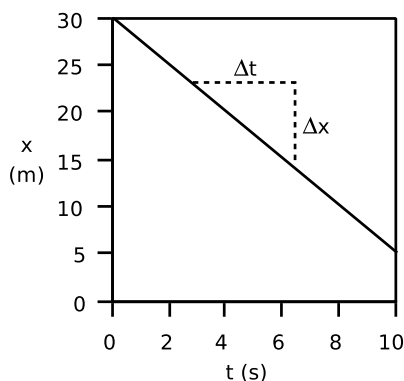
In algebra notation, we'd say that the graph of  $x$  vs.  $t$  shows the same change in position,  $\Delta x = 5.0$  m, over each interval of  $\Delta t = 2.0$  s. The object's velocity or speed is obtained by calculating  $v = \Delta x / \Delta t = (5.0 \text{ m}) / (2.0 \text{ s}) = 2.5 \text{ m/s}$ . In graphical terms, the velocity can be interpreted as the slope of the line. Since the graph is a straight line, it wouldn't have mattered if we'd taken a longer time interval and calculated  $v = \Delta x / \Delta t = (10.0 \text{ m}) / (4.0 \text{ s})$ . The answer would still have been the same, 2.5 m/s.

Note that when we divide a number that has units of meters by another number that has units of seconds, we get units of meters per second, which can be written m/s. This is another case where we treat units as if they were algebra symbols, even though they're not.

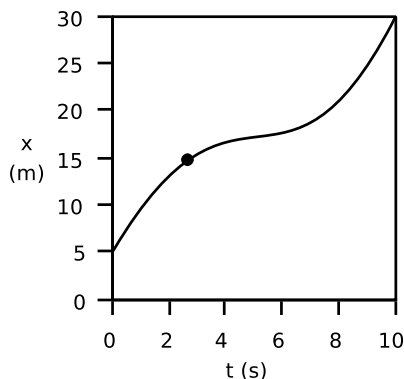
In example p, the object is moving in the opposite direction: as time progresses, its  $x$  coordinate decreases. Recalling the definition of the  $\Delta$  notation as "after minus before," we find that  $\Delta t$  is still positive, but  $\Delta x$  must be negative. The slope of the line is therefore



o / Motion with constant velocity.



p / Motion that decreases  $x$  is represented with negative values of  $\Delta x$  and  $v$ .



q / Motion with changing velocity. How can we find the velocity at the time indicated by the dot?

negative, and we say that the object has a negative velocity,  $v = \Delta x / \Delta t = (-5.0 \text{ m}) / (2.0 \text{ s}) = -2.5 \text{ m/s}$ . We've already seen that the plus and minus signs of  $\Delta x$  values have the interpretation of telling us which direction the object moved. Since  $\Delta t$  is always positive, dividing by  $\Delta t$  doesn't change the plus or minus sign, and the plus and minus signs of velocities are to be interpreted in the same way. In graphical terms, a positive slope characterizes a line that goes up as we go to the right, and a negative slope tells us that the line went down as we went to the right.

▷ *Solved problem: light-years*

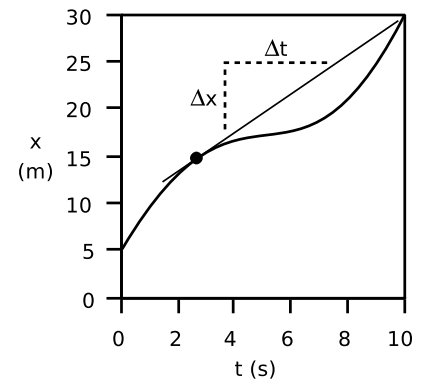
*page 88, problem 4*

## Motion with changing velocity

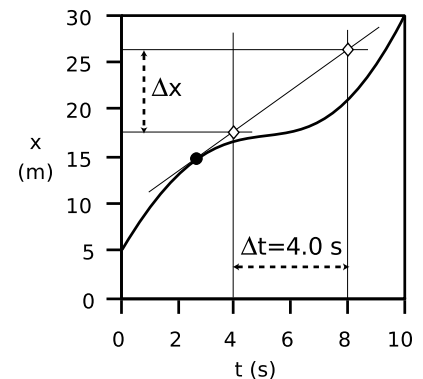
Now what about a graph like figure q? This might be a graph of a car's motion as the driver cruises down the freeway, then slows down to look at a car crash by the side of the road, and then speeds up again, disappointed that there is nothing dramatic going on such as flames or babies trapped in their car seats. (Note that we are still talking about one-dimensional motion. Just because the graph is curvy doesn't mean that the car's path is curvy. The graph is not like a map, and the horizontal direction of the graph represents the passing of time, not distance.)

Example q is similar to example o in that the object moves a total of 25.0 m in a period of 10.0 s, but it is no longer true that it makes the same amount of progress every second. There is no way to characterize the entire graph by a certain velocity or slope, because the velocity is different at every moment. It would be incorrect to say that because the car covered 25.0 m in 10.0 s, its velocity was 2.5 m/s. It moved faster than that at the beginning and end, but slower in the middle. There may have been certain instants at which the car was indeed going 2.5 m/s, but the speedometer swept past that value without "sticking," just as it swung through various other values of speed. (I definitely want my next car to have a speedometer calibrated in m/s and showing both negative and positive values.)

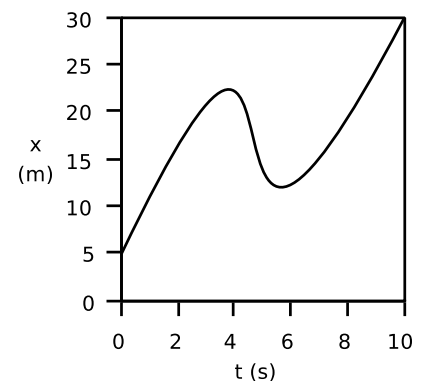
We assume that our speedometer tells us what is happening to the speed of our car at every instant, but how can we define speed mathematically in a case like this? We can't just define it as the slope of the curvy graph, because a curve doesn't have a single well-defined slope as does a line. A mathematical definition that corresponded to the speedometer reading would have to be one that attached a different velocity value to a single point on the curve, i.e., a single instant in time, rather than to the entire graph. If we wish to define the speed at one instant such as the one marked with a dot, the best way to proceed is illustrated in r, where we have drawn the line through that point called the tangent line, the line that "hugs the curve." We can then adopt the following definition of velocity:



r / The velocity at any given moment is defined as the slope of the tangent line through the relevant point on the graph.



s / Example: finding the velocity at the point indicated with the dot.



t / Reversing the direction of motion.

### definition of velocity

The velocity of an object at any given moment is the slope of the tangent line through the relevant point on its  $x - t$  graph.

One interpretation of this definition is that the velocity tells us how many meters the object would have traveled in one second, if it had continued moving at the same speed for at least one second. To some people the graphical nature of this definition seems “inaccurate” or “not mathematical.” The equation by itself, however, is only valid if the velocity is constant, and so cannot serve as a general definition.

#### *The slope of the tangent line*

*example 2*

▷ What is the velocity at the point shown with a dot on the graph?

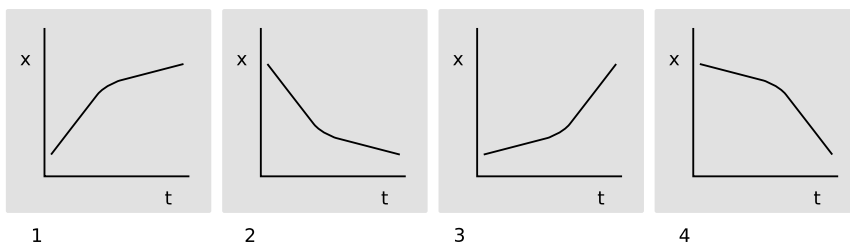
▷ First we draw the tangent line through that point. To find the slope of the tangent line, we need to pick two points on it. Theoretically, the slope should come out the same regardless of which two points we pick, but in practical terms we’ll be able to measure more accurately if we pick two points fairly far apart, such as the two white diamonds. To save work, we pick points that are directly above labeled points on the  $t$  axis, so that  $\Delta t = 4.0$  s is easy to read off. One diamond lines up with  $x \approx 17.5$  m, the other with  $x \approx 26.5$  m, so  $\Delta x = 9.0$  m. The velocity is  $\Delta x / \Delta t = 2.2$  m/s.

### Conventions about graphing

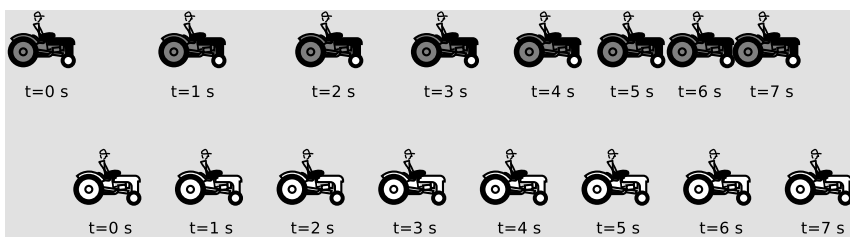
The placement of  $t$  on the horizontal axis and  $x$  on the upright axis may seem like an arbitrary convention, or may even have disturbed you, since your algebra teacher always told you that  $x$  goes on the horizontal axis and  $y$  goes on the upright axis. There is a reason for doing it this way, however. In example s, we have an object that reverses its direction of motion twice. It can only be in one place at any given time, but there can be more than one time when it is at a given place. For instance, this object passed through  $x = 17$  m on three separate occasions, but there is no way it could have been in more than one place at  $t = 5.0$  s. Resurrecting some terminology you learned in your trigonometry course, we say that  $x$  is a function of  $t$ , but  $t$  is not a function of  $x$ . In situations such as this, there is a useful convention that the graph should be oriented so that any vertical line passes through the curve at only one point. Putting the  $x$  axis across the page and  $t$  upright would have violated this convention. To people who are used to interpreting graphs, a graph that violates this convention is as annoying as fingernails scratching on a chalkboard. We say that this is a graph of “ $x$  versus  $t$ .” If the axes were the other way around, it would be a graph of “ $t$  versus  $x$ .” I remember the “versus” terminology by visualizing the labels on the  $x$  and  $t$  axes and remembering that when you read, you go from left to right and from top to bottom.

## Discussion questions

**A** Park is running slowly in gym class, but then he notices Jenna watching him, so he speeds up to try to impress her. Which of the graphs could represent his motion?



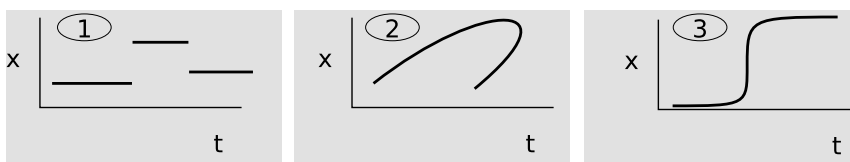
**B** The figure shows a sequence of positions for two racing tractors. Compare the tractors' velocities as the race progresses. When do they have the same velocity? [Based on a question by Lillian McDermott.]



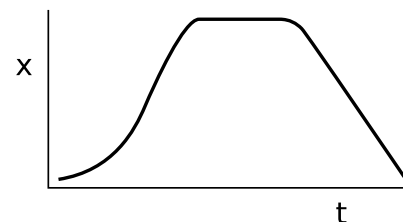
**C** If an object had an  $x - t$  graph that was a straight line with  $\Delta x = 0$  and  $\Delta t \neq 0$ , what would you say about its velocity? Sketch an example of such a graph. What about  $\Delta t = 0$  and  $\Delta x \neq 0$ ?

**D** If an object has a wavy motion graph like the one in figure t on p. 75, what are the times at which the object reverses its direction? Describe the object's velocity at these points.

**E** Discuss anything unusual about the following three graphs.



**F** I have been using the term "velocity" and avoiding the more common English word "speed," because introductory physics texts typically define them to mean different things. They use the word "speed," and the symbol " $s$ " to mean the absolute value of the velocity,  $s = |v|$ . Although I've chosen not to emphasize this distinction in technical vocabulary, there are clearly two different concepts here. Can you think of an example of a graph of  $x$ -versus- $t$  in which the object has constant speed, but not constant velocity?



**G** For the graph shown in the figure, describe how the object's velocity changes.

Discussion question G.



**H** Two physicists duck out of a boring scientific conference. On the street, they witness an accident in which a pedestrian is injured by a hit-and-run driver. A criminal trial results, and they must testify. In her testimony, Dr. Transverz Waive says, "The car was moving along pretty fast, I'd say the velocity was +40 mi/hr. They saw the old lady too late, and even though they slammed on the brakes they still hit her before they stopped. Then they made a *U* turn and headed off at a velocity of about -20 mi/hr, I'd say." Dr. Longitud N.L. Vibrasheun says, "He was really going too fast, maybe his velocity was -35 or -40 mi/hr. After he hit Mrs. Hapless, he turned around and left at a velocity of, oh, I'd guess maybe +20 or +25 mi/hr." Is their testimony contradictory? Explain.

## 2.4 The principle of inertia

### Physical effects relate only to a change in velocity

Consider two statements of a kind that was at one time made with the utmost seriousness:

People like Galileo and Copernicus who say the earth is rotating must be crazy. We know the earth can't be moving. Why, if the earth was really turning once every day, then our whole city would have to be moving hundreds of leagues in an hour. That's impossible! Buildings would shake on their foundations. Gale-force winds would knock us over. Trees would fall down. The Mediterranean would come sweeping across the east coasts of Spain and Italy. And furthermore, what force would be making the world turn?

All this talk of passenger trains moving at forty miles an hour is sheer hogwash! At that speed, the air in a passenger compartment would all be forced against the back wall. People in the front of the car would suffocate, and people at the back would die because in such concentrated air, they wouldn't be able to expel a breath.

Some of the effects predicted in the first quote are clearly just based on a lack of experience with rapid motion that is smooth and free of vibration. But there is a deeper principle involved. In each case, the speaker is assuming that the mere fact of motion must have dramatic physical effects. More subtly, they also believe that a force is needed to keep an object in motion: the first person thinks a force would be needed to maintain the earth's rotation, and the second apparently thinks of the rear wall as pushing on the air to keep it moving.

Common modern knowledge and experience tell us that these people's predictions must have somehow been based on incorrect reasoning, but it is not immediately obvious where the fundamental flaw lies. It's one of those things a four-year-old could infuriate you by demanding a clear explanation of. One way of getting at the fundamental principle involved is to consider how the modern

concept of the universe differs from the popular conception at the time of the Italian Renaissance. To us, the word “earth” implies a planet, one of the nine planets of our solar system, a small ball of rock and dirt that is of no significance to anyone in the universe except for members of our species, who happen to live on it. To Galileo’s contemporaries, however, the earth was the biggest, most solid, most important thing in all of creation, not to be compared with the wandering lights in the sky known as planets. To us, the earth is just another object, and when we talk loosely about “how fast” an object such as a car “is going,” we really mean the car-object’s velocity relative to the earth-object.



u / This Air Force doctor volunteered to ride a rocket sled as a medical experiment. The obvious effects on his head and face are not because of the sled’s speed but because of its rapid *changes* in speed: increasing in 2 and 3, and decreasing in 5 and 6. In 4 his speed is greatest, but because his speed is not increasing or decreasing very much at this moment, there is little effect on him.

### Motion is relative

According to our modern world-view, it really isn’t that reasonable to expect that a special force should be required to make the air in the train have a certain velocity relative to our planet. After all, the “moving” air in the “moving” train might just happen to have zero velocity relative to some other planet we don’t even know about. Aristotle claimed that things “naturally” wanted to be at rest, lying on the surface of the earth. But experiment after exper-



v / Why does Aristotle look so sad? Has he realized that his entire system of physics is wrong?



w / The earth spins. People in Shanghai say they’re at rest and people in Los Angeles are moving. Angelenos say the same about the Shanghainese.



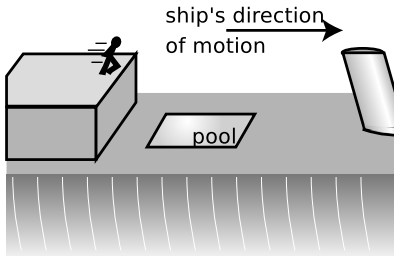
x / The jets are at rest. The Empire State Building is moving.

iment has shown that there is really nothing so special about being at rest relative to the earth. For instance, if a mattress falls out of the back of a truck on the freeway, the reason it rapidly comes to rest with respect to the planet is simply because of friction forces exerted by the asphalt, which happens to be attached to the planet.

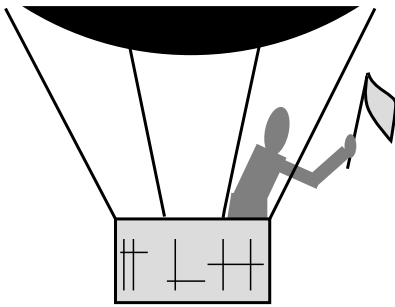
Galileo's insights are summarized as follows:

### The principle of inertia

No force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects.



Discussion question A.



Discussion question B.

There are many examples of situations that seem to disprove the principle of inertia, but these all result from forgetting that friction is a force. For instance, it seems that a force is needed to keep a sailboat in motion. If the wind stops, the sailboat stops too. But the wind's force is not the only force on the boat; there is also a frictional force from the water. If the sailboat is cruising and the wind suddenly disappears, the backward frictional force still exists, and since it is no longer being counteracted by the wind's forward force, the boat stops. To disprove the principle of inertia, we would have to find an example where a moving object slowed down even though no forces whatsoever were acting on it.

#### self-check E

What is incorrect about the following supposed counterexamples to the principle of inertia?

(1) When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies — they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.

(2) When you're driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion. ▸

Answer, p. 531

▸ Solved problem: a bug on a wheel

page 88, problem 7

### Discussion questions

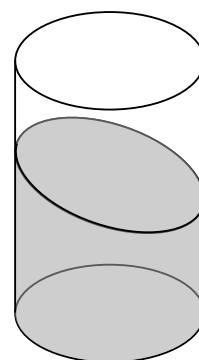
**A** A passenger on a cruise ship finds, while the ship is docked, that he can leap off of the upper deck and just barely make it into the pool on the lower deck. If the ship leaves dock and is cruising rapidly, will this adrenaline junkie still be able to make it?

**B** You are a passenger in the open basket hanging under a helium balloon. The balloon is being carried along by the wind at a constant velocity. If you are holding a flag in your hand, will the flag wave? If so, which way? [Based on a question from PSSC Physics.]

**C** Aristotle stated that all objects naturally wanted to come to rest, with the unspoken implication that “rest” would be interpreted relative to the

surface of the earth. Suppose we go back in time and transport Aristotle to the moon. Aristotle knew, as we do, that the moon circles the earth; he said it didn't fall down because, like everything else in the heavens, it was made out of some special substance whose "natural" behavior was to go in circles around the earth. We land, put him in a space suit, and kick him out the door. What would he expect his fate to be in this situation? If intelligent creatures inhabited the moon, and one of them independently came up with the equivalent of Aristotelian physics, what would they think about objects coming to rest?

**D** The glass is sitting on a level table in a train's dining car, but the surface of the water is tilted. What can you infer about the motion of the train?



Discussion question D.

## 2.5 Addition of velocities

### Addition of velocities to describe relative motion

Since absolute motion cannot be unambiguously measured, the only way to describe motion unambiguously is to describe the motion of one object relative to another. Symbolically, we can write  $v_{PQ}$  for the velocity of object  $P$  relative to object  $Q$ .

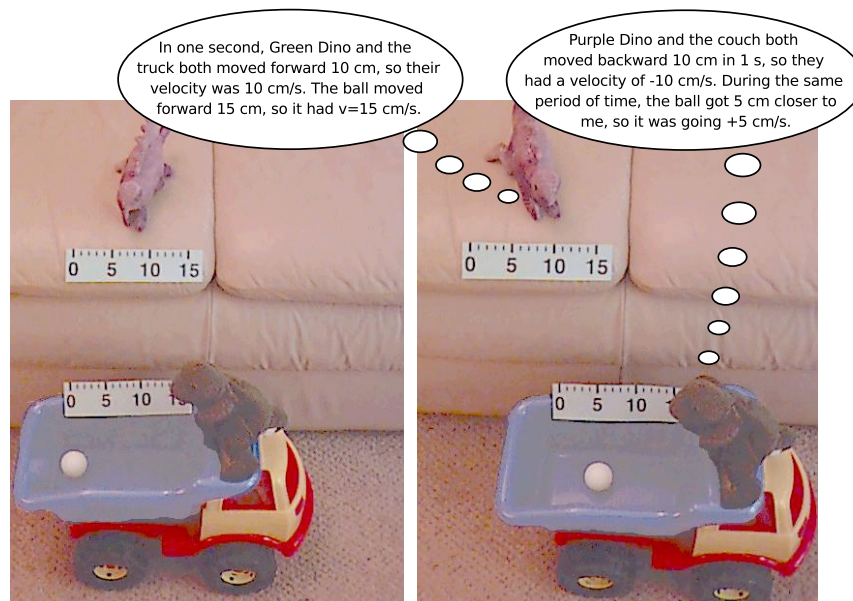
Velocities measured with respect to different reference points can be compared by addition. In the figure below, the ball's velocity relative to the couch equals the ball's velocity relative to the truck plus the truck's velocity relative to the couch:

$$\begin{aligned} v_{BC} &= v_{BT} + v_{TC} \\ &= 5 \text{ cm/s} + 10 \text{ cm/s} \\ &= 15 \text{ cm/s} \end{aligned}$$

The same equation can be used for any combination of three objects, just by substituting the relevant subscripts for B, T, and C. Just remember to write the equation so that the velocities being added have the same subscript twice in a row. In this example, if you read off the subscripts going from left to right, you get  $BC \dots = \dots BTTC$ . The fact that the two "inside" subscripts on the right are the same means that the equation has been set up correctly. Notice how subscripts on the left look just like the subscripts on the right, but with the two T's eliminated.

### Negative velocities in relative motion

My discussion of how to interpret positive and negative signs of velocity may have left you wondering why we should bother. Why not just make velocity positive by definition? The original reason why negative numbers were invented was that bookkeepers decided it would be convenient to use the negative number concept for payments to distinguish them from receipts. It was just plain easier than writing receipts in black and payments in red ink. After adding up



y / These two highly competent physicists disagree on absolute velocities, but they would agree on relative velocities. Purple Dino considers the couch to be at rest, while Green Dino thinks of the truck as being at rest. They agree, however, that the truck's velocity relative to the couch is  $v_{TC} = 10$  cm/s, the ball's velocity relative to the truck is  $v_{BT} = 5$  cm/s, and the ball's velocity relative to the couch is  $v_{BC} = v_{BT} + v_{TC} = 15$  cm/s.

your month's positive receipts and negative payments, you either got a positive number, indicating profit, or a negative number, showing a loss. You could then show that total with a high-tech "+" or "-" sign, instead of looking around for the appropriate bottle of ink.

Nowadays we use positive and negative numbers for all kinds of things, but in every case the point is that it makes sense to add and subtract those things according to the rules you learned in grade school, such as "minus a minus makes a plus, why this is true we need not discuss." Adding velocities has the significance of comparing relative motion, and with this interpretation negative and positive velocities can be used within a consistent framework. For example, the truck's velocity relative to the couch equals the truck's velocity relative to the ball plus the ball's velocity relative to the couch:

$$\begin{aligned} v_{TC} &= v_{TB} + v_{BC} \\ &= -5 \text{ cm/s} + 15 \text{ cm/s} \\ &= 10 \text{ cm/s} \end{aligned}$$

If we didn't have the technology of negative numbers, we would have had to remember a complicated set of rules for adding velocities: (1) if the two objects are both moving forward, you add, (2) if one is moving forward and one is moving backward, you subtract, but (3)



if they're both moving backward, you add. What a pain that would have been.

▷ *Solved problem: two dimensions*

*page 89, problem 10*

### *Airspeed*

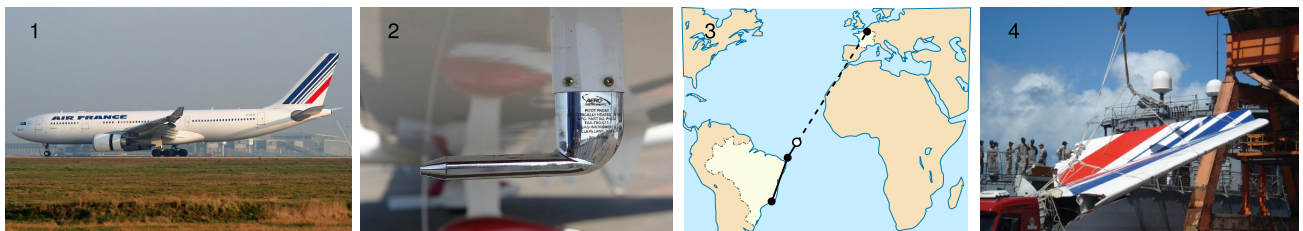
### *example 3*

On June 1, 2009, Air France flight 447 disappeared without warning over the Atlantic Ocean. All 232 people aboard were killed. Investigators believe the disaster was triggered because the pilots lost the ability to accurately determine their speed relative to the air. This is done using sensors called Pitot tubes, mounted outside the plane on the wing. Automated radio signals showed that these sensors gave conflicting readings before the crash, possibly because they iced up. For fuel efficiency, modern passenger jets fly at a very high altitude, but in the thin air they can only fly within a very narrow range of speeds. If the speed is too low, the plane stalls, and if it's too high, it breaks up. If the pilots can't tell what their airspeed is, they can't keep it in the safe range.

Many people's reaction to this story is to wonder why planes don't just use GPS to measure their speed. One reason is that GPS tells you your speed relative to the ground, not relative to the air. Letting  $P$  be the plane,  $A$  the air, and  $G$  the ground, we have

$$\mathbf{v}_{PG} = \mathbf{v}_{PA} + \mathbf{v}_{AG} \quad ,$$

where  $\mathbf{v}_{PG}$  (the “true ground speed”) is what GPS would measure,  $\mathbf{v}_{PA}$  (“airspeed”) is what's critical for stable flight, and  $\mathbf{v}_{AG}$  is the velocity of the wind relative to the ground 9000 meters below. Knowing  $\mathbf{v}_{PG}$  isn't enough to determine  $\mathbf{v}_{PA}$  unless  $\mathbf{v}_{AG}$  is also known.



z / Example 3. 1. The aircraft before the disaster. 2. A Pitot tube. 3. The flight path of flight 447. 4. Wreckage being recovered.

### Discussion questions

**A** Interpret the general rule  $\mathbf{v}_{AB} = -\mathbf{v}_{BA}$  in words.

**B** Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols.

## 2.6 Graphs of velocity versus time

Since changes in velocity play such a prominent role in physics, we need a better way to look at changes in velocity than by laboriously drawing tangent lines on  $x$ -versus- $t$  graphs. A good method is to draw a graph of velocity versus time. The examples on the left show the  $x-t$  and  $v-t$  graphs that might be produced by a car starting from a traffic light, speeding up, cruising for a while at constant speed, and finally slowing down for a stop sign. If you have an air freshener hanging from your rear-view mirror, then you will see an effect on the air freshener during the beginning and ending periods when the velocity is changing, but it will not be tilted during the period of constant velocity represented by the flat plateau in the middle of the  $v-t$  graph.

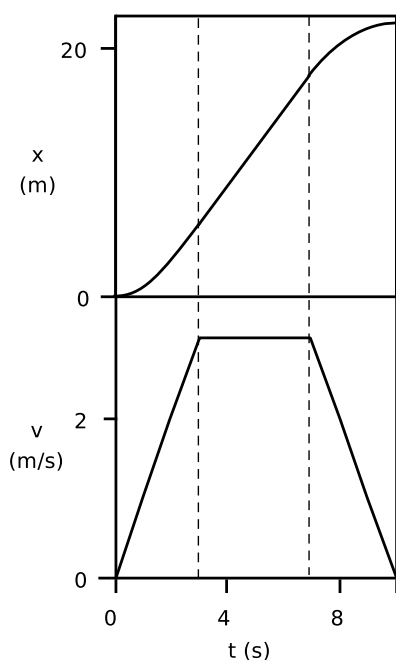
Students often mix up the things being represented on these two types of graphs. For instance, many students looking at the top graph say that the car is speeding up the whole time, since “the graph is becoming greater.” What is getting greater throughout the graph is  $x$ , not  $v$ .

Similarly, many students would look at the bottom graph and think it showed the car backing up, because “it’s going backwards at the end.” But what is decreasing at the end is  $v$ , not  $x$ . Having both the  $x-t$  and  $v-t$  graphs in front of you like this is often convenient, because one graph may be easier to interpret than the other for a particular purpose. Stacking them like this means that corresponding points on the two graphs’ time axes are lined up with each other vertically. However, one thing that is a little counter-intuitive about the arrangement is that in a situation like this one involving a car, one is tempted to visualize the landscape stretching along the horizontal axis of one of the graphs. The horizontal axes, however, represent time, not position. The correct way to visualize the landscape is by mentally rotating the horizon 90 degrees counterclockwise and imagining it stretching along the upright axis of the  $x-t$  graph, which is the only axis that represents different positions in space.

## 2.7 $\int$ Applications of calculus

The integral symbol,  $\int$ , in the heading for this section indicates that it is meant to be read by students in calculus-based physics. Students in an algebra-based physics course should skip these sections. The calculus-related sections in this book are meant to be usable by students who are taking calculus concurrently, so at this early point in the physics course I do not assume you know any calculus yet. This section is therefore not much more than a quick preview of calculus, to help you relate what you’re learning in the two courses.

Newton was the first person to figure out the tangent-line defi-



aa / Graphs of  $x$  and  $v$  versus  $t$  for a car accelerating away from a traffic light, and then stopping for another red light.

nition of velocity for cases where the  $x - t$  graph is nonlinear. Before Newton, nobody had conceptualized the description of motion in terms of  $x - t$  and  $v - t$  graphs. In addition to the graphical techniques discussed in this chapter, Newton also invented a set of symbolic techniques called calculus. If you have an equation for  $x$  in terms of  $t$ , calculus allows you, for instance, to find an equation for  $v$  in terms of  $t$ . In calculus terms, we say that the function  $v(t)$  is the derivative of the function  $x(t)$ . In other words, the derivative of a function is a new function that tells how rapidly the original function was changing. We now use neither Newton's name for his technique (he called it "the method of fluxions") nor his notation. The more commonly used notation is due to Newton's German contemporary Leibnitz, whom the English accused of plagiarizing the calculus from Newton. In the Leibnitz notation, we write

$$v = \frac{dx}{dt}$$

to indicate that the function  $v(t)$  equals the slope of the tangent line of the graph of  $x(t)$  at every time  $t$ . The Leibnitz notation is meant to evoke the delta notation, but with a very small time interval. Because the  $dx$  and  $dt$  are thought of as very small  $\Delta x$ 's and  $\Delta t$ 's, i.e., very small differences, the part of calculus that has to do with derivatives is called differential calculus.

Differential calculus consists of three things:

- The concept and definition of the derivative, which is covered in this book, but which will be discussed more formally in your math course.
- The Leibnitz notation described above, which you'll need to get more comfortable with in your math course.
- A set of rules that allows you to find an equation for the derivative of a given function. For instance, if you happened to have a situation where the position of an object was given by the equation  $x = 2t^7$ , you would be able to use those rules to find  $dx/dt = 14t^6$ . This bag of tricks is covered in your math course.



# Summary

## Selected vocabulary

center of mass . .	the balance point of an object
velocity . . . . .	the rate of change of position; the slope of the tangent line on an $x - t$ graph.

## Notation

$x$ . . . . .	a point in space
$t$ . . . . .	a point in time, a clock reading
$\Delta$ . . . . .	“change in;” the value of a variable afterwards minus its value before
$\Delta x$ . . . . .	a distance, or more precisely a change in $x$ , which may be less than the distance traveled; its plus or minus sign indicates direction
$\Delta t$ . . . . .	a duration of time
$v$ . . . . .	velocity
$v_{AB}$ . . . . .	the velocity of object A relative to object B

## Other terminology and notation

displacement . .	a name for the symbol $\Delta x$
speed . . . . .	the absolute value of the velocity, i.e., the velocity stripped of any information about its direction

## Summary

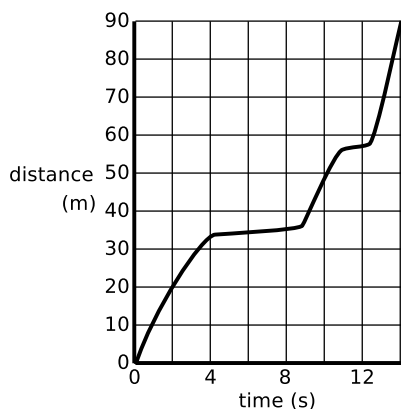
An object’s center of mass is the point at which it can be balanced. For the time being, we are studying the mathematical description only of the motion of an object’s center of mass in cases restricted to one dimension. The motion of an object’s center of mass is usually far simpler than the motion of any of its other parts.

It is important to distinguish location,  $x$ , from distance,  $\Delta x$ , and clock reading,  $t$ , from time interval  $\Delta t$ . When an object’s  $x - t$  graph is linear, we define its velocity as the slope of the line,  $\Delta x / \Delta t$ . When the graph is curved, we generalize the definition so that the velocity is the slope of the tangent line at a given point on the graph.

Galileo’s principle of inertia states that no force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects. Things typically tend to reduce their velocity relative to the surface of our planet only because they are physically rubbing against the planet (or something attached to the planet), not because there is anything special about being at rest with respect to the earth’s surface. When it seems, for instance, that a force is required to keep a book sliding across a table, in fact the force is only serving to cancel the contrary force of friction.

Absolute motion is not a well-defined concept, and if two observers are not at rest relative to one another they will disagree about the absolute velocities of objects. They will, however, agree

about relative velocities. If object  $A$  is in motion relative to object  $B$ , and  $B$  is in motion relative to  $C$ , then  $A$ 's velocity relative to  $C$  is given by  $v_{AC} = v_{AB} + v_{BC}$ . Positive and negative signs are used to indicate the direction of an object's motion.



Problem 1.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** The graph shows the motion of a car stuck in stop-and-go freeway traffic. (a) If you only knew how far the car had gone during this entire time period, what would you think its velocity was? (b) What is the car's maximum velocity? ✓

**2** (a) Let  $\theta$  be the latitude of a point on the Earth's surface. Derive an algebra equation for the distance,  $L$ , traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of  $\theta$  and  $R$ , the radius of the earth. Check: Your equation should give  $L = 0$  for the North Pole.

(b) At what speed is Fullerton, at latitude  $\theta = 34^\circ$ , moving with the rotation of the Earth about its axis? Give your answer in units of mi/h. [See the table in the back of the book for the relevant data.] ✓

**3** A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by the equation

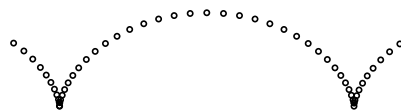
$$y = (10000 \text{ m}) - (50 \text{ m/s}) \left[ t + (5.0 \text{ s})e^{-t/5.0 \text{ s}} \right] .$$

Find her velocity at  $t = 7.0 \text{ s}$ . (This can be done on a calculator, without knowing calculus.) Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum. ✓ ★

**4** A light-year is a unit of distance used in astronomy, and defined as the distance light travels in one year. The speed of light is  $3.0 \times 10^8 \text{ m/s}$ . Find how many meters there are in one light-year, expressing your answer in scientific notation. ▷ Solution, p. 517

**5** You're standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train's velocity? Explain. ▷ Solution, p. 517

**6** A honeybee's position as a function of time is given by  $x = 10t - t^3$ , where  $t$  is in seconds and  $x$  in meters. What is its velocity at  $t = 3.0 \text{ s}$ ? ✓ ∫



Problem 7.

**7** The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor. Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally

difficult?

▷ Solution, p. 518

**8** Peanut plants fold up their leaves at night. Estimate the top speed of the tip of one of the leaves shown in the figure, expressing your result in scientific notation in SI units. ✓

**9** (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.5. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

**10** Our full discussion of two- and three-dimensional motion is postponed until the second half of the book, but here is a chance to use a little mathematical creativity in anticipation of that generalization. Suppose a ship is sailing east at a certain speed  $v$ , and a passenger is walking across the deck at the same speed  $v$ , so that his track across the deck is perpendicular to the ship's center-line. What is his speed relative to the water, and in what direction is he moving relative to the water? ▷ Solution, p. 518

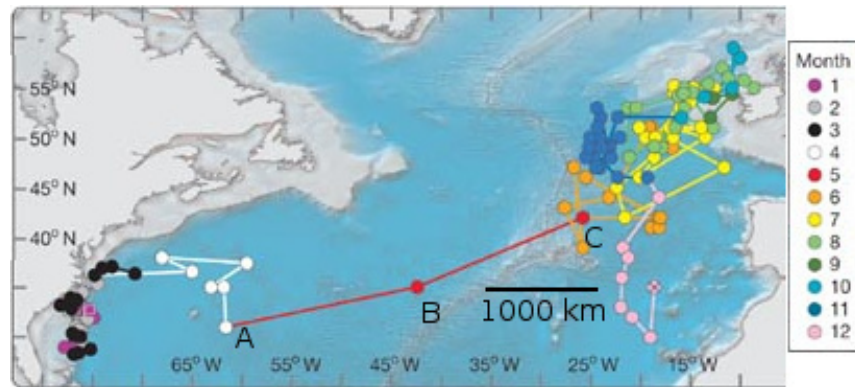
**11** Freddi Fish<sup>(TM)</sup> has a position as a function of time given by  $x = a/(b + t^2)$ . (a) Infer the units of the constants  $a$  and  $b$ . (b) Find her maximum speed. (c) Check that your answer has the right units. ✓ ∫

**12** Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe a frame of reference in which your car was *speeding up* during that same period of time. (The frame of reference should be defined by an observer who, although perhaps in motion relative to the earth, is not changing her own speed or direction of motion.)



Problem 8.

Problem 13.



**13** The figure shows the motion of a bluefin tuna, as measured by a radio tag (Block et al., Nature, v. 434, p. 1121, 2005), over the course of several years. Until this study, it had been believed that the populations of the fish in the eastern and western Atlantic were separate, but this particular fish was observed to cross the entire Atlantic Ocean, from Virginia to Ireland. Points A, B, and C show a period of one month, during which the fish made the most rapid progress. Estimate its speed during that month, in units of kilometers per hour.  $\checkmark$

**14** Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion like  $x = cte^{-bt}$ , where  $x$  is the position of some point on the door, and  $c$  and  $b$  are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.)

(a) Infer the units of the constants  $b$  and  $c$ .

(b) Find the door's maximum speed (i.e., the greatest absolute value of its velocity) for  $t > 0$ .  $\checkmark$

(c) Show that your answer has units that make sense.

$\int$



Galileo's contradiction of Aristotle had serious consequences. He was interrogated by the Church authorities and convicted of teaching that the earth went around the sun as a matter of fact and not, as he had promised previously, as a mere mathematical hypothesis. He was placed under permanent house arrest, and forbidden to write about or teach his theories. Immediately after being forced to recant his claim that the earth revolved around the sun, the old man is said to have muttered defiantly "and yet it does move." The story is dramatic, but there are some omissions in the commonly taught heroic version. There was a rumor that the Simplicio character represented the Pope. Also, some of the ideas Galileo advocated had controversial religious overtones. He believed in the existence of atoms, and atomism was thought by some people to contradict the Church's doctrine of transubstantiation, which said that in the Catholic mass, the blessing of the bread and wine literally transformed them into the flesh and blood of Christ. His support for a cosmology in which the earth circled the sun was also disreputable because one of its supporters, Giordano Bruno, had also proposed a bizarre synthesis of Christianity with the ancient Egyptian religion.

## Chapter 3

# Acceleration and free fall

### 3.1 The motion of falling objects

The motion of falling objects is the simplest and most common example of motion with changing velocity. The early pioneers of

physics had a correct intuition that the way things drop was a message directly from Nature herself about how the universe worked. Other examples seem less likely to have deep significance. A walking person who speeds up is making a conscious choice. If one stretch of a river flows more rapidly than another, it may be only because the channel is narrower there, which is just an accident of the local geography. But there is something impressively consistent, universal, and inexorable about the way things fall.

Stand up now and simultaneously drop a coin and a bit of paper side by side. The paper takes much longer to hit the ground. That's why Aristotle wrote that heavy objects fell more rapidly. Europeans believed him for two thousand years.

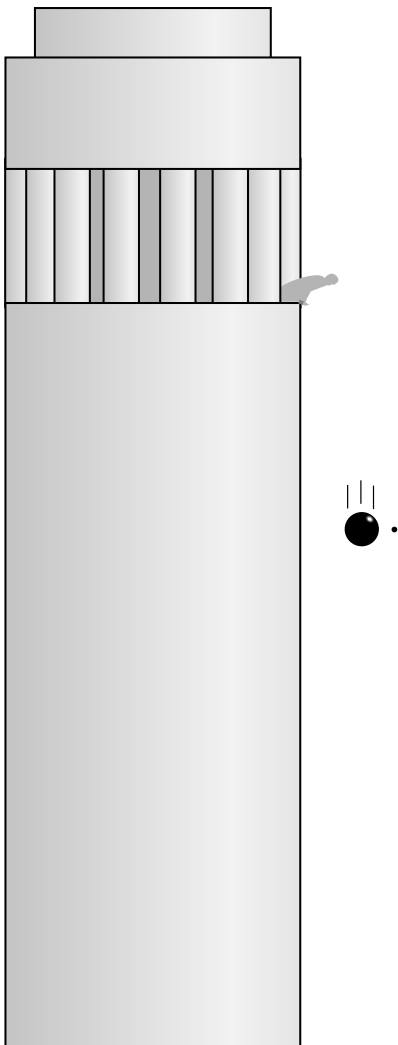
Now repeat the experiment, but make it into a race between the coin and your shoe. My own shoe is about 50 times heavier than the nickel I had handy, but it looks to me like they hit the ground at exactly the same moment. So much for Aristotle! Galileo, who had a flair for the theatrical, did the experiment by dropping a bullet and a heavy cannonball from a tall tower. Aristotle's observations had been incomplete, his interpretation a vast oversimplification.

It is inconceivable that Galileo was the first person to observe a discrepancy with Aristotle's predictions. Galileo was the one who changed the course of history because he was able to assemble the observations into a coherent pattern, and also because he carried out systematic quantitative (numerical) measurements rather than just describing things qualitatively.

Why is it that some objects, like the coin and the shoe, have similar motion, but others, like a feather or a bit of paper, are different? Galileo speculated that in addition to the force that always pulls objects down, there was an upward force exerted by the air. Anyone can speculate, but Galileo went beyond speculation and came up with two clever experiments to probe the issue. First, he experimented with objects falling in water, which probed the same issues but made the motion slow enough that he could take time measurements with a primitive pendulum clock. With this technique, he established the following facts:

- All heavy, streamlined objects (for example a steel rod dropped point-down) reach the bottom of the tank in about the same amount of time, only slightly longer than the time they would take to fall the same distance in air.
- Objects that are lighter or less streamlined take a longer time to reach the bottom.

This supported his hypothesis about two contrary forces. He imagined an idealized situation in which the falling object did not

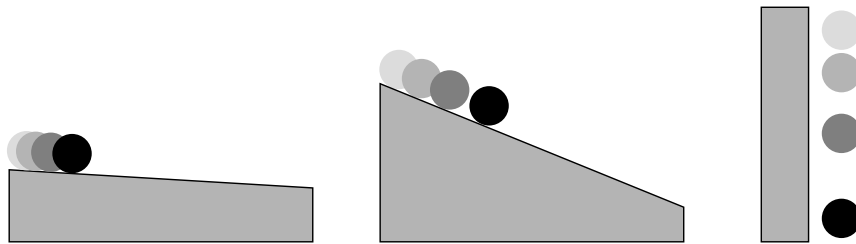


a / Galileo dropped a cannonball and a musketball simultaneously from a tower, and observed that they hit the ground at nearly the same time. This contradicted Aristotle's long-accepted idea that heavier objects fell faster.

have to push its way through any substance at all. Falling in air would be more like this ideal case than falling in water, but even a thin, sparse medium like air would be sufficient to cause obvious effects on feathers and other light objects that were not streamlined. Today, we have vacuum pumps that allow us to suck nearly all the air out of a chamber, and if we drop a feather and a rock side by side in a vacuum, the feather does not lag behind the rock at all.

### How the speed of a falling object increases with time

Galileo's second stroke of genius was to find a way to make quantitative measurements of how the speed of a falling object increased as it went along. Again it was problematic to make sufficiently accurate time measurements with primitive clocks, and again he found a tricky way to slow things down while preserving the essential physical phenomena: he let a ball roll down a slope instead of dropping it vertically. The steeper the incline, the more rapidly the ball would gain speed. Without a modern video camera, Galileo had invented a way to make a slow-motion version of falling.



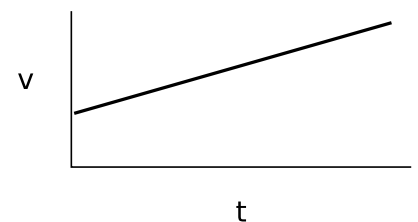
b / Velocity increases more gradually on the gentle slope, but the motion is otherwise the same as the motion of a falling object.

Although Galileo's clocks were only good enough to do accurate experiments at the smaller angles, he was confident after making a systematic study at a variety of small angles that his basic conclusions were generally valid. Stated in modern language, what he found was that the velocity-versus-time graph was a line. In the language of algebra, we know that a line has an equation of the form  $y = ax + b$ , but our variables are  $v$  and  $t$ , so it would be  $v = at + b$ . (The constant  $b$  can be interpreted simply as the initial velocity of the object, i.e., its velocity at the time when we started our clock, which we conventionally write as  $v_0$ .)

#### self-check A

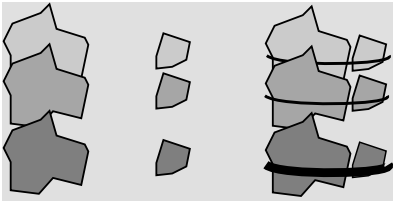
An object is rolling down an incline. After it has been rolling for a short time, it is found to travel 13 cm during a certain one-second interval. During the second after that, it goes 16 cm. How many cm will it travel in the second after that?

▷ Answer, p. 531

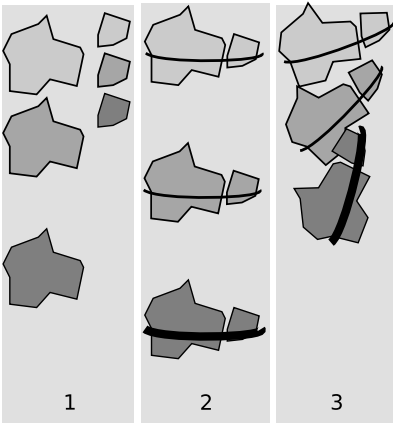


c / The  $v - t$  graph of a falling object is a line.





d / Galileo's experiments show that all falling objects have the same motion if air resistance is negligible.



e / 1. Aristotle said that heavier objects fell faster than lighter ones. 2. If two rocks are tied together, that makes an extra-heavy rock, which should fall faster. 3. But Aristotle's theory would also predict that the light rock would hold back the heavy rock, resulting in a slower fall.

## A contradiction in Aristotle's reasoning

Galileo's inclined-plane experiment disproved the long-accepted claim by Aristotle that a falling object had a definite "natural falling speed" proportional to its weight. Galileo had found that the speed just kept on increasing, and weight was irrelevant as long as air friction was negligible. Not only did Galileo prove experimentally that Aristotle had been wrong, but he also pointed out a logical contradiction in Aristotle's own reasoning. Simplicio, the stupid character, mouths the accepted Aristotelian wisdom:

SIMPLICIO: There can be no doubt but that a particular body . . . has a fixed velocity which is determined by nature. . .

SALVIATI: If then we take two bodies whose natural speeds are different, it is clear that, [according to Aristotle], on uniting the two, the more rapid one will be partly held back by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMPLICIO: You are unquestionably right.

SALVIATI: But if this is true, and if a large stone moves with a speed of, say, eight [unspecified units] while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.

## What is gravity?

The physicist Richard Feynman liked to tell a story about how when he was a little kid, he asked his father, "Why do things fall?" As an adult, he praised his father for answering, "Nobody knows why things fall. It's a deep mystery, and the smartest people in the world don't know the basic reason for it." Contrast that with the average person's off-the-cuff answer, "Oh, it's because of gravity." Feynman liked his father's answer, because his father realized that simply giving a name to something didn't mean that you understood it. The radical thing about Galileo's and Newton's approach to science was that they concentrated first on describing mathematically what really did happen, rather than spending a lot of time on untestable speculation such as Aristotle's statement that "Things fall because they are trying to reach their natural place in contact with the earth." That doesn't mean that science can never answer the "why" questions. Over the next month or two as you delve deeper into physics, you will learn that there are more fundamental reasons why all falling objects have  $v - t$  graphs with the same slope, regardless

of their mass. Nevertheless, the methods of science always impose limits on how deep our explanation can go.

## 3.2 Acceleration

### Definition of acceleration for linear $v - t$ graphs

Galileo's experiment with dropping heavy and light objects from a tower showed that all falling objects have the same motion, and his inclined-plane experiments showed that the motion was described by  $v = at + v_0$ . The initial velocity  $v_0$  depends on whether you drop the object from rest or throw it down, but even if you throw it down, you cannot change the slope,  $a$ , of the  $v - t$  graph.

Since these experiments show that all falling objects have linear  $v - t$  graphs with the same slope, the slope of such a graph is apparently an important and useful quantity. We use the word acceleration, and the symbol  $a$ , for the slope of such a graph. In symbols,  $a = \Delta v / \Delta t$ . The acceleration can be interpreted as the amount of speed gained in every second, and it has units of velocity divided by time, i.e., "meters per second per second," or m/s/s. Continuing to treat units as if they were algebra symbols, we simplify "m/s/s" to read "m/s<sup>2</sup>." Acceleration can be a useful quantity for describing other types of motion besides falling, and the word and the symbol " $a$ " can be used in a more general context. We reserve the more specialized symbol " $g$ " for the acceleration of falling objects, which on the surface of our planet equals 9.8 m/s<sup>2</sup>. Often when doing approximate calculations or merely illustrative numerical examples it is good enough to use  $g = 10 \text{ m/s}^2$ , which is off by only 2%.

#### *Finding final speed, given time* example 1

▷ A despondent physics student jumps off a bridge, and falls for three seconds before hitting the water. How fast is he going when he hits the water?

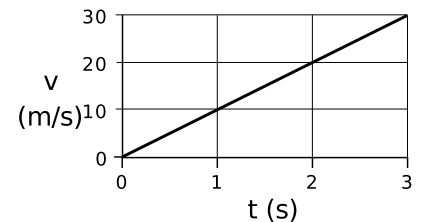
▷ Approximating  $g$  as  $10 \text{ m/s}^2$ , he will gain  $10 \text{ m/s}$  of speed each second. After one second, his velocity is  $10 \text{ m/s}$ , after two seconds it is  $20 \text{ m/s}$ , and on impact, after falling for three seconds, he is moving at  $30 \text{ m/s}$ .

#### *Extracting acceleration from a graph* example 2

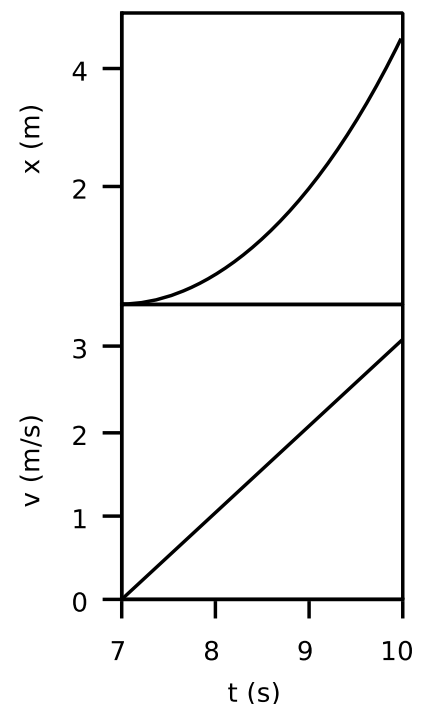
▷ The  $x - t$  and  $v - t$  graphs show the motion of a car starting from a stop sign. What is the car's acceleration?

▷ Acceleration is defined as the slope of the  $v$ - $t$  graph. The graph rises by  $3 \text{ m/s}$  during a time interval of  $3 \text{ s}$ , so the acceleration is  $(3 \text{ m/s}) / (3 \text{ s}) = 1 \text{ m/s}^2$ .

Incorrect solution #1: The final velocity is  $3 \text{ m/s}$ , and acceleration is velocity divided by time, so the acceleration is  $(3 \text{ m/s}) / (10 \text{ s}) = 0.3 \text{ m/s}^2$ .



f / Example 1.



g / Example 2.

✗ The solution is incorrect because you can't find the slope of a graph from one point. This person was just using the point at the right end of the v-t graph to try to find the slope of the curve.

Incorrect solution #2: Velocity is distance divided by time so  $v = (4.5 \text{ m})/(3 \text{ s}) = 1.5 \text{ m/s}$ . Acceleration is velocity divided by time, so  $a = (1.5 \text{ m/s})/(3 \text{ s}) = 0.5 \text{ m/s}^2$ .

✗ The solution is incorrect because velocity is the slope of the tangent line. In a case like this where the velocity is changing, you can't just pick two points on the x-t graph and use them to find the velocity.

---

*Converting  $g$  to different units* *example 3*

▷ What is  $g$  in units of  $\text{cm/s}^2$ ?

▷ The answer is going to be how many  $\text{cm/s}$  of speed a falling object gains in one second. If it gains  $9.8 \text{ m/s}$  in one second, then it gains  $980 \text{ cm/s}$  in one second, so  $g = 980 \text{ cm/s}^2$ . Alternatively, we can use the method of fractions that equal one:

$$\frac{9.8 \text{ m}}{\text{s}^2} \times \frac{100 \text{ cm}}{1 \text{ m}} = \frac{980 \text{ cm}}{\text{s}^2}$$

▷ What is  $g$  in units of  $\text{miles/hour}^2$ ?

▷

$$\frac{9.8 \text{ m}}{\text{s}^2} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \left( \frac{3600 \text{ s}}{1 \text{ hour}} \right)^2 = 7.9 \times 10^4 \text{ mile/hour}^2$$

This large number can be interpreted as the speed, in miles per hour, that you would gain by falling for one hour. Note that we had to square the conversion factor of  $3600 \text{ s/hour}$  in order to cancel out the units of seconds squared in the denominator.

▷ What is  $g$  in units of  $\text{miles/hour/s}$ ?

▷

$$\frac{9.8 \text{ m}}{\text{s}^2} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ hour}} = 22 \text{ mile/hour/s}$$

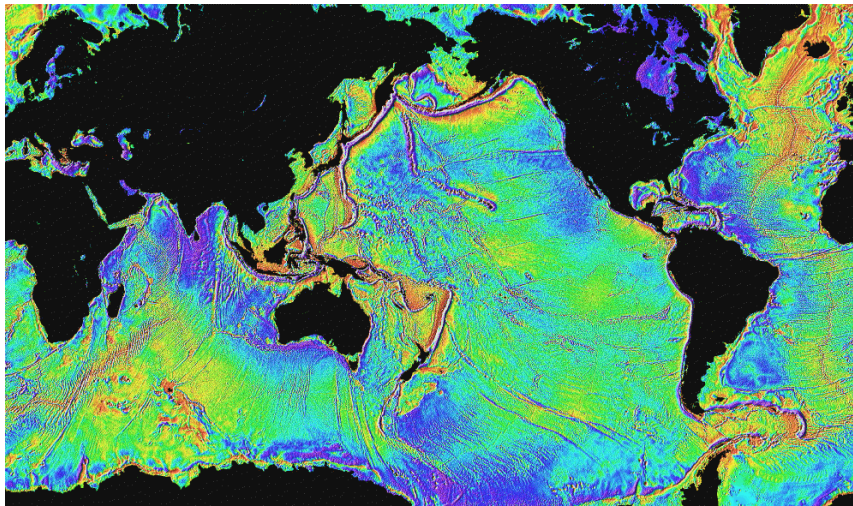
This is a figure that Americans will have an intuitive feel for. If your car has a forward acceleration equal to the acceleration of a falling object, then you will gain 22 miles per hour of speed every second. However, using mixed time units of hours and seconds like this is usually inconvenient for problem-solving. It would be like using units of foot-inches for area instead of  $\text{ft}^2$  or  $\text{in}^2$ .

### **The acceleration of gravity is different in different locations.**

Everyone knows that gravity is weaker on the moon, but actually it is not even the same everywhere on Earth, as shown by the sampling of numerical data in the following table.

location	latitude	elevation (m)	$g$ (m/s <sup>2</sup> )
north pole	90°N	0	9.8322
Reykjavik, Iceland	64°N	0	9.8225
Guayaquil, Ecuador	2°S	0	9.7806
Mt. Cotopaxi, Ecuador	1°S	5896	9.7624
Mt. Everest	28°N	8848	9.7643

The main variables that relate to the value of  $g$  on Earth are latitude and elevation. Although you have not yet learned how  $g$  would be calculated based on any deeper theory of gravity, it is not too hard to guess why  $g$  depends on elevation. Gravity is an attraction between things that have mass, and the attraction gets weaker with increasing distance. As you ascend from the seaport of Guayaquil to the nearby top of Mt. Cotopaxi, you are distancing yourself from the mass of the planet. The dependence on latitude occurs because we are measuring the acceleration of gravity relative to the earth's surface, but the earth's rotation causes the earth's surface to fall out from under you. (We will discuss both gravity and rotation in more detail later in the course.)



This false-color map shows variations in the strength of the earth's gravity. Purple areas have the strongest gravity, yellow the weakest. The overall trend toward weaker gravity at the equator and stronger gravity at the poles has been artificially removed to allow the weaker local variations to show up. The map covers only the oceans because of the technique used to make it: satellites look for bulges and depressions in the surface of the ocean. A very slight bulge will occur over an undersea mountain, for instance, because the mountain's gravitational attraction pulls water toward it. The US government originally began collecting data like these for military use, to correct for the deviations in the paths of missiles. The data have recently been released for scientific and commercial use (e.g., searching for sites for off-shore oil wells).

Much more spectacular differences in the strength of gravity can be observed away from the Earth's surface:

location	$g$ (m/s <sup>2</sup> )
asteroid Vesta (surface)	0.3
Earth's moon (surface)	1.6
Mars (surface)	3.7
Earth (surface)	9.8
Jupiter (cloud-tops)	26
Sun (visible surface)	270
typical neutron star (surface)	$10^{12}$
black hole (center)	infinite according to some theories, on the order of $10^{52}$ according to others

A typical neutron star is not so different in size from a large asteroid, but is orders of magnitude more massive, so the mass of a body definitely correlates with the  $g$  it creates. On the other hand, a neutron star has about the same mass as our Sun, so why is its  $g$  billions of times greater? If you had the misfortune of being on the surface of a neutron star, you'd be within a few thousand miles of all its mass, whereas on the surface of the Sun, you'd still be millions of miles from most of its mass.

### Discussion questions

**A** What is wrong with the following definitions of  $g$ ?

- (1) " $g$  is gravity."
- (2) " $g$  is the speed of a falling object."
- (3) " $g$  is how hard gravity pulls on things."

**B** When advertisers specify how much acceleration a car is capable of, they do not give an acceleration as defined in physics. Instead, they usually specify how many seconds are required for the car to go from rest to 60 miles/hour. Suppose we use the notation " $a$ " for the acceleration as defined in physics, and " $a_{\text{car ad}}$ " for the quantity used in advertisements for cars. In the US's non-metric system of units, what would be the units of  $a$  and  $a_{\text{car ad}}$ ? How would the use and interpretation of large and small, positive and negative values be different for  $a$  as opposed to  $a_{\text{car ad}}$ ?

**C** Two people stand on the edge of a cliff. As they lean over the edge, one person throws a rock down, while the other throws one straight up with an exactly opposite initial velocity. Compare the speeds of the rocks on impact at the bottom of the cliff.

## 3.3 Positive and negative acceleration

Gravity always pulls down, but that does not mean it always speeds things up. If you throw a ball straight up, gravity will first slow it down to  $v = 0$  and then begin increasing its speed. When I took physics in high school, I got the impression that positive signs of acceleration indicated speeding up, while negative accelerations represented slowing down, i.e., deceleration. Such a definition would be inconvenient, however, because we would then have to say that the same downward tug of gravity could produce either a positive

or a negative acceleration. As we will see in the following example, such a definition also would not be the same as the slope of the  $v-t$  graph

Let's study the example of the rising and falling ball. In the example of the person falling from a bridge, I assumed positive velocity values without calling attention to it, which meant I was assuming a coordinate system whose  $x$  axis pointed down. In this example, where the ball is reversing direction, it is not possible to avoid negative velocities by a tricky choice of axis, so let's make the more natural choice of an axis pointing up. The ball's velocity will initially be a positive number, because it is heading up, in the same direction as the  $x$  axis, but on the way back down, it will be a negative number. As shown in the figure, the  $v-t$  graph does not do anything special at the top of the ball's flight, where  $v$  equals 0. Its slope is always negative. In the left half of the graph, there is a negative slope because the positive velocity is getting closer to zero. On the right side, the negative slope is due to a negative velocity that is getting farther from zero, so we say that the ball is speeding up, but its velocity is decreasing!

To summarize, what makes the most sense is to stick with the original definition of acceleration as the slope of the  $v-t$  graph,  $\Delta v/\Delta t$ . By this definition, it just isn't necessarily true that things speeding up have positive acceleration while things slowing down have negative acceleration. The word "deceleration" is not used much by physicists, and the word "acceleration" is used unblushingly to refer to slowing down as well as speeding up: "There was a red light, and we accelerated to a stop."

#### *Numerical calculation of a negative acceleration* example 4

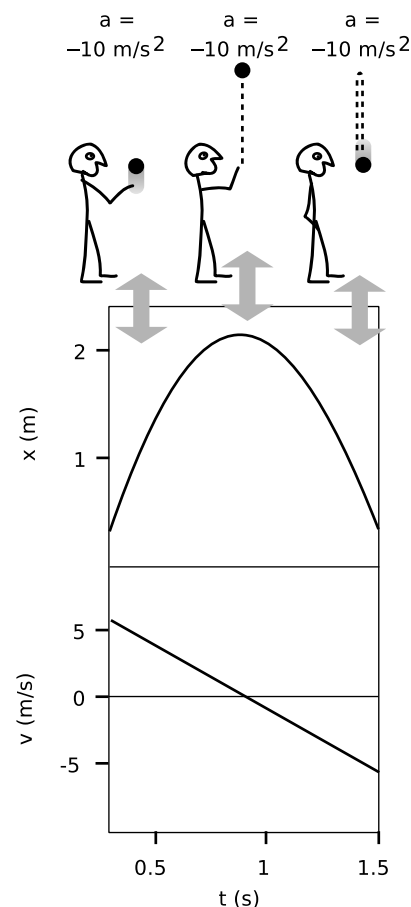
▷ In figure i, what happens if you calculate the acceleration between  $t = 1.0$  and  $1.5$  s?

▷ Reading from the graph, it looks like the velocity is about  $-1$  m/s at  $t = 1.0$  s, and around  $-6$  m/s at  $t = 1.5$  s. The acceleration, figured between these two points, is

$$a = \frac{\Delta v}{\Delta t} = \frac{(-6 \text{ m/s}) - (-1 \text{ m/s})}{(1.5 \text{ s}) - (1.0 \text{ s})} = -10 \text{ m/s}^2.$$

Even though the ball is speeding up, it has a negative acceleration.

Another way of convincing you that this way of handling the plus and minus signs makes sense is to think of a device that measures acceleration. After all, physics is supposed to use operational definitions, ones that relate to the results you get with actual measuring devices. Consider an air freshener hanging from the rear-view mirror of your car. When you speed up, the air freshener swings backward. Suppose we define this as a positive reading. When you slow down, the air freshener swings forward, so we'll call this a negative reading



i / The ball's acceleration stays the same — on the way up, at the top, and on the way back down. It's always negative.

on our accelerometer. But what if you put the car in reverse and start speeding up backwards? Even though you're speeding up, the accelerometer responds in the same way as it did when you were going forward and slowing down. There are four possible cases:

motion of car	accelerometer swings	slope of v-t graph	direction of force acting on car
forward, speeding up	backward	+	forward
forward, slowing down	forward	−	backward
backward, speeding up	forward	−	backward
backward, slowing down	backward	+	forward

Note the consistency of the three right-hand columns — nature is trying to tell us that this is the right system of classification, not the left-hand column.

Because the positive and negative signs of acceleration depend on the choice of a coordinate system, the acceleration of an object under the influence of gravity can be either positive or negative. Rather than having to write things like “ $g = 9.8 \text{ m/s}^2$  or  $-9.8 \text{ m/s}^2$ ” every time we want to discuss  $g$ 's numerical value, we simply define  $g$  as the absolute value of the acceleration of objects moving under the influence of gravity. We consistently let  $g = 9.8 \text{ m/s}^2$ , but we may have either  $a = g$  or  $a = -g$ , depending on our choice of a coordinate system.

#### *Acceleration with a change in direction of motion*      *example 5*

▷ A person kicks a ball, which rolls up a sloping street, comes to a halt, and rolls back down again. The ball has constant acceleration. The ball is initially moving at a velocity of  $4.0 \text{ m/s}$ , and after  $10.0 \text{ s}$  it has returned to where it started. At the end, it has sped back up to the same speed it had initially, but in the opposite direction. What was its acceleration?

▷ By giving a positive number for the initial velocity, the statement of the question implies a coordinate axis that points up the slope of the hill. The “same” speed in the opposite direction should therefore be represented by a negative number,  $-4.0 \text{ m/s}$ . The acceleration is

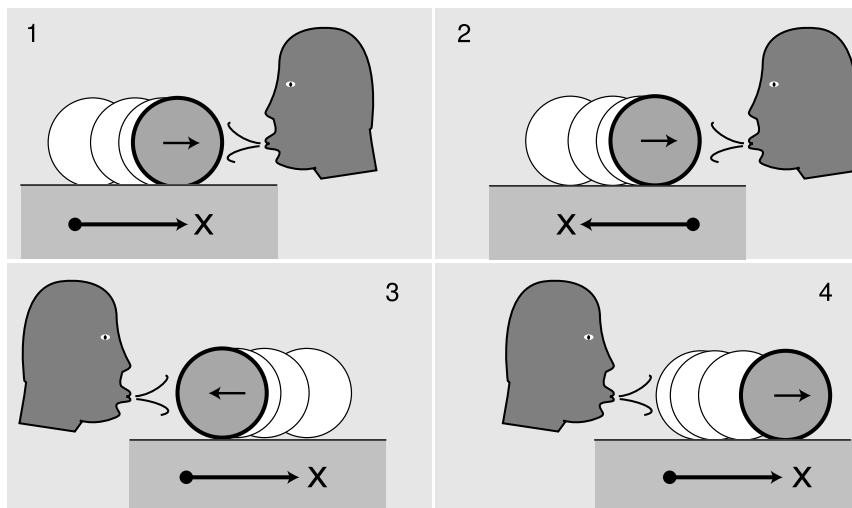
$$\begin{aligned}
 a &= \Delta v / \Delta t \\
 &= (v_f - v_0) / 10.0 \text{ s} \\
 &= [(-4.0 \text{ m/s}) - (4.0 \text{ m/s})] / 10.0 \text{ s} \\
 &= -0.80 \text{ m/s}^2 .
 \end{aligned}$$

The acceleration was no different during the upward part of the roll than on the downward part of the roll.

Incorrect solution: Acceleration is  $\Delta v / \Delta t$ , and at the end it's not moving any faster or slower than when it started, so  $\Delta v = 0$  and

$a = 0$ .

x The velocity does change, from a positive number to a negative number.



Discussion question B.

### Discussion questions

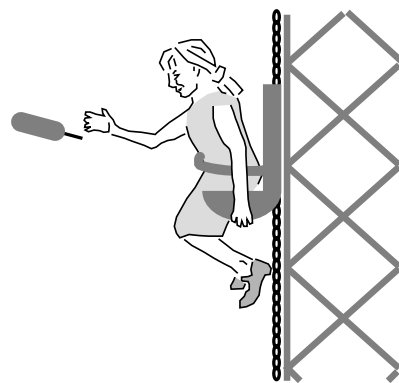
**A** A child repeatedly jumps up and down on a trampoline. Discuss the sign and magnitude of his acceleration, including both the time when he is in the air and the time when his feet are in contact with the trampoline.

**B** The figure shows a refugee from a Picasso painting blowing on a rolling water bottle. In some cases the person's blowing is speeding the bottle up, but in others it is slowing it down. The arrow inside the bottle shows which direction it is going, and a coordinate system is shown at the bottom of each figure. In each case, figure out the plus or minus signs of the velocity and acceleration. It may be helpful to draw a  $v - t$  graph in each case.

**C** Sally is on an amusement park ride which begins with her chair being hoisted straight up a tower at a constant speed of 60 miles/hour. Despite stern warnings from her father that he'll take her home the next time she misbehaves, she decides that as a scientific experiment she really needs to release her corndog over the side as she's on the way up. She does not throw it. She simply sticks it out of the car, lets it go, and watches it against the background of the sky, with no trees or buildings as reference points. What does the corndog's motion look like as observed by Sally? Does its speed ever appear to her to be zero? What acceleration does she observe it to have: is it ever positive? negative? zero? What would her enraged father answer if asked for a similar description of its motion as it appears to him, standing on the ground?

**D** Can an object maintain a constant acceleration, but meanwhile reverse the direction of its velocity?

**E** Can an object have a velocity that is positive and increasing at the same time that its acceleration is decreasing?



Discussion question C.

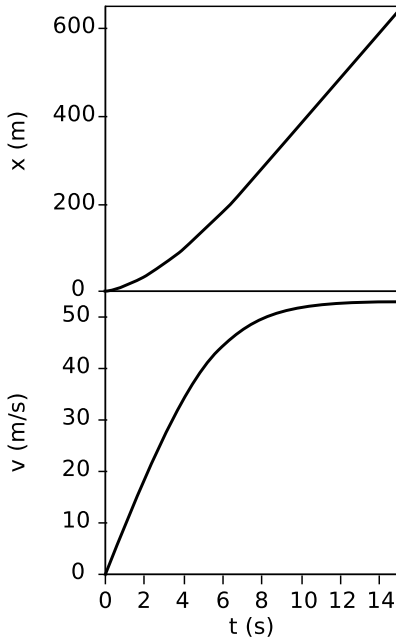


### 3.4 Varying acceleration

So far we have only been discussing examples of motion for which the  $v - t$  graph is linear. If we wish to generalize our definition to  $v - t$  graphs that are more complex curves, the best way to proceed is similar to how we defined velocity for curved  $x - t$  graphs:

#### definition of acceleration

The acceleration of an object at any instant is the slope of the tangent line passing through its  $v$ -versus- $t$  graph at the relevant point.



k / Example 6.

#### A skydiver

#### example 6

▷ The graphs in figure k show the results of a fairly realistic computer simulation of the motion of a skydiver, including the effects of air friction. The  $x$  axis has been chosen pointing down, so  $x$  is increasing as she falls. Find (a) the skydiver's acceleration at  $t = 3.0$  s, and also (b) at  $t = 7.0$  s.

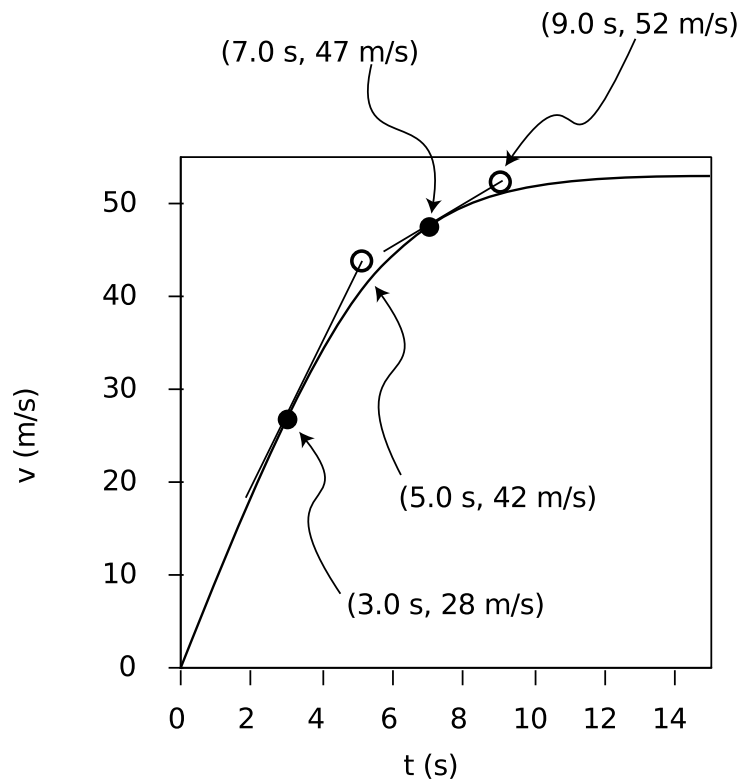
▷ The solution is shown in figure l. I've added tangent lines at the two points in question.

(a) To find the slope of the tangent line, I pick two points on the line (not necessarily on the actual curve): (3.0 s, 28 m/s) and (5.0 s, 42 m/s). The slope of the tangent line is  $(42 \text{ m/s} - 28 \text{ m/s}) / (5.0 \text{ s} - 3.0 \text{ s}) = 7.0 \text{ m/s}^2$ .

(b) Two points on this tangent line are (7.0 s, 47 m/s) and (9.0 s, 52 m/s). The slope of the tangent line is  $(52 \text{ m/s} - 47 \text{ m/s}) / (9.0 \text{ s} - 7.0 \text{ s}) = 2.5 \text{ m/s}^2$ .

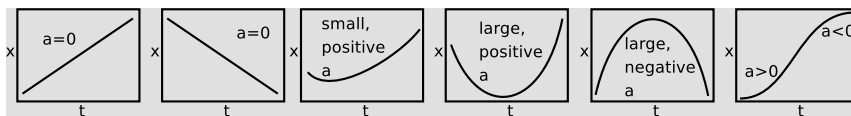
Physically, what's happening is that at  $t = 3.0$  s, the skydiver is not yet going very fast, so air friction is not yet very strong. She therefore has an acceleration almost as great as  $g$ . At  $t = 7.0$  s, she is moving almost twice as fast (about 100 miles per hour), and air friction is extremely strong, resulting in a significant departure from the idealized case of no air friction.

In example 6, the  $x - t$  graph was not even used in the solution of the problem, since the definition of acceleration refers to the slope of the  $v - t$  graph. It is possible, however, to interpret an  $x - t$  graph to find out something about the acceleration. An object with zero acceleration, i.e., constant velocity, has an  $x - t$  graph that is a straight line. A straight line has no curvature. A change in velocity requires a change in the slope of the  $x - t$  graph, which means that it is a curve rather than a line. Thus acceleration relates to the curvature of the  $x - t$  graph. Figure m shows some examples.



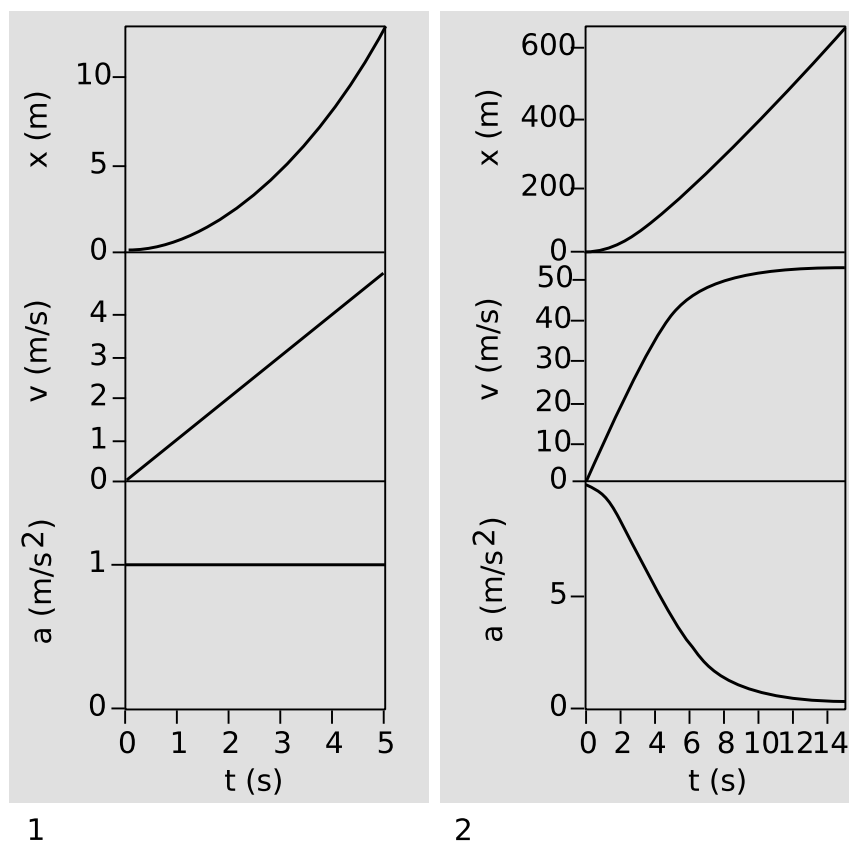
l / The solution to example 6.

In example 6, the  $x - t$  graph was more strongly curved at the beginning, and became nearly straight at the end. If the  $x - t$  graph is nearly straight, then its slope, the velocity, is nearly constant, and the acceleration is therefore small. We can thus interpret the acceleration as representing the curvature of the  $x - t$  graph, as shown in figure m. If the “cup” of the curve points up, the acceleration is positive, and if it points down, the acceleration is negative.

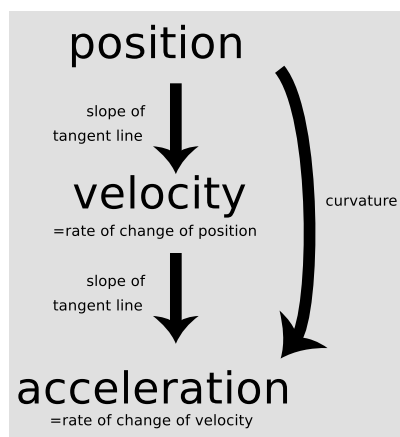


m / Acceleration relates to the curvature of the  $x - t$  graph.

Since the relationship between  $a$  and  $v$  is analogous to the relationship between  $v$  and  $x$ , we can also make graphs of acceleration as a function of time, as shown in figure n.



n / Examples of graphs of  $x$ ,  $v$ , and  $a$  versus  $t$ . 1. An object in free fall, with no friction. 2. A continuation of example 6, the skydiver.



o / How position, velocity, and acceleration are related.

▷ *Solved problem: Drawing a  $v - t$  graph.* page 115, problem 14

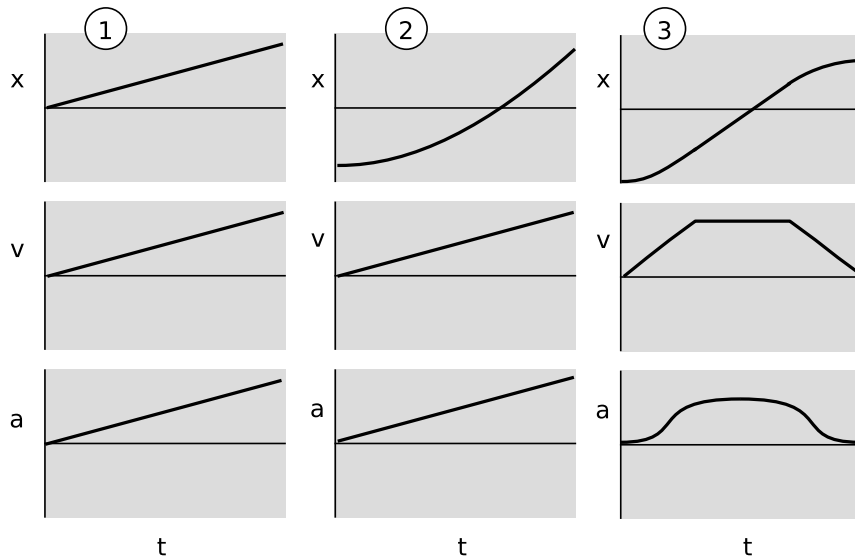
▷ *Solved problem: Drawing  $v - t$  and  $a - t$  graphs.* page 116, problem 20

Figure o summarizes the relationships among the three types of graphs.

### Discussion questions

**A** Describe in words how the changes in the  $a - t$  graph in figure n/2 relate to the behavior of the  $v - t$  graph.

**B** Explain how each set of graphs contains inconsistencies, and fix them.



**C** In each case, pick a coordinate system and draw  $x - t$ ,  $v - t$ , and  $a - t$  graphs. Picking a coordinate system means picking where you want  $x = 0$  to be, and also picking a direction for the positive  $x$  axis.

(1) An ocean liner is cruising in a straight line at constant speed.

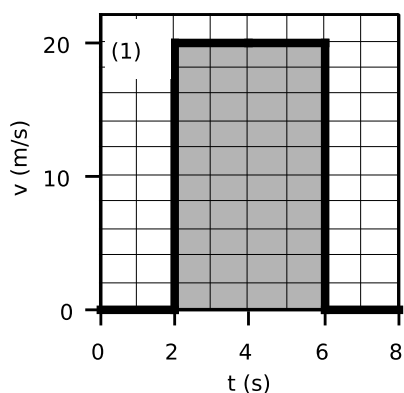
(2) You drop a ball. Draw two different sets of graphs (a total of 6), with one set's positive  $x$  axis pointing in the opposite direction compared to the other's.

(3) You're driving down the street looking for a house you've never been to before. You realize you've passed the address, so you slow down, put the car in reverse, back up, and stop in front of the house.

### 3.5 The area under the velocity-time graph

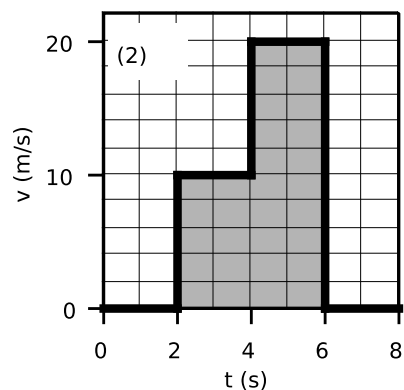
A natural question to ask about falling objects is how fast they fall, but Galileo showed that the question has no answer. The physical law that he discovered connects a cause (the attraction of the planet Earth's mass) to an effect, but the effect is predicted in terms of an acceleration rather than a velocity. In fact, no physical law predicts a definite velocity as a result of a specific phenomenon, because velocity cannot be measured in absolute terms, and only changes in velocity relate directly to physical phenomena.

The unfortunate thing about this situation is that the definitions of velocity and acceleration are stated in terms of the tangent-line technique, which lets you go from  $x$  to  $v$  to  $a$ , but not the other way around. Without a technique to go backwards from  $a$  to  $v$  to  $x$ , we cannot say anything quantitative, for instance, about the  $x - t$  graph of a falling object. Such a technique does exist, and I used it to make the  $x - t$  graphs in all the examples above.

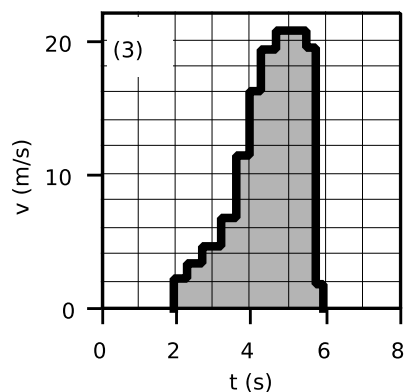


First let's concentrate on how to get  $x$  information out of a  $v - t$  graph. In example p/1, an object moves at a speed of 20 m/s for a period of 4.0 s. The distance covered is  $\Delta x = v\Delta t = (20 \text{ m/s}) \times (4.0 \text{ s}) = 80 \text{ m}$ . Notice that the quantities being multiplied are the width and the height of the shaded rectangle — or, strictly speaking, the time represented by its width and the velocity represented by its height. The distance of  $\Delta x = 80 \text{ m}$  thus corresponds to the area of the shaded part of the graph.

The next step in sophistication is an example like p/2, where the object moves at a constant speed of 10 m/s for two seconds, then for two seconds at a different constant speed of 20 m/s. The shaded region can be split into a small rectangle on the left, with an area representing  $\Delta x = 20 \text{ m}$ , and a taller one on the right, corresponding to another 40 m of motion. The total distance is thus 60 m, which corresponds to the total area under the graph.



An example like p/3 is now just a trivial generalization; there is simply a large number of skinny rectangular areas to add up. But notice that graph p/3 is quite a good approximation to the smooth curve p/4. Even though we have no formula for the area of a funny shape like p/4, we can approximate its area by dividing it up into smaller areas like rectangles, whose area is easier to calculate. If someone hands you a graph like p/4 and asks you to find the area under it, the simplest approach is just to count up the little rectangles on the underlying graph paper, making rough estimates of fractional rectangles as you go along.

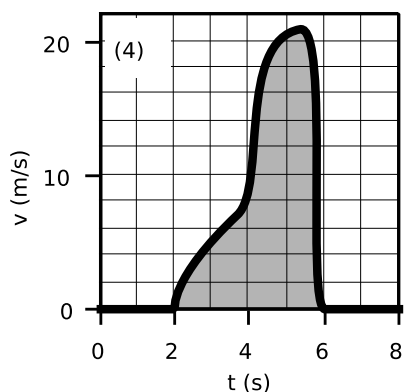


That's what I've done in figure q. Each rectangle on the graph paper is 1.0 s wide and 2 m/s tall, so it represents 2 m. Adding up all the numbers gives  $\Delta x = 41 \text{ m}$ . If you needed better accuracy, you could use graph paper with smaller rectangles.

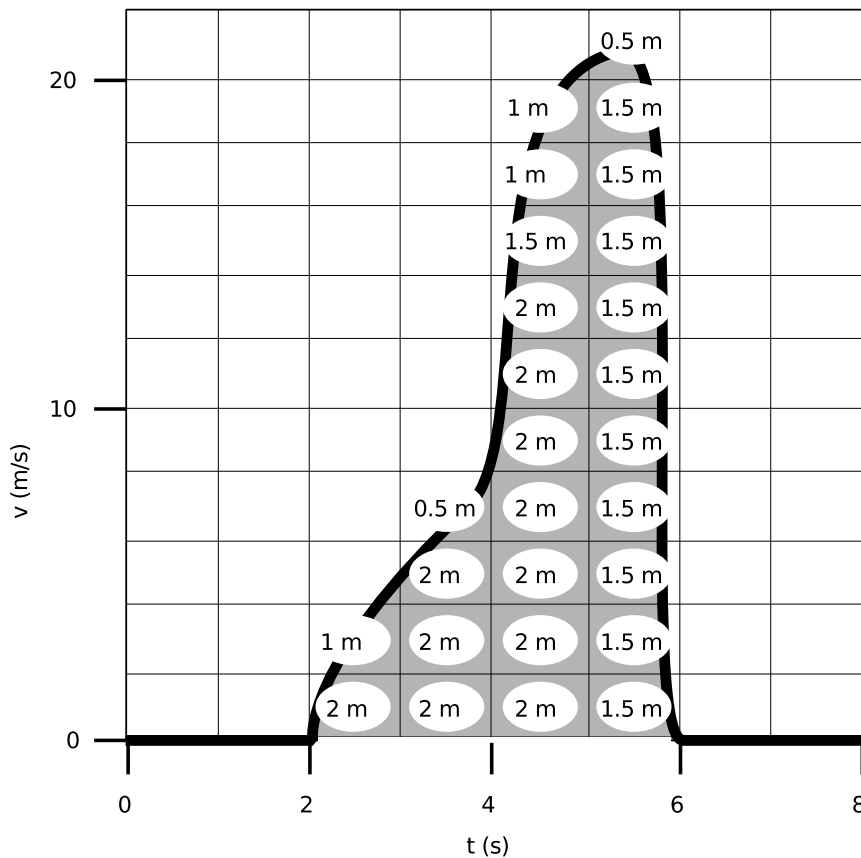
It's important to realize that this technique gives you  $\Delta x$ , not  $x$ . The  $v - t$  graph has no information about where the object was when it started.

The following are important points to keep in mind when applying this technique:

- If the range of  $v$  values on your graph does not extend down to zero, then you will get the wrong answer unless you compensate by adding in the area that is not shown.
- As in the example, one rectangle on the graph paper does not necessarily correspond to one meter of distance.
- Negative velocity values represent motion in the opposite direction, so as suggested by figure r, area under the  $t$  axis should be subtracted, i.e., counted as “negative area.”



p/ The area under the  $v - t$  graph gives  $\Delta x$ .



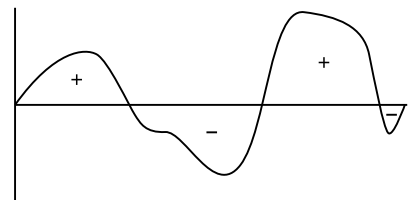
q / An example using estimation of fractions of a rectangle.

- Since the result is a  $\Delta x$  value, it only tells you  $x_{after} - x_{before}$ , which may be less than the actual distance traveled. For instance, the object could come back to its original position at the end, which would correspond to  $\Delta x = 0$ , even though it had actually moved a nonzero distance.

Finally, note that one can find  $\Delta v$  from an  $a - t$  graph using an entirely analogous method. Each rectangle on the  $a - t$  graph represents a certain amount of velocity change.

### Discussion question

**A** Roughly what would a pendulum's  $v - t$  graph look like? What would happen when you applied the area-under-the-curve technique to find the pendulum's  $\Delta x$  for a time period covering many swings?



r / Area underneath the axis is considered negative.

## 3.6 Algebraic results for constant acceleration

Although the area-under-the-curve technique can be applied to any graph, no matter how complicated, it may be laborious to carry out, and if fractions of rectangles must be estimated the result will only be approximate. In the special case of motion with constant acceleration, it is possible to find a convenient shortcut which produces exact results. When the acceleration is constant, the  $v - t$  graph

is a straight line, as shown in the figure. The area under the curve can be divided into a triangle plus a rectangle, both of whose areas can be calculated exactly:  $A = bh$  for a rectangle and  $A = bh/2$  for a triangle. The height of the rectangle is the initial velocity,  $v_o$ , and the height of the triangle is the change in velocity from beginning to end,  $\Delta v$ . The object's  $\Delta x$  is therefore given by the equation  $\Delta x = v_o \Delta t + \Delta v \Delta t/2$ . This can be simplified a little by using the definition of acceleration,  $a = \Delta v / \Delta t$ , to eliminate  $\Delta v$ , giving

$$\Delta x = v_o \Delta t + \frac{1}{2} a \Delta t^2 \quad . \quad \begin{array}{l} \text{[motion with} \\ \text{constant acceleration]} \end{array}$$

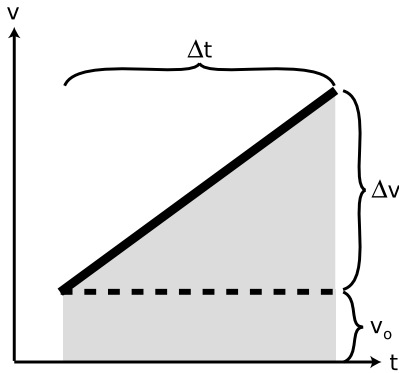
Since this is a second-order polynomial in  $\Delta t$ , the graph of  $\Delta x$  versus  $\Delta t$  is a parabola, and the same is true of a graph of  $x$  versus  $t$  — the two graphs differ only by shifting along the two axes. Although I have derived the equation using a figure that shows a positive  $v_o$ , positive  $a$ , and so on, it still turns out to be true regardless of what plus and minus signs are involved.

Another useful equation can be derived if one wants to relate the change in velocity to the distance traveled. This is useful, for instance, for finding the distance needed by a car to come to a stop. For simplicity, we start by deriving the equation for the special case of  $v_o = 0$ , in which the final velocity  $v_f$  is a synonym for  $\Delta v$ . Since velocity and distance are the variables of interest, not time, we take the equation  $\Delta x = \frac{1}{2} a \Delta t^2$  and use  $\Delta t = \Delta v / a$  to eliminate  $\Delta t$ . This gives  $\Delta x = (\Delta v)^2 / 2a$ , which can be rewritten as

$$v_f^2 = 2a\Delta x \quad . \quad \text{[motion with constant acceleration, } v_o = 0\text{]}$$

For the more general case where  $v_o \neq 0$ , we skip the tedious algebra leading to the more general equation,

$$v_f^2 = v_o^2 + 2a\Delta x \quad . \quad \text{[motion with constant acceleration]}$$



s / The shaded area tells us how far an object moves while accelerating at a constant rate.

To help get this all organized in your head, first let's categorize the variables as follows:

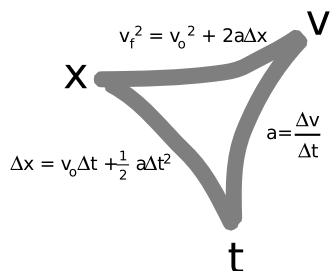
Variables that change during motion with constant acceleration:

$$x, v, t$$

Variable that doesn't change:

$$a$$

If you know one of the changing variables and want to find another, there is always an equation that relates those two:



The symmetry among the three variables is imperfect only because the equation relating  $x$  and  $t$  includes the initial velocity.

There are two main difficulties encountered by students in applying these equations:

- The equations apply only to motion with constant acceleration. You can't apply them if the acceleration is changing.
- Students are often unsure of which equation to use, or may cause themselves unnecessary work by taking the longer path around the triangle in the chart above. Organize your thoughts by listing the variables you are given, the ones you want to find, and the ones you aren't given and don't care about.

#### *Saving an old lady*

#### *example 7*

▷ You are trying to pull an old lady out of the way of an oncoming truck. You are able to give her an acceleration of  $20 \text{ m/s}^2$ . Starting from rest, how much time is required in order to move her 2 m?

▷ First we organize our thoughts:

Variables given:  $\Delta x$ ,  $a$ ,  $v_o$

Variables desired:  $\Delta t$

Irrelevant variables:  $v_f$

Consulting the triangular chart above, the equation we need is clearly  $\Delta x = v_o\Delta t + \frac{1}{2}a\Delta t^2$ , since it has the four variables of interest and omits the irrelevant one. Eliminating the  $v_o$  term and solving for  $\Delta t$  gives  $\Delta t = \sqrt{2\Delta x/a} = 0.4 \text{ s}$ .



- ▷ *Solved problem: A stupid celebration*                      page 115, problem 15
- ▷ *Solved problem: Dropping a rock on Mars*                  page 115, problem 16
- ▷ *Solved problem: The Dodge Viper*                              page 116, problem 18
- ▷ *Solved problem: Half-way sped up*                            page 116, problem 22

### Discussion questions

**A** In chapter 1, I gave examples of correct and incorrect reasoning about proportionality, using questions about the scaling of area and volume. Try to translate the incorrect modes of reasoning shown there into mistakes about the following question: If the acceleration of gravity on Mars is  $1/3$  that on Earth, how many times longer does it take for a rock to drop the same distance on Mars?

**B** Check that the units make sense in the three equations derived in this section.

## 3.7 $\int$ Applications of calculus

In section 2.7, I discussed how the slope-of-the-tangent-line idea related to the calculus concept of a derivative, and the branch of calculus known as differential calculus. The other main branch of calculus, integral calculus, has to do with the area-under-the-curve concept discussed in section 3.5. Again there is a concept, a notation, and a bag of tricks for doing things symbolically rather than graphically. In calculus, the area under the  $v - t$  graph between  $t = t_1$  and  $t = t_2$  is notated like this:

$$\text{area under curve} = \Delta x = \int_{t_1}^{t_2} v dt \quad .$$

The expression on the right is called an integral, and the s-shaped symbol, the integral sign, is read as “integral of ...”

Integral calculus and differential calculus are closely related. For instance, if you take the derivative of the function  $x(t)$ , you get the function  $v(t)$ , and if you integrate the function  $v(t)$ , you get  $x(t)$  back again. In other words, integration and differentiation are inverse operations. This is known as the fundamental theorem of calculus.

On an unrelated topic, there is a special notation for taking the derivative of a function twice. The acceleration, for instance, is the second (i.e., double) derivative of the position, because differentiating  $x$  once gives  $v$ , and then differentiating  $v$  gives  $a$ . This is written as

$$a = \frac{d^2 x}{dt^2} \quad .$$

The seemingly inconsistent placement of the twos on the top and bottom confuses all beginning calculus students. The motivation for this funny notation is that acceleration has units of  $\text{m/s}^2$ , and

the notation correctly suggests that: the top looks like it has units of meters, the bottom seconds<sup>2</sup>. The notation is not meant, however, to suggest that  $t$  is really squared.

# Summary

## Selected vocabulary

gravity . . . . .	A general term for the phenomenon of attraction between things having mass. The attraction between our planet and a human-sized object causes the object to fall.
acceleration . . .	The rate of change of velocity; the slope of the tangent line on a $v - t$ graph.

## Notation

$v_o$ . . . . .	initial velocity
$v_f$ . . . . .	final velocity
$a$ . . . . .	acceleration
$g$ . . . . .	the acceleration of objects in free fall; the strength of the local gravitational field

## Summary

Galileo showed that when air resistance is negligible all falling bodies have the same motion regardless of mass. Moreover, their  $v - t$  graphs are straight lines. We therefore define a quantity called acceleration as the slope,  $\Delta v / \Delta t$ , of an object's  $v - t$  graph. In cases other than free fall, the  $v - t$  graph may be curved, in which case the definition is generalized as the slope of a tangent line on the  $v - t$  graph. The acceleration of objects in free fall varies slightly across the surface of the earth, and greatly on other planets.

Positive and negative signs of acceleration are defined according to whether the  $v - t$  graph slopes up or down. This definition has the advantage that a force with a given sign, representing its direction, always produces an acceleration with the same sign.

The area under the  $v - t$  graph gives  $\Delta x$ , and analogously the area under the  $a - t$  graph gives  $\Delta v$ .

For motion with constant acceleration, the following three equations hold:

$$\begin{aligned}\Delta x &= v_o \Delta t + \frac{1}{2} a \Delta t^2 \\ v_f^2 &= v_o^2 + 2a \Delta x \\ a &= \frac{\Delta v}{\Delta t}\end{aligned}$$

They are not valid if the acceleration is changing.

## Problems

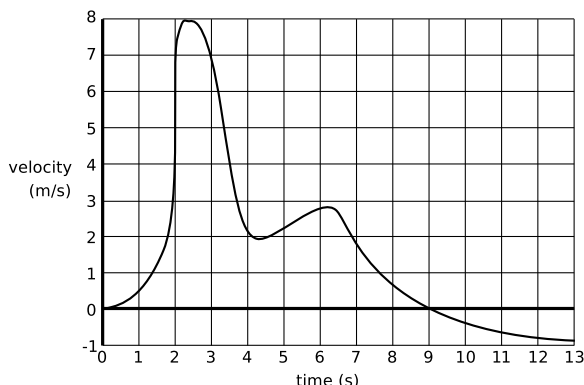
### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

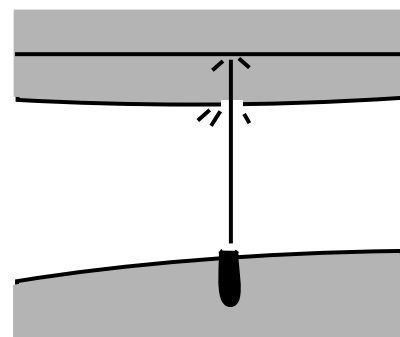
★ A difficult problem.

**1** The graph represents the velocity of a bee along a straight line. At  $t = 0$ , the bee is at the hive. (a) When is the bee farthest from the hive? (b) How far is the bee at its farthest point from the hive? (c) At  $t = 13$  s, how far is the bee from the hive? [Hint: Try problem 19 first.] ✓



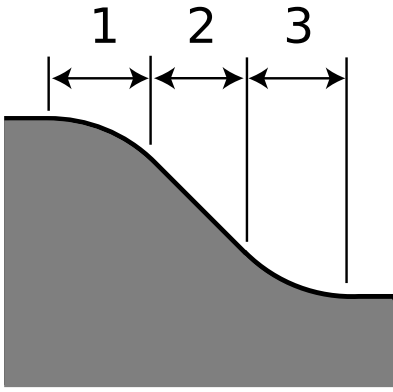
**2** A rock is dropped into a pond. Draw plots of its position versus time, velocity versus time, and acceleration versus time. Include its whole motion, starting from the moment it is dropped, and continuing while it falls through the air, passes through the water, and ends up at rest on the bottom of the pond. Do your work on a photocopy or a printout of page 121.

**3** In an 18th-century naval battle, a cannon ball is shot horizontally, passes through the side of an enemy ship's hull, flies across the galley, and lodges in a bulkhead. Draw plots of its horizontal position, velocity, and acceleration as functions of time, starting while it is inside the cannon and has not yet been fired, and ending when it comes to rest. There is not any significant amount of friction from the air. Although the ball may rise and fall, you are only concerned with its horizontal motion, as seen from above. Do your work on a photocopy or a printout of page 121.



Problem 3.

**4** Draw graphs of position, velocity, and acceleration as functions of time for a person bungee jumping. (In bungee jumping, a person has a stretchy elastic cord tied to his/her ankles, and jumps off of a high platform. At the bottom of the fall, the cord brings the person up short. Presumably the person bounces up a little.) Do your work on a photocopy or a printout of page 121.



Problem 5.

**5** A ball rolls down the ramp shown in the figure, consisting of a curved knee, a straight slope, and a curved bottom. For each part of the ramp, tell whether the ball's velocity is increasing, decreasing, or constant, and also whether the ball's acceleration is increasing, decreasing, or constant. Explain your answers. Assume there is no air friction or rolling resistance. Hint: Try problem 20 first. [Based on a problem by Hewitt.]

**6** A toy car is released on one side of a piece of track that is bent into an upright  $U$  shape. The car goes back and forth. When the car reaches the limit of its motion on one side, its velocity is zero. Is its acceleration also zero? Explain using a  $v - t$  graph. [Based on a problem by Serway and Faughn.]

**7** What is the acceleration of a car that moves at a steady velocity of 100 km/h for 100 seconds? Explain your answer. [Based on a problem by Hewitt.]

**8** A physics homework question asks, "If you start from rest and accelerate at  $1.54 \text{ m/s}^2$  for 3.29 s, how far do you travel by the end of that time?" A student answers as follows:

$$1.54 \times 3.29 = 5.07 \text{ m}$$

His Aunt Wanda is good with numbers, but has never taken physics. She doesn't know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?

**9** You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well? ✓

**10** You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get half-way there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.

(a) Find a formula for the time,  $T$ , required for the round trip, in terms of  $d$ , the distance from our sun to the star, and  $a$ , the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.

(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of  $d = 4 \times 10^{16} \text{ m}$ . Suppose you use an acceleration of  $a = 10 \text{ m/s}^2$ , just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?

(c) Using the same numbers for  $d$  and  $a$ , find your maximum speed. Compare this to the speed of light, which is  $3.0 \times 10^8 \text{ m/s}$ . (Later in this course, you will learn that there are some new things going

on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you've learned so far.) ✓ ★

**11** You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)

**12** Alice drops a rock off a cliff. Bubba shoots a gun straight down from the edge of the same cliff. Compare the accelerations of the rock and the bullet while they are in the air on the way down. [Based on a problem by Serway and Faughn.]

**13** A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by an equation of the form

$$y = b - c \left( t + k e^{-t/k} \right) \quad ,$$

where  $e$  is the base of natural logarithms, and  $b$ ,  $c$ , and  $k$  are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.

(a) What units would  $b$ ,  $c$ , and  $k$  have to have for the equation to make sense?

(b) Find the person's velocity,  $v$ , as a function of time. [You will need to use the chain rule, and the fact that  $d(e^x)/dx = e^x$ .] ✓

(c) Use your answer from part (b) to get an interpretation of the constant  $c$ . [Hint:  $e^{-x}$  approaches zero for large values of  $x$ .]

(d) Find the person's acceleration,  $a$ , as a function of time. ✓

(e) Use your answer from part (b) to show that if she waits long enough to open her chute, her acceleration will become very small.

∫

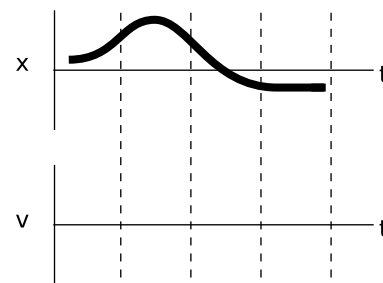
**14** The top part of the figure shows the position-versus-time graph for an object moving in one dimension. On the bottom part of the figure, sketch the corresponding  $v$ -versus- $t$  graph.

▷ Solution, p. 518

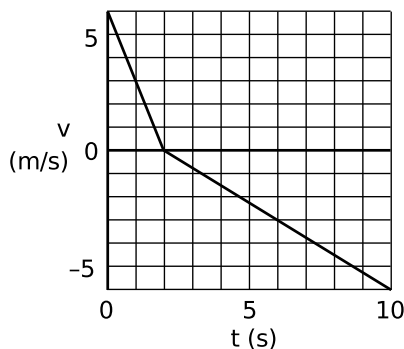
**15** On New Year's Eve, a stupid person fires a pistol straight up. The bullet leaves the gun at a speed of 100 m/s. How long does it take before the bullet hits the ground? ▷ Solution, p. 518

**16** If the acceleration of gravity on Mars is  $1/3$  that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance. ▷ Solution, p. 518

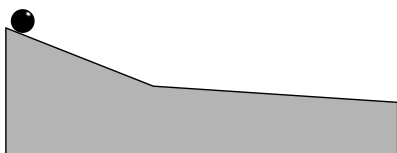
**17** A honeybee's position as a function of time is given by  $x = 10t - t^3$ , where  $t$  is in seconds and  $x$  in meters. What is its acceleration at  $t = 3.0$  s? ▷ Solution, p. 518 ∫



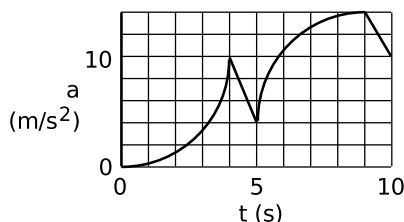
Problem 14.



Problem 19.



Problem 20.



Problem 23.

**18** In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car's top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with *constant* acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part b and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time? ▷ Solution, p. 518

**19** The graph represents the motion of a ball that rolls up a hill and then back down. When does the ball return to the location it had at  $t = 0$ ? ▷ Solution, p. 519

**20** (a) The ball is released at the top of the ramp shown in the figure. Friction is negligible. Use physical reasoning to draw  $v - t$  and  $a - t$  graphs. Assume that the ball doesn't bounce at the point where the ramp changes slope. (b) Do the same for the case where the ball is rolled up the slope from the right side, but doesn't quite have enough speed to make it over the top. ▷ Solution, p. 519

**21** You throw a rubber ball up, and it falls and bounces several times. Draw graphs of position, velocity, and acceleration as functions of time. ▷ Solution, p. 519

**22** Starting from rest, a ball rolls down a ramp, traveling a distance  $L$  and picking up a final speed  $v$ . How much of the distance did the ball have to cover before achieving a speed of  $v/2$ ? [Based on a problem by Arnold Arons.] ▷ Solution, p. 520

**23** The graph shows the acceleration of a chipmunk in a TV cartoon. It consists of two circular arcs and two line segments. At  $t = 0.00$  s, the chipmunk's velocity is  $-3.10$  m/s. What is its velocity at  $t = 10.00$  s? ✓

**24** Find the error in the following calculation. A student wants to find the distance traveled by a car that accelerates from rest for 5.0 s with an acceleration of  $2.0$  m/s<sup>2</sup>. First he solves  $a = \Delta v / \Delta t$  for  $\Delta v = 10$  m/s. Then he multiplies to find  $(10 \text{ m/s})(5.0 \text{ s}) = 50$  m. Do not just recalculate the result by a different method; if that was all you did, you'd have no way of knowing which calculation was correct, yours or his.

**25** Acceleration could be defined either as  $\Delta v/\Delta t$  or as the slope of the tangent line on the  $v - t$  graph. Is either one superior as a definition, or are they equivalent? If you say one is better, give an example of a situation where it makes a difference which one you use.

**26** If an object starts accelerating from rest, we have  $v^2 = 2a\Delta x$  for its speed after it has traveled a distance  $\Delta x$ . Explain in words why it makes sense that the equation has velocity squared, but distance only to the first power. Don't recapitulate the derivation in the book, or give a justification based on units. The point is to explain what this feature of the equation tells us about the way speed increases as more distance is covered.

**27** The figure shows a practical, simple experiment for determining  $g$  to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights  $\Delta x_1$  and  $\Delta x_2$ . A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity  $T$  defined as  $T = \Delta t_2 - \Delta t_1$ , i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound when they are released, we have no way of measuring the individual times  $\Delta t_2$  and  $\Delta t_1$ .

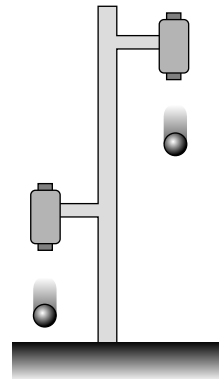
(a) Find an equation for  $g$  in terms of the measured quantities  $T$ ,  $\Delta x_1$  and  $\Delta x_2$ . ✓

(b) Check the units of your equation.

(c) Check that your equation gives the correct result in the case where  $\Delta x_1$  is very close to zero. However, is this case realistic?

(d) What happens when  $\Delta x_1 = \Delta x_2$ ? Discuss this both mathematically and physically.

**28** The speed required for a low-earth orbit is  $7.9 \times 10^3$  m/s (see ch. 10). When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to  $3g$  to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn't necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth's surface.



Problem 27.





**Problem 29.** This spectacular series of photos from a 2011 paper by Burrows and Sutton (“Biomechanics of jumping in the flea,” *J. Exp. Biology* 214:836) shows the flea jumping at about a 45-degree angle, but for the sake of this estimate just consider the case of a flea jumping vertically.

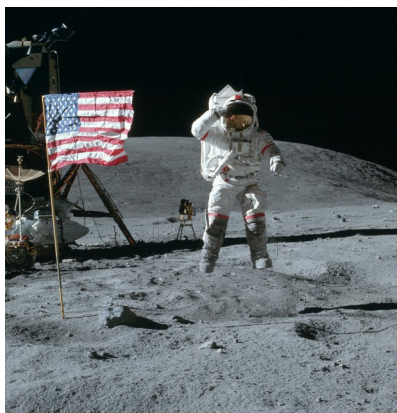
**29** Some fleas can jump as high as 30 cm. The flea only has a short time to build up speed — the time during which its center of mass is accelerating upward but its feet are still in contact with the ground. Make an order-of-magnitude estimate of the acceleration the flea needs to have while straightening its legs, and state your answer in units of  $g$ , i.e., how many “ $g$ ’s it pulls.” (For comparison, fighter pilots black out or die if they exceed about 5 or 10  $g$ ’s.)

**30** Consider the following passage from *Alice in Wonderland*, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall *never* come to an end? “I wonder how many miles I’ve fallen by this time?” she said aloud. “I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think” (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a *very* good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)...

Alice doesn’t know much physics, but let’s try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of  $10 \text{ m/s}^2$ . This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth,  $g$  is zero, because the earth is pulling you equally in every direction at once). ✓

**31** The photo shows Apollo 16 astronaut John Young jumping on the moon and saluting at the top of his jump. The video footage of the jump shows him staying aloft for 1.45 seconds. Gravity on the moon is  $1/6$  as strong as on the earth. Compute the height of the jump. ✓



**Problem 31.**

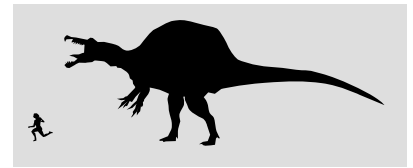
**32** Most people don't know that *Spinosaurus aegyptiacus*, not *Tyrannosaurus rex*, was the biggest tyrannosaur. We can't put a dinosaur on a track and time it in the 100 meter dash, so we can only infer from physical models how fast it could have run. When an animal walks at a normal pace, typically its legs swing more or less like pendulums of the same length  $\ell$ . As a further simplification of this model, let's imagine that the leg simply moves at a fixed acceleration as it falls to the ground. That is, we model the time for a quarter of a stride cycle as being the same as the time required for free fall from a height  $\ell$ . *S. aegyptiacus* had legs about four times longer than those of a human. (a) Compare the time required for a human's stride cycle to that for *S. aegyptiacus*. ✓

(b) Compare their running speeds. ✓

**33** Engineering professor Qingming Li used sensors and video cameras to study punches delivered in the lab by British former welterweight boxing champion Ricky "the Hitman" Hatton. For comparison, Li also let a TV sports reporter put on the gloves and throw punches. The time it took for Hatton's best punch to arrive, i.e., the time his opponent would have had to react, was about 0.47 of that for the reporter. Let's assume that the fist starts from rest and moves with constant acceleration all the way up until impact, at some fixed distance (arm's length). Compare Hatton's acceleration to the reporter's. ✓

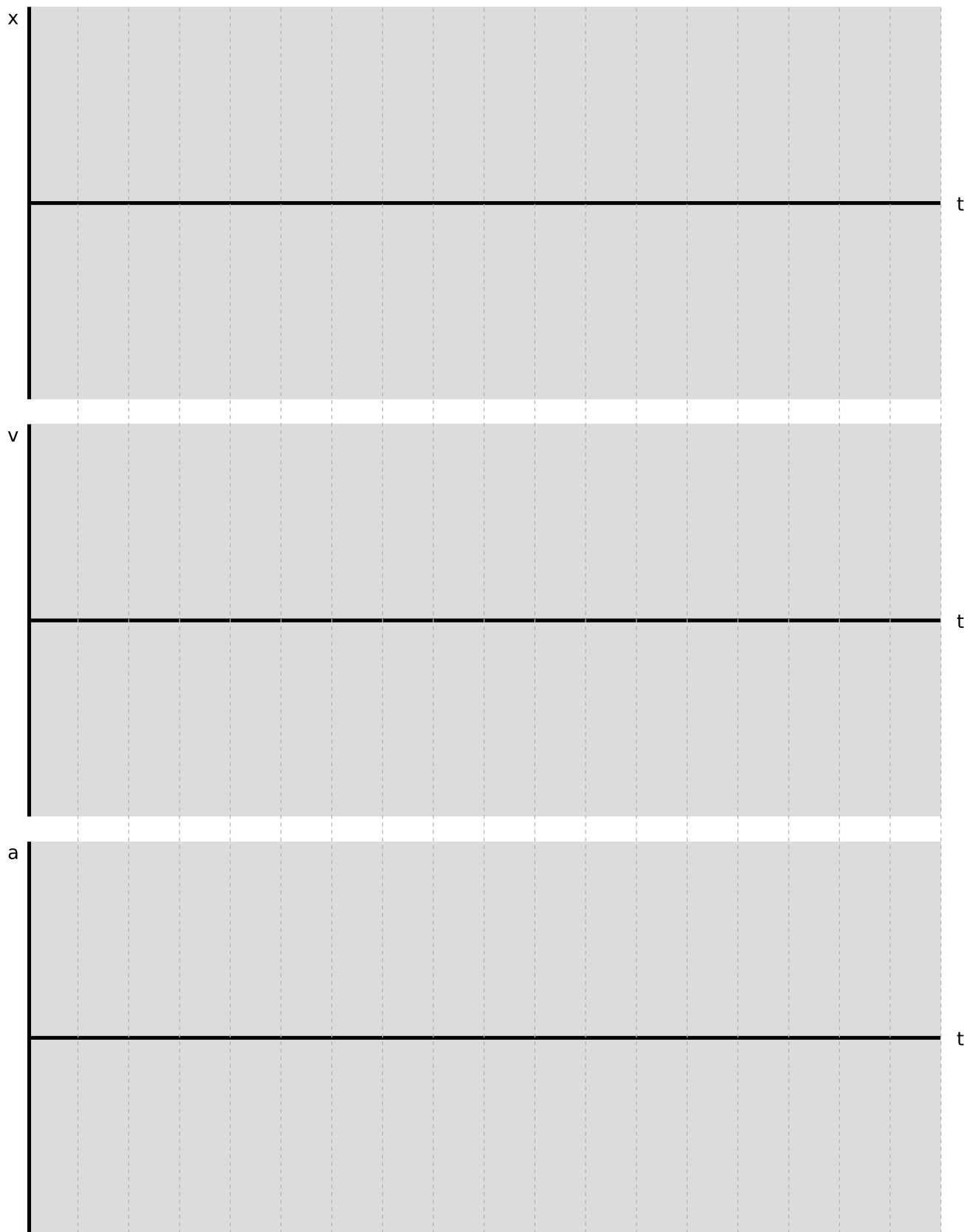
**34** Aircraft carriers originated in World War I, and the first landing on a carrier was performed by E.H. Dunning in a Sopwith Pup biplane, landing on HMS Furious. (Dunning was killed the second time he attempted the feat.) In such a landing, the pilot slows down to just above the plane's stall speed, which is the minimum speed at which the plane can fly without stalling. The plane then lands and is caught by cables and decelerated as it travels the length of the flight deck. Comparing a modern US F-14 fighter jet landing on an Enterprise-class carrier to Dunning's original exploit, the stall speed is greater by a factor of 4.8, and to accomodate this, the length of the flight deck is greater by a factor of 1.9. Which deceleration is greater, and by what factor? ✓

**35** In college-level women's softball in the U.S., typically a pitcher is expected to be at least 1.75 m tall, but Virginia Tech pitcher Jasmin Harrell is 1.62 m. Although a pitcher actually throws by stepping forward and swinging her arm in a circle, let's make a simplified physical model to estimate how much of a disadvantage Harrell has had to overcome due to her height. We'll pretend that the pitcher gives the ball a constant acceleration in a straight line, and that the length of this line is proportional to the pitcher's height. Compare the acceleration Harrell would have to supply with the acceleration that would suffice for a pitcher of the nominal minimum height, if both were to throw a pitch at the same speed. ✓



Problem 32.

**36** When the police engage in a high-speed chase on city streets, it can be extremely dangerous both to the police and to other motorists and pedestrians. Suppose that the police car must travel at a speed that is limited by the need to be able to stop before hitting a baby carriage, and that the distance at which the driver first sees the baby carriage is fixed. Tests show that in a panic stop from high speed, a police car based on a Chevy Impala has a deceleration 9% greater than that of a Dodge Intrepid. Compare the maximum safe speeds for the two cars. ✓







Isaac Newton

## Chapter 4

# Force and motion

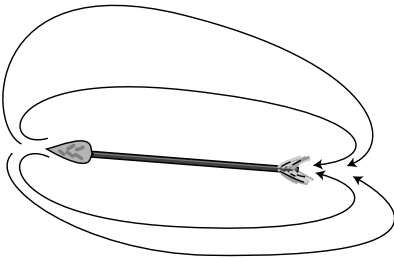
If I have seen farther than others, it is because I have stood on the shoulders of giants.

*Newton, referring to Galileo*

Even as great and skeptical a genius as Galileo was unable to make much progress on the causes of motion. It was not until a generation later that Isaac Newton (1642-1727) was able to attack the problem successfully. In many ways, Newton's personality was the opposite of Galileo's. Where Galileo aggressively publicized his ideas,

Newton had to be coaxed by his friends into publishing a book on his physical discoveries. Where Galileo's writing had been popular and dramatic, Newton originated the stilted, impersonal style that most people think is standard for scientific writing. (Scientific journals today encourage a less ponderous style, and papers are often written in the first person.) Galileo's talent for arousing animosity among the rich and powerful was matched by Newton's skill at making himself a popular visitor at court. Galileo narrowly escaped being burned at the stake, while Newton had the good fortune of being on the winning side of the revolution that replaced King James II with William and Mary of Orange, leading to a lucrative post running the English royal mint.

Newton discovered the relationship between force and motion, and revolutionized our view of the universe by showing that the same physical laws applied to all matter, whether living or nonliving, on or off of our planet's surface. His book on force and motion, the **Mathematical Principles of Natural Philosophy**, was uncontradicted by experiment for 200 years, but his other main work, **Optics**, was on the wrong track, asserting that light was composed of particles rather than waves. Newton was also an avid alchemist, a fact that modern scientists would like to forget.



a / Aristotle said motion had to be caused by a force. To explain why an arrow kept flying after the bowstring was no longer pushing on it, he said the air rushed around behind the arrow and pushed it forward. We know this is wrong, because an arrow shot in a vacuum chamber does not instantly drop to the floor as it leaves the bow. Galileo and Newton realized that a force would only be needed to change the arrow's motion, not to make its motion continue.

## 4.1 Force

### We need only explain changes in motion, not motion itself.

So far you've studied the measurement of motion in some detail, but not the reasons why a certain object would move in a certain way. This chapter deals with the "why" questions. Aristotle's ideas about the causes of motion were completely wrong, just like all his other ideas about physical science, but it will be instructive to start with them, because they amount to a road map of modern students' incorrect preconceptions.

Aristotle thought he needed to explain both why motion occurs and why motion might change. Newton inherited from Galileo the important counter-Aristotelian idea that motion needs no explanation, that it is only *changes* in motion that require a physical cause. Aristotle's needlessly complex system gave three reasons for motion:

Natural motion, such as falling, came from the tendency of objects to go to their "natural" place, on the ground, and come to rest.

Voluntary motion was the type of motion exhibited by animals, which moved because they chose to.

Forced motion occurred when an object was acted on by some other object that made it move.

## Motion changes due to an interaction between two objects.

In the Aristotelian theory, natural motion and voluntary motion are one-sided phenomena: the object causes its own motion. Forced motion is supposed to be a two-sided phenomenon, because one object imposes its “commands” on another. Where Aristotle conceived of some of the phenomena of motion as one-sided and others as two-sided, Newton realized that a change in motion was always a two-sided relationship of a force acting between two physical objects.

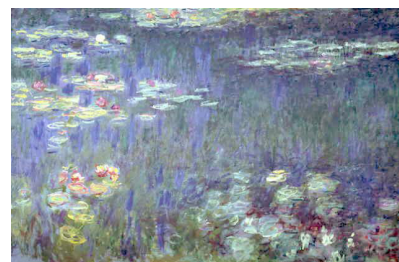
The one-sided “natural motion” description of falling makes a crucial omission. The acceleration of a falling object is not caused by its own “natural” tendencies but by an attractive force between it and the planet Earth. Moon rocks brought back to our planet do not “want” to fly back up to the moon because the moon is their “natural” place. They fall to the floor when you drop them, just like our homegrown rocks. As we’ll discuss in more detail later in this course, gravitational forces are simply an attraction that occurs between any two physical objects. Minute gravitational forces can even be measured between human-scale objects in the laboratory.

The idea of natural motion also explains incorrectly why things come to rest. A basketball rolling across a beach slows to a stop because it is interacting with the sand via a frictional force, not because of its own desire to be at rest. If it was on a frictionless surface, it would never slow down. Many of Aristotle’s mistakes stemmed from his failure to recognize friction as a force.

The concept of voluntary motion is equally flawed. You may have been a little uneasy about it from the start, because it assumes a clear distinction between living and nonliving things. Today, however, we are used to having the human body likened to a complex machine. In the modern world-view, the border between the living and the inanimate is a fuzzy no-man’s land inhabited by viruses, prions, and silicon chips. Furthermore, Aristotle’s statement that you can take a step forward “because you choose to” inappropriately mixes two levels of explanation. At the physical level of explanation, the reason your body steps forward is because of a frictional force acting between your foot and the floor. If the floor was covered with a puddle of oil, no amount of “choosing to” would enable you to take a graceful stride forward.

## Forces can all be measured on the same numerical scale.

In the Aristotelian-scholastic tradition, the description of motion as natural, voluntary, or forced was only the broadest level of classification, like splitting animals into birds, reptiles, mammals, and amphibians. There might be thousands of types of motion, each of which would follow its own rules. Newton’s realization that all changes in motion were caused by two-sided interactions made



b / “Our eyes receive blue light reflected from this painting because Monet wanted to represent water with the color blue.” This is a valid statement at one level of explanation, but physics works at the physical level of explanation, in which blue light gets to your eyes because it is reflected by blue pigments in the paint.



it seem that the phenomena might have more in common than had been apparent. In the Newtonian description, there is only one cause for a change in motion, which we call force. Forces may be of different types, but they all produce changes in motion according to the same rules. Any acceleration that can be produced by a magnetic force can equally well be produced by an appropriately controlled stream of water. We can speak of two forces as being equal if they produce the same change in motion when applied in the same situation, which means that they pushed or pulled equally hard in the same direction.

The idea of a numerical scale of force and the newton unit were introduced in chapter 0. To recapitulate briefly, a force is when a pair of objects push or pull on each other, and one newton is the force required to accelerate a 1-kg object from rest to a speed of 1 m/s in 1 second.

### **More than one force on an object**

As if we hadn't kicked poor Aristotle around sufficiently, his theory has another important flaw, which is important to discuss because it corresponds to an extremely common student misconception. Aristotle conceived of forced motion as a relationship in which one object was the boss and the other "followed orders." It therefore would only make sense for an object to experience one force at a time, because an object couldn't follow orders from two sources at once. In the Newtonian theory, forces are numbers, not orders, and if more than one force acts on an object at once, the result is found by adding up all the forces. It is unfortunate that the use of the English word "force" has become standard, because to many people it suggests that you are "forcing" an object to do something. The force of the earth's gravity cannot "force" a boat to sink, because there are other forces acting on the boat. Adding them up gives a total of zero, so the boat accelerates neither up nor down.

### **Objects can exert forces on each other at a distance.**

Aristotle declared that forces could only act between objects that were touching, probably because he wished to avoid the type of occult speculation that attributed physical phenomena to the influence of a distant and invisible pantheon of gods. He was wrong, however, as you can observe when a magnet leaps onto your refrigerator or when the planet earth exerts gravitational forces on objects that are in the air. Some types of forces, such as friction, only operate between objects in contact, and are called contact forces. Magnetism, on the other hand, is an example of a noncontact force. Although the magnetic force gets stronger when the magnet is closer to your refrigerator, touching is not required.

## Weight

In physics, an object's weight,  $F_W$ , is defined as the earth's gravitational force on it. The SI unit of weight is therefore the Newton. People commonly refer to the kilogram as a unit of weight, but the kilogram is a unit of mass, not weight. Note that an object's weight is not a fixed property of that object. Objects weigh more in some places than in others, depending on the local strength of gravity. It is their mass that always stays the same. A baseball pitcher who can throw a 90-mile-per-hour fastball on earth would not be able to throw any faster on the moon, because the ball's inertia would still be the same.

## Positive and negative signs of force

We'll start by considering only cases of one-dimensional center-of-mass motion in which all the forces are parallel to the direction of motion, i.e., either directly forward or backward. In one dimension, plus and minus signs can be used to indicate directions of forces, as shown in figure c. We can then refer generically to addition of forces, rather than having to speak sometimes of addition and sometimes of subtraction. We add the forces shown in the figure and get 11 N. In general, we should choose a one-dimensional coordinate system with its  $x$  axis parallel the direction of motion. Forces that point along the positive  $x$  axis are positive, and forces in the opposite direction are negative. Forces that are not directly along the  $x$  axis cannot be immediately incorporated into this scheme, but that's OK, because we're avoiding those cases for now.

## Discussion questions

**A** In chapter 0, I defined 1 N as the force that would accelerate a 1-kg mass from rest to 1 m/s in 1 s. Anticipating the following section, you might guess that 2 N could be defined as the force that would accelerate the same mass to twice the speed, or twice the mass to the same speed. Is there an easier way to define 2 N based on the definition of 1 N?

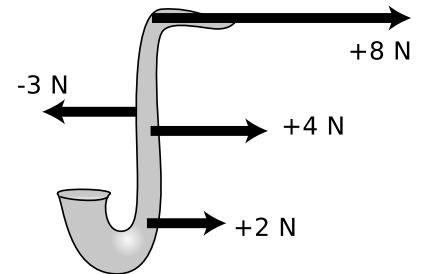
## 4.2 Newton's first law

We are now prepared to make a more powerful restatement of the principle of inertia.

### Newton's first law

If the total force acting on an object is zero, its center of mass continues in the same state of motion.

In other words, an object initially at rest is predicted to remain at rest if the total force on it is zero, and an object in motion remains in motion with the same velocity in the same direction. The converse of Newton's first law is also true: if we observe an object moving with constant velocity along a straight line, then the total force on



c / Forces are applied to a saxophone. In this example, positive signs have been used consistently for forces to the right, and negative signs for forces to the left. (The forces are being applied to different places on the saxophone, but the numerical value of a force carries no information about that.)

it must be zero.

In a future physics course or in another textbook, you may encounter the term “net force,” which is simply a synonym for total force.

What happens if the total force on an object is not zero? It accelerates. Numerical prediction of the resulting acceleration is the topic of Newton’s second law, which we’ll discuss in the following section.

This is the first of Newton’s three laws of motion. It is not important to memorize which of Newton’s three laws are numbers one, two, and three. If a future physics teacher asks you something like, “Which of Newton’s laws are you thinking of?” a perfectly acceptable answer is “The one about constant velocity when there’s zero total force.” The concepts are more important than any specific formulation of them. Newton wrote in Latin, and I am not aware of any modern textbook that uses a verbatim translation of his statement of the laws of motion. Clear writing was not in vogue in Newton’s day, and he formulated his three laws in terms of a concept now called momentum, only later relating it to the concept of force. Nearly all modern texts, including this one, start with force and do momentum later.

---

*An elevator*

*example 1*

▷ An elevator has a weight of 5000 N. Compare the forces that the cable must exert to raise it at constant velocity, lower it at constant velocity, and just keep it hanging.

▷ In all three cases the cable must pull up with a force of exactly 5000 N. Most people think you’d need at least a little more than 5000 N to make it go up, and a little less than 5000 N to let it down, but that’s incorrect. Extra force from the cable is only necessary for speeding the car up when it starts going up or slowing it down when it finishes going down. Decreased force is needed to speed the car up when it gets going down and to slow it down when it finishes going up. But when the elevator is cruising at constant velocity, Newton’s first law says that you just need to cancel the force of the earth’s gravity.

To many students, the statement in the example that the cable’s upward force “cancels” the earth’s downward gravitational force implies that there has been a contest, and the cable’s force has won, vanquishing the earth’s gravitational force and making it disappear. That is incorrect. Both forces continue to exist, but because they add up numerically to zero, the elevator has no center-of-mass acceleration. We know that both forces continue to exist because they both have side-effects other than their effects on the car’s center-of-mass motion. The force acting between the cable and the car continues to produce tension in the cable and keep the cable taut. The

earth's gravitational force continues to keep the passengers (whom we are considering as part of the elevator-object) stuck to the floor and to produce internal stresses in the walls of the car, which must hold up the floor.

---

*Terminal velocity for falling objects*

*example 2*

▷ An object like a feather that is not dense or streamlined does not fall with constant acceleration, because air resistance is nonnegligible. In fact, its acceleration tapers off to nearly zero within a fraction of a second, and the feather finishes dropping at constant speed (known as its terminal velocity). Why does this happen?

▷ Newton's first law tells us that the total force on the feather must have been reduced to nearly zero after a short time. There are two forces acting on the feather: a downward gravitational force from the planet earth, and an upward frictional force from the air. As the feather speeds up, the air friction becomes stronger and stronger, and eventually it cancels out the earth's gravitational force, so the feather just continues with constant velocity without speeding up any more.

The situation for a skydiver is exactly analogous. It's just that the skydiver experiences perhaps a million times more gravitational force than the feather, and it is not until she is falling very fast that the force of air friction becomes as strong as the gravitational force. It takes her several seconds to reach terminal velocity, which is on the order of a hundred miles per hour.

### **More general combinations of forces**

It is too constraining to restrict our attention to cases where all the forces lie along the line of the center of mass's motion. For one thing, we can't analyze any case of horizontal motion, since any object on earth will be subject to a vertical gravitational force! For instance, when you are driving your car down a straight road, there are both horizontal forces and vertical forces. However, the vertical forces have no effect on the center of mass motion, because the road's upward force simply counteracts the earth's downward gravitational force and keeps the car from sinking into the ground.

Later in the book we'll deal with the most general case of many forces acting on an object at any angles, using the mathematical technique of vector addition, but the following slight generalization of Newton's first law allows us to analyze a great many cases of interest:

Suppose that an object has two sets of forces acting on it, one set along the line of the object's initial motion and another set perpendicular to the first set. If both sets of forces cancel, then the object's center of mass continues in the same state of motion.

### *A passenger riding the subway*

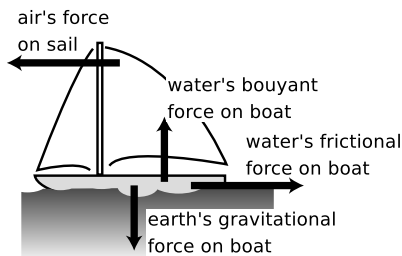
*example 3*

- ▷ Describe the forces acting on a person standing in a subway train that is cruising at constant velocity.
- ▷ No force is necessary to keep the person moving relative to the ground. He will not be swept to the back of the train if the floor is slippery. There are two vertical forces on him, the earth's downward gravitational force and the floor's upward force, which cancel. There are no horizontal forces on him at all, so of course the total horizontal force is zero.

### *Forces on a sailboat*

*example 4*

- ▷ If a sailboat is cruising at constant velocity with the wind coming from directly behind it, what must be true about the forces acting on it?
- ▷ The forces acting on the boat must be canceling each other out. The boat is not sinking or leaping into the air, so evidently the vertical forces are canceling out. The vertical forces are the downward gravitational force exerted by the planet earth and an upward force from the water.



d / Example 4.

The air is making a forward force on the sail, and if the boat is not accelerating horizontally then the water's backward frictional force must be canceling it out.

Contrary to Aristotle, more force is not needed in order to maintain a higher speed. Zero total force is always needed to maintain constant velocity. Consider the following made-up numbers:

	boat moving at a low, constant velocity	boat moving at a high, constant velocity
forward force of the wind on the sail . . .	10,000 N	20,000 N
backward force of the water on the hull . . .	−10,000 N	−20,000 N
total force on the boat . . .	0 N	0 N

The faster boat still has zero total force on it. The forward force on it is greater, and the backward force smaller (more negative), but that's irrelevant because Newton's first law has to do with the total force, not the individual forces.

This example is quite analogous to the one about terminal velocity of falling objects, since there is a frictional force that increases with speed. After casting off from the dock and raising the sail, the boat will accelerate briefly, and then reach its terminal velocity, at which the water's frictional force has become as great as the wind's force on the sail.

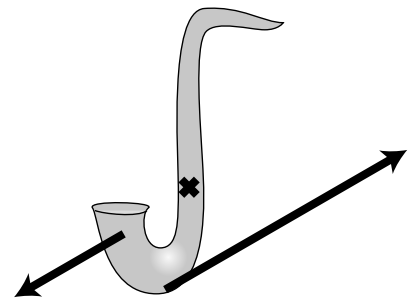
### A car crash

### example 5

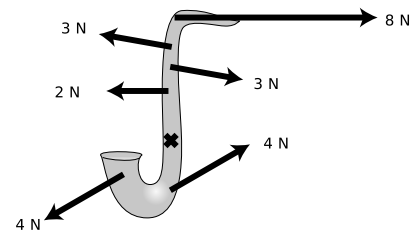
- ▷ If you drive your car into a brick wall, what is the mysterious force that slams your face into the steering wheel?
- ▷ Your surgeon has taken physics, so she is not going to believe your claim that a mysterious force is to blame. She knows that your face was just following Newton's first law. Immediately after your car hit the wall, the only forces acting on your head were the same canceling-out forces that had existed previously: the earth's downward gravitational force and the upward force from your neck. There were no forward or backward forces on your head, but the car did experience a backward force from the wall, so the car slowed down and your face caught up.

### Discussion questions

- A** Newton said that objects continue moving if no forces are acting on them, but his predecessor Aristotle said that a force was necessary to keep an object moving. Why does Aristotle's theory seem more plausible, even though we now believe it to be wrong? What insight was Aristotle missing about the reason why things seem to slow down naturally? Give an example.
- B** In the figure what would have to be true about the saxophone's initial motion if the forces shown were to result in continued one-dimensional motion of its center of mass?
- C** This figure requires an ever further generalization of the preceding discussion. After studying the forces, what does your physical intuition tell you will happen? Can you state in words how to generalize the conditions for one-dimensional motion to include situations like this one?



Discussion question B.



Discussion question C.

## 4.3 Newton's second law

What about cases where the total force on an object is not zero, so that Newton's first law doesn't apply? The object will have an acceleration. The way we've defined positive and negative signs of force and acceleration guarantees that positive forces produce positive accelerations, and likewise for negative values. How much acceleration will it have? It will clearly depend on both the object's mass and on the amount of force.

Experiments with any particular object show that its acceleration is directly proportional to the total force applied to it. This may seem wrong, since we know of many cases where small amounts of force fail to move an object at all, and larger forces get it going. This apparent failure of proportionality actually results from forgetting that there is a frictional force in addition to the force we apply to move the object. The object's acceleration is exactly proportional to the total force on it, not to any individual force on it. In the absence of friction, even a very tiny force can slowly change the velocity of a very massive object.

Experiments also show that the acceleration is inversely proportional to the object's mass, and combining these two proportionalities gives the following way of predicting the acceleration of any object:

### Newton's second law

$$a = F_{total}/m \quad ,$$

where

$m$  is an object's mass

$F_{total}$  is the sum of the forces acting on it, and

$a$  is the acceleration of the object's center of mass.

We are presently restricted to the case where the forces of interest are parallel to the direction of motion.

#### *An accelerating bus*

*example 6*

▷ A VW bus with a mass of 2000 kg accelerates from 0 to 25 m/s (freeway speed) in 34 s. Assuming the acceleration is constant, what is the total force on the bus?

▷ We solve Newton's second law for  $F_{total} = ma$ , and substitute  $\Delta v/\Delta t$  for  $a$ , giving

$$\begin{aligned} F_{total} &= m\Delta v/\Delta t \\ &= (2000 \text{ kg})(25 \text{ m/s} - 0 \text{ m/s})/(34 \text{ s}) \\ &= 1.5 \text{ kN} \quad . \end{aligned}$$

### A generalization

As with the first law, the second law can be easily generalized to include a much larger class of interesting situations:

Suppose an object is being acted on by two sets of forces, one set lying parallel to the object's initial direction of motion and another set acting along a perpendicular line. If the forces perpendicular to the initial direction of motion cancel out, then the object accelerates along its original line of motion according to  $a = F_{\parallel}/m$ , where  $F_{\parallel}$  is the sum of the forces parallel to the line.

#### *A coin sliding across a table*

*example 7*

Suppose a coin is sliding to the right across a table, e, and let's choose a positive  $x$  axis that points to the right. The coin's velocity

is positive, and we expect based on experience that it will slow down, i.e., its acceleration should be negative.

Although the coin's motion is purely horizontal, it feels both vertical and horizontal forces. The Earth exerts a downward gravitational force  $F_2$  on it, and the table makes an upward force  $F_3$  that prevents the coin from sinking into the wood. In fact, without these vertical forces the horizontal frictional force wouldn't exist: surfaces don't exert friction against one another unless they are being pressed together.

Although  $F_2$  and  $F_3$  contribute to the physics, they do so only indirectly. The only thing that directly relates to the acceleration along the horizontal direction is the horizontal force:  $a = F_1/m$ .

### The relationship between mass and weight

Mass is different from weight, but they're related. An apple's mass tells us how hard it is to change its motion. Its weight measures the strength of the gravitational attraction between the apple and the planet earth. The apple's weight is less on the moon, but its mass is the same. Astronauts assembling the International Space Station in zero gravity cannot just pitch massive modules back and forth with their bare hands; the modules are weightless, but not massless.

We have already seen the experimental evidence that when weight (the force of the earth's gravity) is the only force acting on an object, its acceleration equals the constant  $g$ , and  $g$  depends on where you are on the surface of the earth, but not on the mass of the object. Applying Newton's second law then allows us to calculate the magnitude of the gravitational force on any object in terms of its mass:

$$|F_W| = mg \quad .$$

(The equation only gives the magnitude, i.e. the absolute value, of  $F_W$ , because we're defining  $g$  as a positive number, so it equals the absolute value of a falling object's acceleration.)

▷ *Solved problem: Decelerating a car*

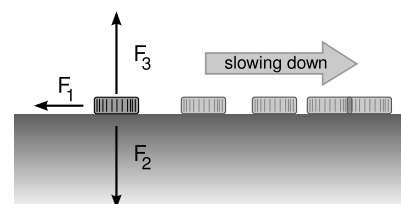
*page 143, problem 7*

#### Weight and mass

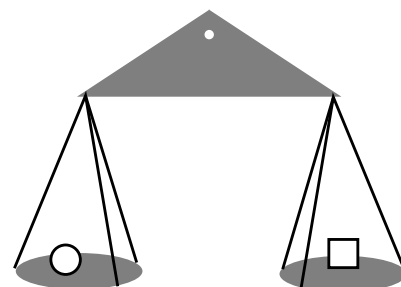
*example 8*

▷ Figure g shows masses of one and two kilograms hung from a spring scale, which measures force in units of newtons. Explain the readings.

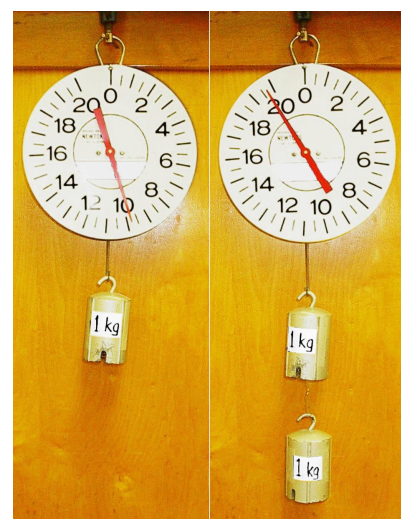
▷ Let's start with the single kilogram. It's not accelerating, so evidently the total force on it is zero: the spring scale's upward force on it is canceling out the earth's downward gravitational force. The spring scale tells us how much force it is being obliged to supply, but since the two forces are equal in strength, the spring scale's reading can also be interpreted as measuring the strength of the gravitational force, i.e., the weight of the one-



e / A coin slides across a table. Even for motion in one dimension, some of the forces may not lie along the line of the motion.



f / A simple double-pan balance works by comparing the weight forces exerted by the earth on the contents of the two pans. Since the two pans are at almost the same location on the earth's surface, the value of  $g$  is essentially the same for each one, and equality of weight therefore also implies equality of mass.



g / Example 8.



kilogram mass. The weight of a one-kilogram mass should be

$$\begin{aligned}F_W &= mg \\&= (1.0 \text{ kg})(9.8 \text{ m/s}^2) = 9.8 \text{ N} \quad ,\end{aligned}$$

and that's indeed the reading on the spring scale.

Similarly for the two-kilogram mass, we have

$$\begin{aligned}F_W &= mg \\&= (2.0 \text{ kg})(9.8 \text{ m/s}^2) = 19.6 \text{ N} \quad .\end{aligned}$$

---

*Calculating terminal velocity* *example 9*

▷ Experiments show that the force of air friction on a falling object such as a skydiver or a feather can be approximated fairly well with the equation  $|F_{air}| = c\rho Av^2$ , where  $c$  is a constant,  $\rho$  is the density of the air,  $A$  is the cross-sectional area of the object as seen from below, and  $v$  is the object's velocity. Predict the object's terminal velocity, i.e., the final velocity it reaches after a long time.

▷ As the object accelerates, its greater  $v$  causes the upward force of the air to increase until finally the gravitational force and the force of air friction cancel out, after which the object continues at constant velocity. We choose a coordinate system in which positive is up, so that the gravitational force is negative and the force of air friction is positive. We want to find the velocity at which

$$\begin{aligned}F_{air} + F_W &= 0 \quad , \quad \text{i.e.,} \\c\rho Av^2 - mg &= 0 \quad .\end{aligned}$$

Solving for  $v$  gives

$$v_{terminal} = \sqrt{\frac{mg}{c\rho A}}$$

*self-check A*

It is important to get into the habit of interpreting equations. This may be difficult at first, but eventually you will get used to this kind of reasoning.

- (1) Interpret the equation  $v_{\text{terminal}} = \sqrt{mg/c\rho A}$  in the case of  $\rho=0$ .
  - (2) How would the terminal velocity of a 4-cm steel ball compare to that of a 1-cm ball?
  - (3) In addition to teasing out the *mathematical* meaning of an equation, we also have to be able to place it in its *physical* context. How generally important is this equation?
- ▷ Answer, p. 531

**Discussion questions**

**A** Show that the Newton can be reexpressed in terms of the three basic mks units as the combination  $\text{kg}\cdot\text{m}/\text{s}^2$ .

**B** What is wrong with the following statements?

- (1) “g is the force of gravity.”
- (2) “Mass is a measure of how much space something takes up.”

**C** Criticize the following incorrect statement:

“If an object is at rest and the total force on it is zero, it stays at rest. There can also be cases where an object is moving and keeps on moving without having any total force on it, but that can only happen when there’s no friction, like in outer space.”

**D** Table h gives laser timing data for Ben Johnson’s 100 m dash at the 1987 World Championship in Rome. (His world record was later revoked because he tested positive for steroids.) How does the total force on him change over the duration of the race?

$x$ (m)	$t$ (s)
10	1.84
20	2.86
30	3.80
40	4.67
50	5.53
60	6.38
70	7.23
80	8.10
90	8.96
100	9.83

h / Discussion question D.

## 4.4 What force is not

Violin teachers have to endure their beginning students' screeching. A frown appears on the woodwind teacher's face as she watches her student take a breath with an expansion of his ribcage but none in his belly. What makes physics teachers cringe is their students' verbal statements about forces. Below I have listed six dicta about what force is not.

### 1. Force is not a property of one object.

A great many of students' incorrect descriptions of forces could be cured by keeping in mind that a force is an interaction of two objects, not a property of one object.

*Incorrect statement:* "That magnet has a lot of force."

X If the magnet is one millimeter away from a steel ball bearing, they may exert a very strong attraction on each other, but if they were a meter apart, the force would be virtually undetectable. The magnet's strength can be rated using certain electrical units (ampere – meters<sup>2</sup>), but not in units of force.

### 2. Force is not a measure of an object's motion.

If force is not a property of a single object, then it cannot be used as a measure of the object's motion.

*Incorrect statement:* "The freight train rumbled down the tracks with awesome force."

X Force is not a measure of motion. If the freight train collides with a stalled cement truck, then some awesome forces will occur, but if it hits a fly the force will be small.

### 3. Force is not energy.

There are two main approaches to understanding the motion of objects, one based on force and one on a different concept, called energy. The SI unit of energy is the Joule, but you are probably more familiar with the calorie, used for measuring food's energy, and the kilowatt-hour, the unit the electric company uses for billing you. Physics students' previous familiarity with calories and kilowatt-hours is matched by their universal unfamiliarity with measuring forces in units of Newtons, but the precise operational definitions of the energy concepts are more complex than those of the force concepts, and textbooks, including this one, almost universally place the force description of physics before the energy description. During the long period after the introduction of force and before the careful definition of energy, students are therefore vulnerable to situations in which, without realizing it, they are imputing the properties of energy to phenomena of force.

*Incorrect statement:* "How can my chair be making an upward force on my rear end? It has no power!"

X Power is a concept related to energy, e.g., a 100-watt lightbulb uses

up 100 joules per second of energy. When you sit in a chair, no energy is used up, so forces can exist between you and the chair without any need for a source of power.

#### 4. Force is not stored or used up.

Because energy can be stored and used up, people think force also can be stored or used up.

*Incorrect statement:* “If you don’t fill up your tank with gas, you’ll run out of force.”

X Energy is what you’ll run out of, not force.

#### 5. Forces need not be exerted by living things or machines.

Transforming energy from one form into another usually requires some kind of living or mechanical mechanism. The concept is not applicable to forces, which are an interaction between objects, not a thing to be transferred or transformed.

*Incorrect statement:* “How can a wooden bench be making an upward force on my rear end? It doesn’t have any springs or anything inside it.”

X No springs or other internal mechanisms are required. If the bench didn’t make any force on you, you would obey Newton’s second law and fall through it. Evidently it does make a force on you!

#### 6. A force is the direct cause of a change in motion.

I can click a remote control to make my garage door change from being at rest to being in motion. My finger’s force on the button, however, was not the force that acted on the door. When we speak of a force on an object in physics, we are talking about a force that acts directly. Similarly, when you pull a reluctant dog along by its leash, the leash and the dog are making forces on each other, not your hand and the dog. The dog is not even touching your hand.

*self-check B*

Which of the following things can be correctly described in terms of force?

- (1) A nuclear submarine is charging ahead at full steam.
- (2) A nuclear submarine’s propellers spin in the water.
- (3) A nuclear submarine needs to refuel its reactor periodically.      ▷

Answer, p. 531

#### Discussion questions

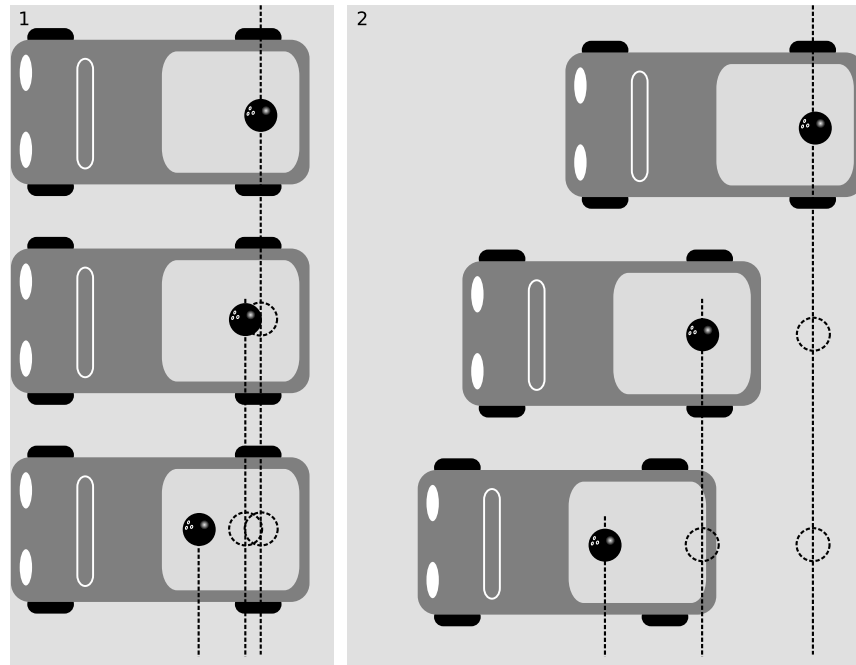
**A** Criticize the following incorrect statement: “If you shove a book across a table, friction takes away more and more of its force, until finally it stops.”

**B** You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: “The ball gets some force from you when you hit it, and when it hits the wall, it loses part of that force, so it doesn’t bounce back as fast. The muscles in your arm are the only things that a force can come from.”

## 4.5 Inertial and noninertial frames of reference

One day, you're driving down the street in your pickup truck, on your way to deliver a bowling ball. The ball is in the back of the truck, enjoying its little jaunt and taking in the fresh air and sunshine. Then you have to slow down because a stop sign is coming up. As you brake, you glance in your rearview mirror, and see your trusty companion accelerating toward you. Did some mysterious force push it forward? No, it only seems that way because you and the car are slowing down. The ball is faithfully obeying Newton's first law, and as it continues at constant velocity it gets ahead relative to the slowing truck. No forces are acting on it (other than the same canceling-out vertical forces that were always acting on it).<sup>1</sup> The ball only appeared to violate Newton's first law because there was something wrong with your frame of reference, which was based on the truck.

i / 1. In a frame of reference that moves with the truck, the bowling ball appears to violate Newton's first law by accelerating despite having no horizontal forces on it. 2. In an inertial frame of reference, which the surface of the earth approximately is, the bowling ball obeys Newton's first law. It moves equal distances in equal time intervals, i.e., maintains constant velocity. In this frame of reference, it is the truck that appears to have a change in velocity, which makes sense, since the road is making a horizontal force on it.



How, then, are we to tell in which frames of reference Newton's laws are valid? It's no good to say that we should avoid moving frames of reference, because there is no such thing as absolute rest or absolute motion. All frames can be considered as being either at rest or in motion. According to an observer in India, the strip mall that constituted the frame of reference in panel (b) of the figure was moving along with the earth's rotation at hundreds of miles per hour.

The reason why Newton's laws fail in the truck's frame of refer-

<sup>1</sup>Let's assume for simplicity that there is no friction.

ence is not because the truck is *moving* but because it is *accelerating*. (Recall that physicists use the word to refer either to speeding up or slowing down.) Newton's laws were working just fine in the moving truck's frame of reference as long as the truck was moving at constant velocity. It was only when its speed changed that there was a problem. How, then, are we to tell which frames are accelerating and which are not? What if you claim that your truck is not accelerating, and the sidewalk, the asphalt, and the Burger King are accelerating? The way to settle such a dispute is to examine the motion of some object, such as the bowling ball, which we know has zero total force on it. Any frame of reference in which the ball appears to obey Newton's first law is then a valid frame of reference, and to an observer in that frame, Mr. Newton assures us that all the other objects in the universe will obey his laws of motion, not just the ball.

Valid frames of reference, in which Newton's laws are obeyed, are called inertial frames of reference. Frames of reference that are not inertial are called noninertial frames. In those frames, objects violate the principle of inertia and Newton's first law. While the truck was moving at constant velocity, both it and the sidewalk were valid inertial frames. The truck became an invalid frame of reference when it began changing its velocity.

You usually assume the ground under your feet is a perfectly inertial frame of reference, and we made that assumption above. It isn't perfectly inertial, however. Its motion through space is quite complicated, being composed of a part due to the earth's daily rotation around its own axis, the monthly wobble of the planet caused by the moon's gravity, and the rotation of the earth around the sun. Since the accelerations involved are numerically small, the earth is approximately a valid inertial frame.

Noninertial frames are avoided whenever possible, and we will seldom, if ever, have occasion to use them in this course. Sometimes, however, a noninertial frame can be convenient. Naval gunners, for instance, get all their data from radars, human eyeballs, and other detection systems that are moving along with the earth's surface. Since their guns have ranges of many miles, the small discrepancies between their shells' actual accelerations and the accelerations predicted by Newton's second law can have effects that accumulate and become significant. In order to kill the people they want to kill, they have to add small corrections onto the equation  $a = F_{total}/m$ . Doing their calculations in an inertial frame would allow them to use the usual form of Newton's second law, but they would have to convert all their data into a different frame of reference, which would require cumbersome calculations.

### Discussion question

**A** If an object has a linear  $x - t$  graph in a certain inertial frame,

what is the effect on the graph if we change to a coordinate system with a different origin? What is the effect if we keep the same origin but reverse the positive direction of the  $x$  axis? How about an inertial frame moving alongside the object? What if we describe the object's motion in a noninertial frame?

## Summary

### Selected vocabulary

weight . . . . .	the force of gravity on an object, equal to $mg$
inertial frame . .	a frame of reference that is not accelerating, one in which Newton's first law is true
noninertial frame	an accelerating frame of reference, in which Newton's first law is violated

### Notation

$F_W$ . . . . .	weight
-----------------	--------

### Other terminology and notation

net force . . . . .	another way of saying "total force"
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### Summary

Newton's first law of motion states that if all the forces on an object cancel each other out, then the object continues in the same state of motion. This is essentially a more refined version of Galileo's principle of inertia, which did not refer to a numerical scale of force.

Newton's second law of motion allows the prediction of an object's acceleration given its mass and the total force on it,  $a_{cm} = F_{total}/m$ . This is only the one-dimensional version of the law; the full-three dimensional treatment will come in chapter 8, Vectors. Without the vector techniques, we can still say that the situation remains unchanged by including an additional set of vectors that cancel among themselves, even if they are not in the direction of motion.

Newton's laws of motion are only true in frames of reference that are not accelerating, known as inertial frames.



## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** An object is observed to be moving at constant speed in a certain direction. Can you conclude that no forces are acting on it? Explain. [Based on a problem by Serway and Faughn.]

**2** At low speeds, every car's acceleration is limited by traction, not by the engine's power. Suppose that at low speeds, a certain car is normally capable of an acceleration of  $3 \text{ m/s}^2$ . If it is towing a trailer with half as much mass as the car itself, what acceleration can it achieve? [Based on a problem from PSSC Physics.]

**3** (a) Let  $T$  be the maximum tension that an elevator's cable can withstand without breaking, i.e., the maximum force it can exert. If the motor is programmed to give the car an acceleration  $a$ , what is the maximum mass that the car can have, including passengers, if the cable is not to break? ✓

(b) Interpret the equation you derived in the special cases of  $a = 0$  and of a downward acceleration of magnitude  $g$ . ("Interpret" means to analyze the behavior of the equation, and connect that to reality, as in the self-check on page 135.)

**4** A helicopter of mass  $m$  is taking off vertically. The only forces acting on it are the earth's gravitational force and the force,  $F_{air}$ , of the air pushing up on the propeller blades.

(a) If the helicopter lifts off at  $t = 0$ , what is its vertical speed at time  $t$ ?

(b) Check that the units of your answer to part a make sense.

(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically*.

(d) Plug numbers into your equation from part a, using  $m = 2300 \text{ kg}$ ,  $F_{air} = 27000 \text{ N}$ , and  $t = 4.0 \text{ s}$ . ✓

**5** In the 1964 Olympics in Tokyo, the best men's high jump was 2.18 m. Four years later in Mexico City, the gold medal in the same event was for a jump of 2.24 m. Because of Mexico City's altitude (2400 m), the acceleration of gravity there is lower than that in Tokyo by about  $0.01 \text{ m/s}^2$ . Suppose a high-jumper has a mass of 72 kg.

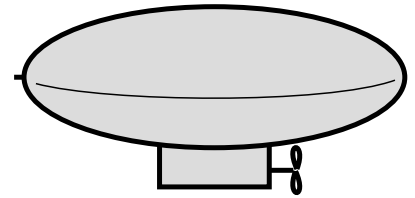
(a) Compare his mass and weight in the two locations.

(b) Assume that he is able to jump with the same initial vertical

velocity in both locations, and that all other conditions are the same except for gravity. How much higher should he be able to jump in Mexico City? ✓

(Actually, the reason for the big change between '64 and '68 was the introduction of the “Fosbury flop.”) ★

**6** A blimp is initially at rest, hovering, when at  $t = 0$  the pilot turns on the motor of the propeller. The motor cannot instantly get the propeller going, but the propeller speeds up steadily. The steadily increasing force between the air and the propeller is given by the equation  $F = kt$ , where  $k$  is a constant. If the mass of the blimp is  $m$ , find its position as a function of time. (Assume that during the period of time you’re dealing with, the blimp is not yet moving fast enough to cause a significant backward force due to air resistance.) ✓ ∫



Problem 6.

**7** A car is accelerating forward along a straight road. If the force of the road on the car’s wheels, pushing it forward, is a constant 3.0 kN, and the car’s mass is 1000 kg, then how long will the car take to go from 20 m/s to 50 m/s? ▷ Solution, p. 520

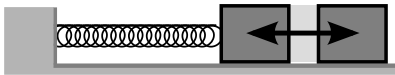
**8** Some garden shears are like a pair of scissors: one sharp blade slices past another. In the “anvil” type, however, a sharp blade presses against a flat one rather than going past it. A gardening book says that for people who are not very physically strong, the anvil type can make it easier to cut tough branches, because it concentrates the force on one side. Evaluate this claim based on Newton’s laws. [Hint: Consider the forces acting on the branch, and the motion of the branch.]

**9** A uranium atom deep in the earth spits out an alpha particle. An alpha particle is a fragment of an atom. This alpha particle has initial speed  $v$ , and travels a distance  $d$  before stopping in the earth. (a) Find the force,  $F$ , from the dirt that stopped the particle, in terms of  $v, d$ , and its mass,  $m$ . Don’t plug in any numbers yet. Assume that the force was constant. ✓

(b) Show that your answer has the right units.

(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you’ve figured out this *mathematical* relationship, show that it makes sense *physically*.

(d) Evaluate your result for  $m = 6.7 \times 10^{-27}$  kg,  $v = 2.0 \times 10^4$  km/s, and  $d = 0.71$  mm. ✓



Problem 10, part c.

**10** You are given a large sealed box, and are not allowed to open it. Which of the following experiments measure its mass, and which measure its weight? [Hint: Which experiments would give different results on the moon?]

- (a) Put it on a frozen lake, throw a rock at it, and see how fast it scoots away after being hit.
- (b) Drop it from a third-floor balcony, and measure how loud the sound is when it hits the ground.
- (c) As shown in the figure, connect it with a spring to the wall, and watch it vibrate.

▷ Solution, p. 520

**11** While escaping from the palace of the evil Martian emperor, Sally Spacehound jumps from a tower of height  $h$  down to the ground. Ordinarily the fall would be fatal, but she fires her blaster rifle straight down, producing an upward force of magnitude  $F_B$ . This force is insufficient to levitate her, but it does cancel out some of the force of gravity. During the time  $t$  that she is falling, Sally is unfortunately exposed to fire from the emperor's minions, and can't dodge their shots. Let  $m$  be her mass, and  $g$  the strength of gravity on Mars.

- (a) Find the time  $t$  in terms of the other variables.
- (b) Check the units of your answer to part a.
- (c) For sufficiently large values of  $F_B$ , your answer to part a becomes nonsense — explain what's going on. ✓

**12** When I cook rice, some of the dry grains always stick to the measuring cup. To get them out, I turn the measuring cup upside-down and hit the “roof” with my hand so that the grains come off of the “ceiling.” (a) Explain why static friction is irrelevant here. (b) Explain why gravity is negligible. (c) Explain why hitting the cup works, and why its success depends on hitting the cup hard enough.

## Exercise 4: Force and motion

Equipment:

1-meter pieces of butcher paper

wood blocks with hooks

string

masses to put on top of the blocks to increase friction

spring scales (preferably calibrated in Newtons)

Suppose a person pushes a crate, sliding it across the floor at a certain speed, and then repeats the same thing but at a higher speed. This is essentially the situation you will act out in this exercise. What do you think is different about her force on the crate in the two situations? Discuss this with your group and write down your hypothesis:

-----  
1. First you will measure the amount of friction between the wood block and the butcher paper when the wood and paper surfaces are slipping over each other. The idea is to attach a spring scale to the block and then slide the butcher paper under the block while using the scale to keep the block from moving with it. Depending on the amount of force your spring scale was designed to measure, you may need to put an extra mass on top of the block in order to increase the amount of friction. It is a good idea to use long piece of string to attach the block to the spring scale, since otherwise one tends to pull at an angle instead of directly horizontally.

First measure the amount of friction force when sliding the butcher paper as slowly as possible:-----

Now measure the amount of friction force at a significantly higher speed, say 1 meter per second. (If you try to go too fast, the motion is jerky, and it is impossible to get an accurate reading.)  
-----

Discuss your results. Why are we justified in assuming that the string's force on the block (i.e., the scale reading) is the same amount as the paper's frictional force on the block?

2. Now try the same thing but with the block moving and the paper standing still. Try two different speeds.

Do your results agree with your original hypothesis? If not, discuss what's going on. How does the block "know" how fast to go?





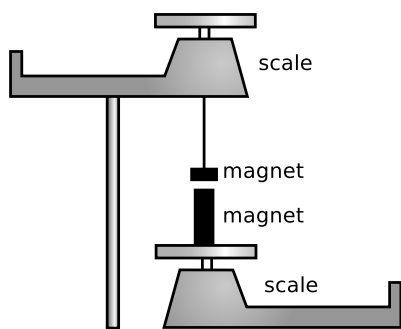
What forces act on the girl?

## Chapter 5

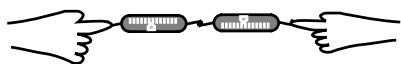
# Analysis of forces

### 5.1 Newton's third law

Newton created the modern concept of force starting from his insight that all the effects that govern motion are interactions between two objects: unlike the Aristotelian theory, Newtonian physics has no phenomena in which an object changes its own motion.



a / Two magnets exert forces on each other.



b / Two people's hands exert forces on each other.



c / Rockets work by pushing exhaust gases out the back. Newton's third law says that if the rocket exerts a backward force on the gases, the gases must make an equal forward force on the rocket. Rocket engines can function above the atmosphere, unlike propellers and jets, which work by pushing against the surrounding air.

Is one object always the “order-giver” and the other the “order-follower”? As an example, consider a batter hitting a baseball. The bat definitely exerts a large force on the ball, because the ball accelerates drastically. But if you have ever hit a baseball, you also know that the ball makes a force on the bat — often with painful results if your technique is as bad as mine!

How does the ball's force on the bat compare with the bat's force on the ball? The bat's acceleration is not as spectacular as the ball's, but maybe we shouldn't expect it to be, since the bat's mass is much greater. In fact, careful measurements of both objects' masses and accelerations would show that  $m_{ball}a_{ball}$  is very nearly equal to  $-m_{bat}a_{bat}$ , which suggests that the ball's force on the bat is of the same magnitude as the bat's force on the ball, but in the opposite direction.

Figures a and b show two somewhat more practical laboratory experiments for investigating this issue accurately and without too much interference from extraneous forces.

In experiment a, a large magnet and a small magnet are weighed separately, and then one magnet is hung from the pan of the top balance so that it is directly above the other magnet. There is an attraction between the two magnets, causing the reading on the top scale to increase and the reading on the bottom scale to decrease. The large magnet is more “powerful” in the sense that it can pick up a heavier paperclip from the same distance, so many people have a strong expectation that one scale's reading will change by a far different amount than the other. Instead, we find that the two changes are equal in magnitude but opposite in direction: the force of the bottom magnet pulling down on the top one has the same strength as the force of the top one pulling up on the bottom one.

In experiment b, two people pull on two spring scales. Regardless of who tries to pull harder, the two forces as measured on the spring scales are equal. Interposing the two spring scales is necessary in order to measure the forces, but the outcome is not some artificial result of the scales' interactions with each other. If one person slaps another hard on the hand, the slapper's hand hurts just as much as the slapped's, and it doesn't matter if the recipient of the slap tries to be inactive. (Punching someone in the mouth causes just as much force on the fist as on the lips. It's just that the lips are more delicate. The forces are equal, but not the levels of pain and injury.)

Newton, after observing a series of results such as these, decided that there must be a fundamental law of nature at work:

### Newton's third law

Forces occur in equal and opposite pairs: whenever object A exerts a force on object B, object B must also be exerting a force on object A. The two forces are equal in magnitude and opposite in direction.

In one-dimensional situations, we can use plus and minus signs to indicate the directions of forces, and Newton's third law can be written succinctly as  $F_{A \text{ on } B} = -F_{B \text{ on } A}$ .

#### self-check A

Figure d analyzes swimming using Newton's third law. Do a similar analysis for a sprinter leaving the starting line. ▷ Answer, p. 531

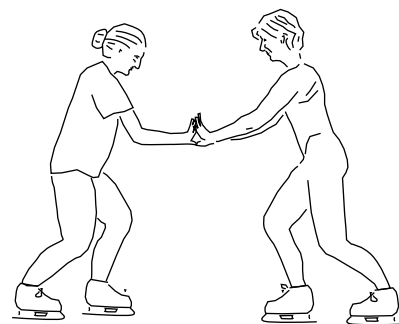
There is no cause and effect relationship between the two forces in Newton's third law. There is no "original" force, and neither one is a response to the other. The pair of forces is a relationship, like marriage, not a back-and-forth process like a tennis match. Newton came up with the third law as a generalization about all the types of forces with which he was familiar, such as frictional and gravitational forces. When later physicists discovered a new type force, such as the force that holds atomic nuclei together, they had to check whether it obeyed Newton's third law. So far, no violation of the third law has ever been discovered, whereas the first and second laws were shown to have limitations by Einstein and the pioneers of atomic physics.

The English vocabulary for describing forces is unfortunately rooted in Aristotelianism, and often implies incorrectly that forces are one-way relationships. It is unfortunate that a half-truth such as "the table exerts an upward force on the book" is so easily expressed, while a more complete and correct description ends up sounding awkward or strange: "the table and the book interact via a force," or "the table and book participate in a force."

To students, it often sounds as though Newton's third law implies nothing could ever change its motion, since the two equal and opposite forces would always cancel. The two forces, however, are always on two different objects, so it doesn't make sense to add them in the first place — we only add forces that are acting on the same object. If two objects are interacting via a force and no other forces are involved, then *both* objects will accelerate — in opposite directions!



d / A swimmer doing the breaststroke pushes backward against the water. By Newton's third law, the water pushes forward on him.



e / Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two figure skaters, initially at rest, push against each other, they will both move.



f / It doesn't make sense for the man to talk about using the woman's money to cancel out his bar tab, because there is no good reason to combine his debts and her assets. Similarly, it doesn't make sense to refer to the equal and opposite forces of Newton's third law as canceling. It only makes sense to add up forces that are acting on the *same* object, whereas two forces related to each other by Newton's third law are always acting on two *different* objects.



### A mnemonic for using Newton's third law correctly

Mnemonics are tricks for memorizing things. For instance, the musical notes that lie between the lines on the treble clef spell the word FACE, which is easy to remember. Many people use the mnemonic "SOHCAHTOA" to remember the definitions of the sine, cosine, and tangent in trigonometry. I have my own modest offering, POFOSTITO, which I hope will make it into the mnemonics hall of fame. It's a way to avoid some of the most common problems with applying Newton's third law correctly:

Pair of  
Opposite  
Forces  
Of the  
Same  
Type  
Involving  
Two  
Objects

*A book lying on a table* *example 1*

▷ A book is lying on a table. What force is the Newton's-third-law partner of the earth's gravitational force on the book?

Answer: Newton's third law works like "B on A, A on B," so the partner must be the book's gravitational force pulling upward on the planet earth. Yes, there is such a force! No, it does not cause the earth to do anything noticeable.

Incorrect answer: The table's upward force on the book is the Newton's-third-law partner of the earth's gravitational force on the book.

x This answer violates two out of three of the commandments of POFOSTITO. The forces are not of the same type, because the table's upward force on the book is not gravitational. Also, three

objects are involved instead of two: the book, the table, and the planet earth.

*Pushing a box up a hill*

*example 2*

▷ A person is pushing a box up a hill. What force is related by Newton's third law to the person's force on the box?

▷ The box's force on the person.

Incorrect answer: The person's force on the box is opposed by friction, and also by gravity.

X This answer fails all three parts of the POFOSTITO test, the most obvious of which is that three forces are referred to instead of a pair.

▷ Solved problem: More about example 2      page 173, problem 20

▷ Solved problem: Why did it accelerate?      page 173, problem 18

### Discussion questions

**A** When you fire a gun, the exploding gases push outward in all directions, causing the bullet to accelerate down the barrel. What third-law pairs are involved? [Hint: Remember that the gases themselves are an object.]

**B** Tam Anh grabs Sarah by the hand and tries to pull her. She tries to remain standing without moving. A student analyzes the situation as follows. "If Tam Anh's force on Sarah is greater than her force on him, he can get her to move. Otherwise, she'll be able to stay where she is." What's wrong with this analysis?

**C** You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: "According to Newton's third law, there has to be a force opposite to your force on the ball. The opposite force is the ball's mass, which resists acceleration, and also air resistance."

## 5.2 Classification and behavior of forces

One of the most basic and important tasks of physics is to classify the forces of nature. I have already referred informally to "types" of forces such as friction, magnetism, gravitational forces, and so on. Classification systems are creations of the human mind, so there is always some degree of arbitrariness in them. For one thing, the level of detail that is appropriate for a classification system depends on what you're trying to find out. Some linguists, the "lumpers," like to emphasize the similarities among languages, and a few extremists have even tried to find signs of similarities between words in languages as different as English and Chinese, lumping the world's languages into only a few large groups. Other linguists, the "splitters," might be more interested in studying the differences in pronunciation between English speakers in New York and Connecticut. The splitters call the lumpers sloppy, but the lumpers say that science

### Optional Topic: Newton's Third Law and Action at a Distance

Newton's third law is completely symmetric in the sense that neither force constitutes a delayed response to the other. Newton's third law does not even mention time, and the forces are supposed to agree at any given instant. This creates an interesting situation when it comes to noncontact forces. Suppose two people are holding magnets, and when one person waves or wiggles her magnet, the other person feels an effect on his. In this way they can send signals to each other from opposite sides of a wall, and if Newton's third law is correct, it would seem that the signals are transmitted instantly, with no time lag. The signals are indeed transmitted quite quickly, but experiments with electrically controlled magnets show that the signals do not leap the gap instantly: they travel at the same speed as light, which is an extremely high speed but not an infinite one.

Is this a contradiction to Newton's third law? Not really. According to current theories, there are no true noncontact forces. Action at a distance does not exist. Although it appears that the wiggling of one magnet affects the other with no need for anything to be in contact with anything, what really happens is that wiggling a magnet creates a ripple in the magnetic field pattern that exists even in empty space. The magnet shoves the ripples out with a kick and receives a kick in return, in strict obedience to Newton's third law. The ripples spread out in all directions, and the ones that hit the other magnet then interact with it, again obeying Newton's third law.

isn't worthwhile unless it can find broad, simple patterns within the seemingly complex universe.

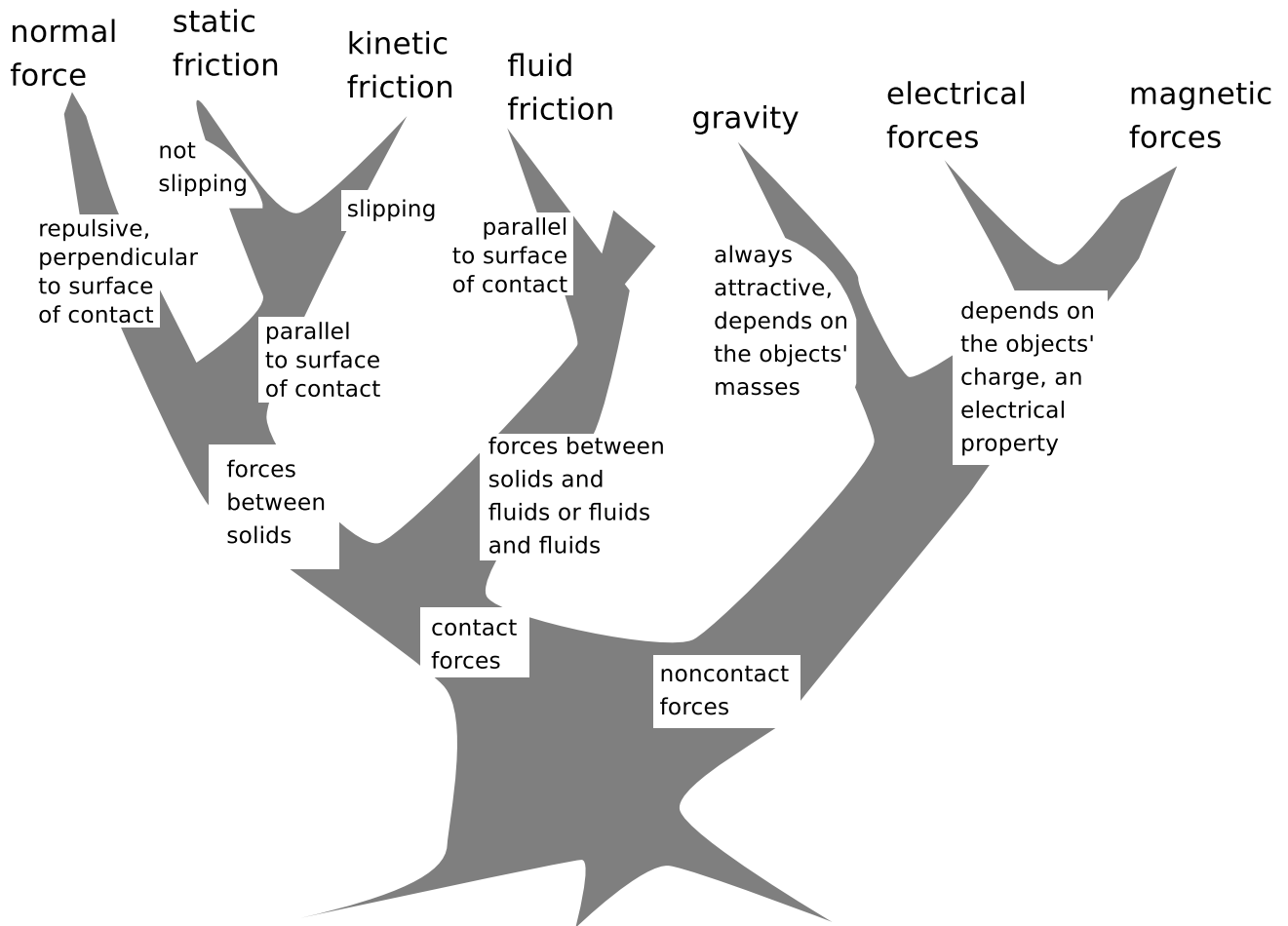
Scientific classification systems are also usually compromises between practicality and naturalness. An example is the question of how to classify flowering plants. Most people think that biological classification is about discovering new species, naming them, and classifying them in the class-order-family-genus-species system according to guidelines set long ago. In reality, the whole system is in a constant state of flux and controversy. One very practical way of classifying flowering plants is according to whether their petals are separate or joined into a tube or cone — the criterion is so clear that it can be applied to a plant seen from across the street. But here practicality conflicts with naturalness. For instance, the begonia has separate petals and the pumpkin has joined petals, but they are so similar in so many other ways that they are usually placed within the same order. Some taxonomists have come up with classification criteria that they claim correspond more naturally to the apparent relationships among plants, without having to make special exceptions, but these may be far less practical, requiring for instance the examination of pollen grains under an electron microscope.



g / A scientific classification system.

In physics, there are two main systems of classification for forces. At this point in the course, you are going to learn one that is very practical and easy to use, and that splits the forces up into a relatively large number of types: seven very common ones that we'll discuss explicitly in this chapter, plus perhaps ten less important ones such as surface tension, which we will not bother with right now.

Physicists, however, are obsessed with finding simple patterns, so recognizing as many as fifteen or twenty types of forces strikes them as distasteful and overly complex. Since about the year 1900, physics has been on an aggressive program to discover ways in which these many seemingly different types of forces arise from a smaller number of fundamental ones. For instance, when you press your hands together, the force that keeps them from passing through each other may seem to have nothing to do with electricity, but at the atomic level, it actually does arise from electrical repulsion between atoms. By about 1950, all the forces of nature had been explained as arising from four fundamental types of forces at the atomic and nuclear level, and the lumping-together process didn't stop there. By the 1960's the length of the list had been reduced to three, and some theorists even believe that they may be able to reduce it to two or one. Although the unification of the forces of nature is one of the most beautiful and important achievements of physics, it makes much more sense to start this course with the more practical and easy system of classification. The unified system of four forces will be one of the highlights of the end of your introductory physics sequence.



h / A practical classification scheme for forces.

The practical classification scheme which concerns us now can be laid out in the form of the tree shown in figure h. The most specific types of forces are shown at the tips of the branches, and it is these types of forces that are referred to in the POFOSTITO mnemonic. For example, electrical and magnetic forces belong to the same general group, but Newton's third law would never relate an electrical force to a magnetic force.

The broadest distinction is that between contact and noncontact forces, which has been discussed in ch. 4. Among the contact forces, we distinguish between those that involve solids only and those that have to do with fluids, a term used in physics to include both gases and liquids.

It should not be necessary to memorize this diagram by rote. It is better to reinforce your memory of this system by calling to mind your commonsense knowledge of certain ordinary phenomena. For instance, we know that the gravitational attraction between us and the planet earth will act even if our feet momentarily leave the

ground, and that although magnets have mass and are affected by gravity, most objects that have mass are nonmagnetic.

---

*Hitting a wall*

*example 3*

▷ A bullet, flying horizontally, hits a steel wall. What type of force is there between the bullet and the wall?

▷ Starting at the bottom of the tree, we determine that the force is a contact force, because it only occurs once the bullet touches the wall. Both objects are solid. The wall forms a vertical plane. If the nose of the bullet was some shape like a sphere, you might imagine that it would only touch the wall at one point. Realistically, however, we know that a lead bullet will flatten out a lot on impact, so there is a surface of contact between the two, and its orientation is vertical. The effect of the force on the bullet is to stop the horizontal motion of the bullet, and this horizontal acceleration must be produced by a horizontal force. The force is therefore perpendicular to the surface of contact, and it's also repulsive (tending to keep the bullet from entering the wall), so it must be a normal force.

Diagram h is meant to be as simple as possible while including most of the forces we deal with in everyday life. If you were an insect, you would be much more interested in the force of surface tension, which allowed you to walk on water. I have not included the nuclear forces, which are responsible for holding the nuclei of atoms, because they are not evident in everyday life.

You should not be afraid to invent your own names for types of forces that do not fit into the diagram. For instance, the force that holds a piece of tape to the wall has been left off of the tree, and if you were analyzing a situation involving scotch tape, you would be absolutely right to refer to it by some commonsense name such as “sticky force.”

On the other hand, if you are having trouble classifying a certain force, you should also consider whether it is a force at all. For instance, if someone asks you to classify the force that the earth has because of its rotation, you would have great difficulty creating a place for it on the diagram. That's because it's a type of motion, not a type of force!

### **Normal forces**

A normal force,  $F_N$ , is a force that keeps one solid object from passing through another. “Normal” is simply a fancy word for “perpendicular,” meaning that the force is perpendicular to the surface of contact. Intuitively, it seems the normal force magically adjusts itself to provide whatever force is needed to keep the objects from occupying the same space. If your muscles press your hands together gently, there is a gentle normal force. Press harder, and the normal

force gets stronger. How does the normal force know how strong to be? The answer is that the harder you jam your hands together, the more compressed your flesh becomes. Your flesh is acting like a spring: more force is required to compress it more. The same is true when you push on a wall. The wall flexes imperceptibly in proportion to your force on it. If you exerted enough force, would it be possible for two objects to pass through each other? No, typically the result is simply to strain the objects so much that one of them breaks.

## Gravitational forces

As we'll discuss in more detail later in the course, a gravitational force exists between any two things that have mass. In everyday life, the gravitational force between two cars or two people is negligible, so the only noticeable gravitational forces are the ones between the earth and various human-scale objects. We refer to these planet-earth-induced gravitational forces as weight forces, and as we have already seen, their magnitude is given by  $|F_W| = mg$ .

▷ *Solved problem: Weight and mass*

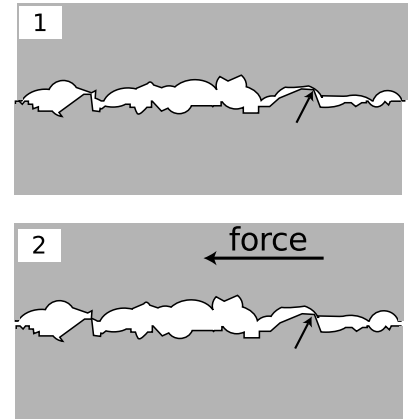
*page 174, problem 26*

## Static and kinetic friction

If you have pushed a refrigerator across a kitchen floor, you have felt a certain series of sensations. At first, you gradually increased your force on the refrigerator, but it didn't move. Finally, you supplied enough force to unstick the fridge, and there was a sudden jerk as the fridge started moving. Once the fridge was unstuck, you could reduce your force significantly and still keep it moving.

While you were gradually increasing your force, the floor's frictional force on the fridge increased in response. The two forces on the fridge canceled, and the fridge didn't accelerate. How did the floor know how to respond with just the right amount of force? Figure i shows one possible *model* of friction that explains this behavior. (A scientific model is a description that we expect to be incomplete, approximate, or unrealistic in some ways, but that nevertheless succeeds in explaining a variety of phenomena.) Figure i/1 shows a microscopic view of the tiny bumps and holes in the surfaces of the floor and the refrigerator. The weight of the fridge presses the two surfaces together, and some of the bumps in one surface will settle as deeply as possible into some of the holes in the other surface. In i/2, your leftward force on the fridge has caused it to ride up a little higher on the bump in the floor labeled with a small arrow. Still more force is needed to get the fridge over the bump and allow it to start moving. Of course, this is occurring simultaneously at millions of places on the two surfaces.

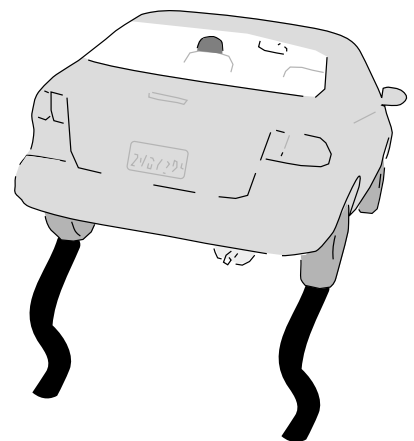
Once you had gotten the fridge moving at constant speed, you found that you needed to exert less force on it. Since zero total force



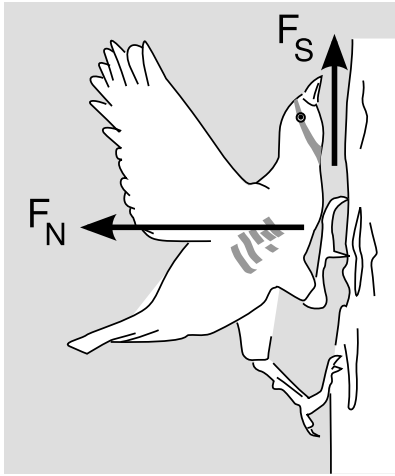
i / A model that correctly explains many properties of friction. The microscopic bumps and holes in two surfaces dig into each other.



j / Static friction: the tray doesn't slip on the waiter's fingers.



k / Kinetic friction: the car skids.



1 / Many landfowl, even those that are competent fliers, prefer to escape from a predator by running upward rather than by flying. This partridge is running up a vertical tree trunk. Humans can't walk up walls because there is no normal force and therefore no frictional force; when  $F_N = 0$ , the maximum force of static friction  $F_{s,max} = \mu_s F_N$  is also zero. The partridge, however, has wings that it can flap in order to create a force between it and the air. Typically when a bird flaps its wings, the resulting force from the air is in the direction that would tend to lift the bird up. In this situation, however, the partridge changes its style of flapping so that the direction is reversed. The normal force between the feet and the tree allows a nonzero static frictional force. The mechanism is similar to that of a spoiler fin on a racing car. Some evolutionary biologists believe that when vertebrate flight first evolved, in dinosaurs, there was first a stage in which the wings were used only as an aid in running up steep inclines, and only later a transition to flight. (Redrawn from a figure by K.P. Dial.)

is needed to make an object move with constant velocity, the floor's rightward frictional force on the fridge has apparently decreased somewhat, making it easier for you to cancel it out. Our model also gives a plausible explanation for this fact: as the surfaces slide past each other, they don't have time to settle down and mesh with one another, so there is less friction.

Even though this model is intuitively appealing and fairly successful, it should not be taken too seriously, and in some situations it is misleading. For instance, fancy racing bikes these days are made with smooth tires that have no tread — contrary to what we'd expect from our model, this does not cause any decrease in friction. Machinists know that two very smooth and clean metal surfaces may stick to each other firmly and be very difficult to slide apart. This cannot be explained in our model, but makes more sense in terms of a model in which friction is described as arising from chemical bonds between the atoms of the two surfaces at their points of contact: very flat surfaces allow more atoms to come in contact.

Since friction changes its behavior dramatically once the surfaces come unstuck, we define two separate types of frictional forces. *Static friction* is friction that occurs between surfaces that are not slipping over each other. Slipping surfaces experience *kinetic friction*. The forces of static and kinetic friction, notated  $F_s$  and  $F_k$ , are always parallel to the surface of contact between the two objects.

#### self-check B

1. When a baseball player slides in to a base, is the friction static, or kinetic?
2. A mattress stays on the roof of a slowly accelerating car. Is the friction static, or kinetic?
3. Does static friction create heat? Kinetic friction? ▷ Answer, p. 531

The maximum possible force of static friction depends on what kinds of surfaces they are, and also on how hard they are being pressed together. The approximate mathematical relationships can be expressed as follows:

$$F_{s,max} = \mu_s F_N \quad ,$$

where  $\mu_s$  is a unitless number, called the coefficient of static friction, which depends on what kinds of surfaces they are. The maximum force that static friction can supply,  $\mu_s F_N$ , represents the boundary between static and kinetic friction. It depends on the normal force, which is numerically equal to whatever force is pressing the two surfaces together. In terms of our model, if the two surfaces are being pressed together more firmly, a greater sideways force will be required in order to make the irregularities in the surfaces ride up

and over each other.

Note that just because we use an adjective such as “applied” to refer to a force, that doesn’t mean that there is some special type of force called the “applied force.” The applied force could be any type of force, or it could be the sum of more than one force trying to make an object move.

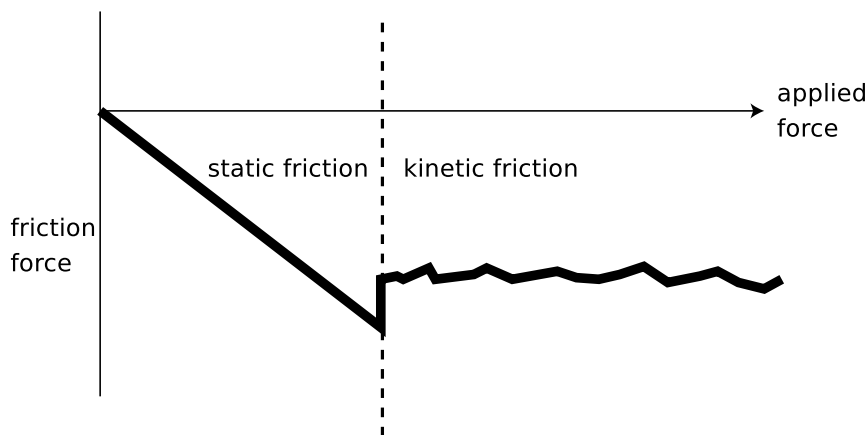
*self-check C*

The arrows in figure I show the forces of the tree trunk on the partridge. Describe the forces the bird makes on the tree.      ▷ Answer, p. 531

The force of kinetic friction on each of the two objects is in the direction that resists the slippage of the surfaces. Its magnitude is usually well approximated as

$$F_k = \mu_k F_N$$

where  $\mu_k$  is the coefficient of kinetic friction. Kinetic friction is usually more or less independent of velocity.



m / We choose a coordinate system in which the applied force, i.e., the force trying to move the objects, is positive. The friction force is then negative, since it is in the opposite direction. As you increase the applied force, the force of static friction increases to match it and cancel it out, until the maximum force of static friction is surpassed. The surfaces then begin slipping past each other, and the friction force becomes smaller in absolute value.

*self-check D*

Can a frictionless surface exert a normal force? Can a frictional force exist without a normal force?      ▷ Answer, p. 532

If you try to accelerate or decelerate your car too quickly, the forces between your wheels and the road become too great, and they begin slipping. This is not good, because kinetic friction is weaker than static friction, resulting in less control. Also, if this occurs while you are turning, the car’s handling changes abruptly because the kinetic friction force is in a different direction than the static friction force had been: contrary to the car’s direction of motion, rather than contrary to the forces applied to the tire.

Most people respond with disbelief when told of the experimental evidence that both static and kinetic friction are approximately



independent of the amount of surface area in contact. Even after doing a hands-on exercise with spring scales to show that it is true, many students are unwilling to believe their own observations, and insist that bigger tires “give more traction.” In fact, the main reason why you would not want to put small tires on a big heavy car is that the tires would burst!

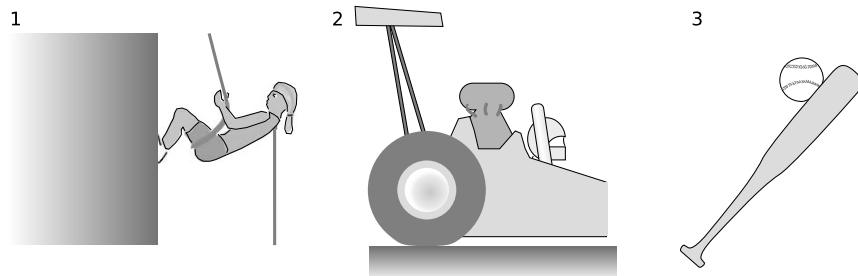
Although many people expect that friction would be proportional to surface area, such a proportionality would make predictions contrary to many everyday observations. A dog’s feet, for example, have very little surface area in contact with the ground compared to a human’s feet, and yet we know that a dog can often win a tug-of-war with a person.

The reason a smaller surface area does not lead to less friction is that the force between the two surfaces is more concentrated, causing their bumps and holes to dig into each other more deeply.

#### *self-check E*

Find the direction of each of the forces in figure n.      ▷ Answer, p. 532

n / 1. The cliff’s normal force on the climber’s feet. 2. The track’s static frictional force on the wheel of the accelerating dragster. 3. The ball’s normal force on the bat.



#### *Locomotives*

#### *example 4*

Looking at a picture of a locomotive, o, we notice two obvious things that are different from an automobile. Where a car typically has two drive wheels, a locomotive normally has many — ten in this example. (Some also have smaller, unpowered wheels in front of and behind the drive wheels, but this example doesn’t.) Also, cars these days are generally built to be as light as possible for their size, whereas locomotives are very massive, and no effort seems to be made to keep their weight low. (The steam locomotive in the photo is from about 1900, but this is true even for modern diesel and electric trains.)



o / Example 4.

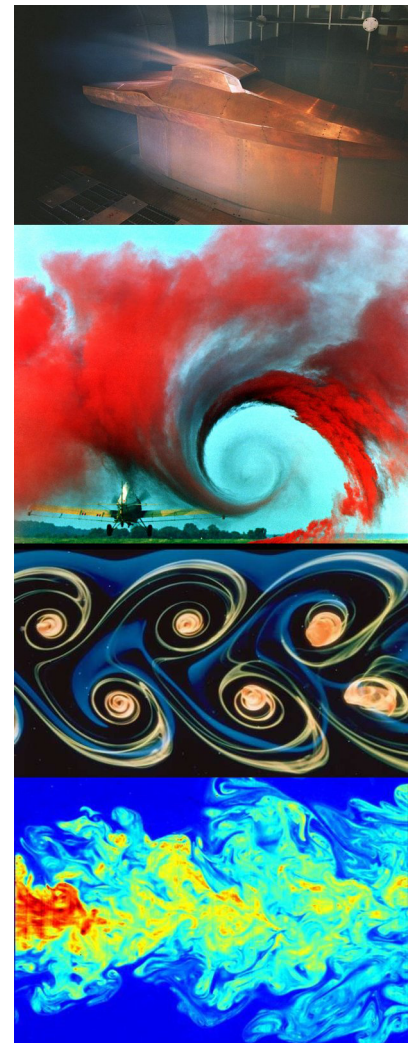
The reason locomotives are built to be so heavy is for traction. The upward normal force of the rails on the wheels,  $F_N$ , cancels the downward force of gravity,  $F_W$ , so ignoring plus and minus signs, these two forces are equal in absolute value,  $F_N = F_W$ . Given this amount of normal force, the maximum force of static friction is  $F_s = \mu_s F_N = \mu_s F_W$ . This static frictional force, of the rails pushing forward on the wheels, is the only force that can accelerate the train, pull it uphill, or cancel out the force of air resistance while cruising at constant speed. The coefficient of static friction for steel on steel is about 1/4, so no locomotive can pull with a force greater than about 1/4 of its own weight. If the engine is capable of supplying more than that amount of force, the result will be simply to break static friction and spin the wheels.

The reason this is all so different from the situation with a car is that a car isn't pulling something else. If you put extra weight in a car, you improve the traction, but you also increase the inertia of the car, and make it just as hard to accelerate. In a train, the inertia is almost all in the cars being pulled, not in the locomotive.

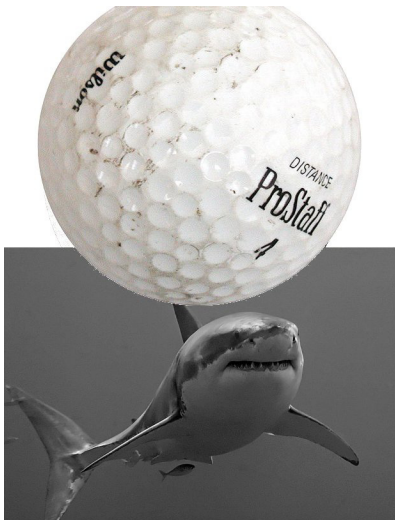
The other fact we have to explain is the large number of driving wheels. First, we have to realize that increasing the number of driving wheels neither increases nor decreases the total amount of static friction, because static friction is independent of the amount of surface area in contact. (The reason four-wheel-drive is good in a car is that if one or more of the wheels is slipping on ice or in mud, the other wheels may still have traction. This isn't typically an issue for a train, since all the wheels experience the same conditions.) The advantage of having more driving wheels on a train is that it allows us to increase the weight of the locomotive without crushing the rails, or damaging bridges.

### Fluid friction

Try to drive a nail into a waterfall and you will be confronted with the main difference between solid friction and fluid friction. Fluid friction is purely kinetic; there is no static fluid friction. The nail in the waterfall may tend to get dragged along by the water flowing past it, but it does not stick in the water. The same is true for gases such as air: recall that we are using the word "fluid" to include both gases and liquids.



p / Fluid friction depends on the fluid's pattern of flow, so it is more complicated than friction between solids, and there are no simple, universally applicable formulas to calculate it. From top to bottom: supersonic wind tunnel, vortex created by a crop duster, series of vortices created by a single object, turbulence.



q / What do the golf ball and the shark have in common? Both use the same trick to reduce fluid friction. The dimples on the golf ball modify the pattern of flow of the air around it, counterintuitively *reducing* friction. Recent studies have shown that sharks can accomplish the same thing by raising, or “bristling,” the scales on their skin at high speeds.



r / The wheelbases of the Hummer H3 and the Toyota Prius are surprisingly similar, differing by only 10%. The main difference in shape is that the Hummer is much taller and wider. It presents a much greater cross-sectional area to the wind, and this is the main reason that it uses about 2.5 times more gas on the freeway.

Unlike kinetic friction between solids, fluid friction increases rapidly with velocity. It also depends on the shape of the object, which is why a fighter jet is more streamlined than a Model T. For objects of the same shape but different sizes, fluid friction typically scales up with the cross-sectional area of the object, which is one of the main reasons that an SUV gets worse mileage on the freeway than a compact car.

### Discussion questions

**A** A student states that when he tries to push his refrigerator, the reason it won’t move is because Newton’s third law says there’s an equal and opposite frictional force pushing back. After all, the static friction force is equal and opposite to the applied force. How would you convince him he is wrong?

**B** Kinetic friction is usually more or less independent of velocity. However, inexperienced drivers tend to produce a jerk at the last moment of deceleration when they stop at a stop light. What does this tell you about the kinetic friction between the brake shoes and the brake drums?

**C** Some of the following are correct descriptions of types of forces that could be added on as new branches of the classification tree. Others are not really types of forces, and still others are not force phenomena at all. In each case, decide what’s going on, and if appropriate, figure out how you would incorporate them into the tree.

sticky force	makes tape stick to things
opposite force	the force that Newton’s third law says relates to every force you make
flowing force	the force that water carries with it as it flows out of a hose
surface tension	lets insects walk on water
horizontal force	a force that is horizontal
motor force	the force that a motor makes on the thing it is turning
canceled force	a force that is being canceled out by some other force

## 5.3 Analysis of forces

Newton’s first and second laws deal with the total of all the forces exerted on a specific object, so it is very important to be able to figure out what forces there are. Once you have focused your attention on one object and listed the forces on it, it is also helpful to describe all the corresponding forces that must exist according to Newton’s third law. We refer to this as “analyzing the forces” in which the object participates.

### A barge

example 5

A barge is being pulled to the right along a canal by teams of horses on the shores. Analyze all the forces in which the barge participates.

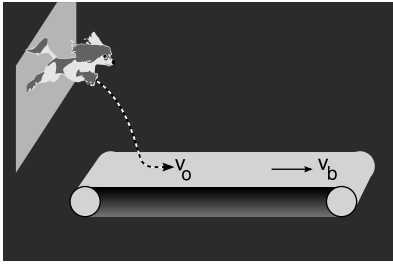
<i>force acting on barge</i>	<i>force related to it by Newton's third law</i>
ropes' normal forces on barge, $\rightarrow$	barge's normal force on ropes, $\leftarrow$
water's fluid friction force on barge, $\leftarrow$	barge's fluid friction force on water, $\rightarrow$
planet earth's gravitational force on barge, $\downarrow$	barge's gravitational force on earth, $\uparrow$
water's "floating" force on barge, $\uparrow$	barge's "floating" force on water, $\downarrow$

Here I've used the word "floating" force as an example of a sensible invented term for a type of force not classified on the tree on p. 153. A more formal technical term would be "hydrostatic force."

Note how the pairs of forces are all structured as "A's force on B, B's force on A": ropes on barge and barge on ropes; water on barge and barge on water. Because all the forces in the left column are forces acting on the barge, all the forces in the right column are forces being exerted by the barge, which is why each entry in the column begins with "barge."

Often you may be unsure whether you have forgotten one of the forces. Here are three strategies for checking your list:

1. See what physical result would come from the forces you've found so far. Suppose, for instance, that you'd forgotten the "floating" force on the barge in the example above. Looking at the forces you'd found, you would have found that there was a downward gravitational force on the barge which was not canceled by any upward force. The barge isn't supposed to sink, so you know you need to find a fourth, upward force.
2. Another technique for finding missing forces is simply to go through the list of all the common types of forces and see if any of them apply.
3. Make a drawing of the object, and draw a dashed boundary line around it that separates it from its environment. Look for points on the boundary where other objects come in contact with your object. This strategy guarantees that you'll find every contact force that acts on the object, although it won't help you to find non-contact forces.



s / Example 6.

*Fifi*

*example 6*

▷ Fifi is an industrial espionage dog who loves doing her job and looks great doing it. She leaps through a window and lands at initial horizontal speed  $v_0$  on a conveyor belt which is itself moving at the greater speed  $v_b$ . Unfortunately the coefficient of kinetic friction  $\mu_k$  between her foot-pads and the belt is fairly low, so she skids for a time  $\Delta t$ , during which the effect on her coiffure are *un désastre*. Find  $\Delta t$ .

▷ We analyze the forces:

<i>force acting on Fifi</i>	<i>force related to it by Newton's third law</i>
planet earth's gravitational force $F_W = mg$ on Fifi, $\downarrow$	Fifi's gravitational force on earth, $\uparrow$
belt's kinetic frictional force $F_k$ on Fifi, $\rightarrow$	Fifi's kinetic frictional force on belt, $\leftarrow$
belt's normal force $F_N$ on Fifi, $\uparrow$	Fifi's normal force on belt, $\downarrow$

Checking the analysis of the forces as described on p. 161:

(1) The physical result makes sense. The left-hand column consists of forces  $\downarrow \rightarrow \uparrow$ . We're describing the time when she's moving horizontally on the belt, so it makes sense that we have two vertical forces that could cancel. The rightward force is what will accelerate her until her speed matches that of the belt.

(2) We've included every relevant type of force from the tree on p. 153.

(3) We've included forces from the belt, which is the only object in contact with Fifi.

The purpose of the analysis is to let us set up equations containing enough information to solve the problem. Let positive  $x$  be to the right. Newton's second law gives

$$(\rightarrow) \quad a = F_k / m$$

Although it's the horizontal motion we care about, the only way to find  $F_k$  is via the relation  $F_k = \mu_k F_N$ , and the only way to find  $F_N$  is from the  $\uparrow \downarrow$  forces. If the vertical forces are to cancel, they must be of equal strength:

$$(\uparrow \downarrow) \quad F_N = mg$$

Using the constant-acceleration equation  $a = \Delta v / \Delta t$ , we have

$$\begin{aligned} \Delta t &= \frac{\Delta v}{a} \\ &= \frac{v_b - v_0}{\mu_k mg / m} \\ &= \frac{v_b - v_0}{\mu_k g} \end{aligned} \quad .$$

The units check out:

$$s = \frac{m/s}{m/s^2},$$

where  $\mu_k$  is omitted as a factor because it's unitless.

We should also check that the dependence on the variables also makes sense. If Fifi puts on her rubber ninja booties, increasing  $\mu_k$ , then dividing by a larger number gives a smaller result for  $\Delta t$ ; this makes sense physically, because the greater friction will cause her to come up to the belt's speed more quickly. The dependence on  $g$  is similar; more gravity would press her harder against the belt, improving her traction. Increasing  $v_b$  increases  $\Delta t$ , which makes sense because it will take her longer to get up to a bigger speed. Since  $v_o$  is subtracted, the dependence of  $\Delta t$  on it is the other way around, and that makes sense too, because if she can land with a greater speed, she has less speeding up left to do.

### Discussion questions

**A** In the example of the barge going down the canal, I referred to a “floating” or “hydrostatic” force that keeps the boat from sinking. If you were adding a new branch on the force-classification tree to represent this force, where would it go?

**B** The earth's gravitational force on you, i.e., your weight, is always equal to  $mg$ , where  $m$  is your mass. So why can you get a shovel to go deeper into the ground by jumping onto it? Just because you're jumping, that doesn't mean your mass or weight is any greater, does it?

## 5.4 Transmission of forces by low-mass objects

You're walking your dog. The dog wants to go faster than you do, and the leash is taut. Does Newton's third law guarantee that your force on your end of the leash is equal and opposite to the dog's force on its end? If they're not exactly equal, is there any reason why they should be approximately equal?

If there was no leash between you, and you were in direct contact with the dog, then Newton's third law would apply, but Newton's third law cannot relate your force on the leash to the dog's force on the leash, because that would involve three separate objects. Newton's third law only says that your force on the leash is equal and opposite to the leash's force on you,

$$F_{yL} = -F_{Ly},$$

and that the dog's force on the leash is equal and opposite to its force on the dog

$$F_{dL} = -F_{Ld}.$$

Still, we have a strong intuitive expectation that whatever force we make on our end of the leash is transmitted to the dog, and vice-versa. We can analyze the situation by concentrating on the forces that act on the leash,  $F_{dL}$  and  $F_{yL}$ . According to Newton's second law, these relate to the leash's mass and acceleration:

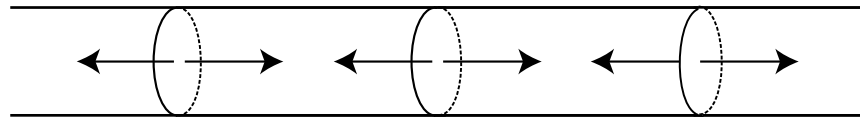
$$F_{dL} + F_{yL} = m_L a_L.$$

The leash is far less massive than any of the other objects involved, and if  $m_L$  is very small, then apparently the total force on the leash is also very small,  $F_{dL} + F_{yL} \approx 0$ , and therefore

$$F_{dL} \approx -F_{yL} \quad .$$

Thus even though Newton's third law does not apply directly to these two forces, we can approximate the low-mass leash as if it was not intervening between you and the dog. It's at least approximately as if you and the dog were acting directly on each other, in which case Newton's third law would have applied.

In general, low-mass objects can be treated approximately as if they simply transmitted forces from one object to another. This can be true for strings, ropes, and cords, and also for rigid objects such as rods and sticks.



t / If we imagine dividing a taut rope up into small segments, then any segment has forces pulling outward on it at each end. If the rope is of negligible mass, then all the forces equal  $+T$  or  $-T$ , where  $T$ , the tension, is a single number.



u / The Golden Gate Bridge's roadway is held up by the tension in the vertical cables.

If you look at a piece of string under a magnifying glass as you pull on the ends more and more strongly, you will see the fibers straightening and becoming taut. Different parts of the string are apparently exerting forces on each other. For instance, if we think of the two halves of the string as two objects, then each half is exerting a force on the other half. If we imagine the string as consisting of many small parts, then each segment is transmitting a force to the next segment, and if the string has very little mass, then all the forces are equal in magnitude. We refer to the magnitude of the forces as the tension in the string,  $T$ . Although the tension is measured in units of Newtons, it is not itself a force. There are many forces within the string, some in one direction and some in the other direction, and their magnitudes are only approximately equal. The concept of tension only makes sense as a general, approximate statement of how big all the forces are.

If a rope goes over a pulley or around some other object, then the tension throughout the rope is approximately equal so long as the pulley has negligible mass and there is not too much friction. A rod or stick can be treated in much the same way as a string, but it is possible to have either compression or tension.

Since tension is not a type of force, the force exerted by a rope on some other object must be of some definite type such as static friction, kinetic friction, or a normal force. If you hold your dog's leash with your hand through the loop, then the force exerted by the leash on your hand is a normal force: it is the force that keeps the leash from occupying the same space as your hand. If you grasp a plain end of a rope, then the force between the rope and your hand is a frictional force.

A more complex example of transmission of forces is the way a car accelerates. Many people would describe the car's engine as making the force that accelerates the car, but the engine is part of the car, so that's impossible: objects can't make forces on themselves. What really happens is that the engine's force is transmitted through the transmission to the axles, then through the tires to the road. By Newton's third law, there will thus be a forward force from the road on the tires, which accelerates the car.

### Discussion question

**A** When you step on the gas pedal, is your foot's force being transmitted in the sense of the word used in this section?

## 5.5 Objects under strain

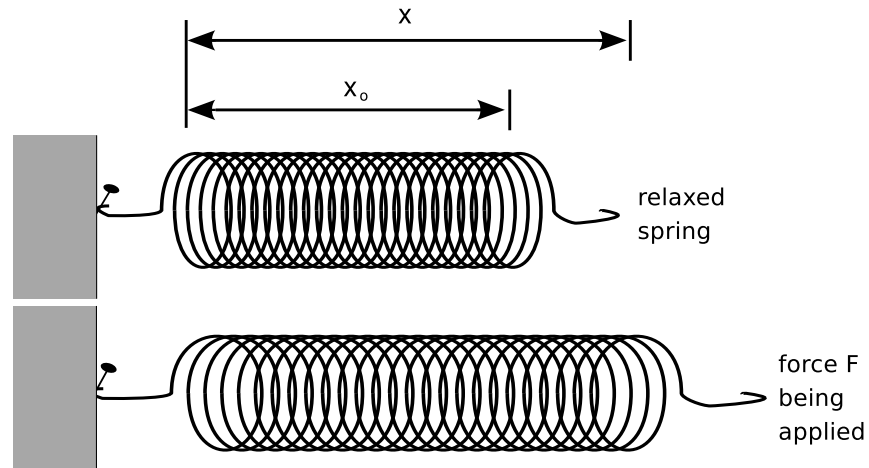
A string lengthens slightly when you stretch it. Similarly, we have already discussed how an apparently rigid object such as a wall is actually flexing when it participates in a normal force. In other cases, the effect is more obvious. A spring or a rubber band visibly elongates when stretched.

Common to all these examples is a change in shape of some kind: lengthening, bending, compressing, etc. The change in shape can be measured by picking some part of the object and measuring its position,  $x$ . For concreteness, let's imagine a spring with one end attached to a wall. When no force is exerted, the unfixed end of the spring is at some position  $x_o$ . If a force acts at the unfixed end, its position will change to some new value of  $x$ . The more force, the greater the departure of  $x$  from  $x_o$ .

Back in Newton's time, experiments like this were considered cutting-edge research, and his contemporary Hooke is remembered today for doing them and for coming up with a simple mathematical



v / Defining the quantities  $F$ ,  $x$ , and  $x_0$  in Hooke's law.



generalization called Hooke's law:

$$F \approx k(x - x_0) \quad . \quad [\text{force required to stretch a spring; valid for small forces only}]$$

Here  $k$  is a constant, called the spring constant, that depends on how stiff the object is. If too much force is applied, the spring exhibits more complicated behavior, so the equation is only a good approximation if the force is sufficiently small. Usually when the force is so large that Hooke's law is a bad approximation, the force ends up permanently bending or breaking the spring.

Although Hooke's law may seem like a piece of trivia about springs, it is actually far more important than that, because all solid objects exert Hooke's-law behavior over some range of sufficiently small forces. For example, if you push down on the hood of a car, it dips by an amount that is directly proportional to the force. (But the car's behavior would not be as mathematically simple if you dropped a boulder on the hood!)

▷ Solved problem: Combining springs page 172, problem 14

▷ Solved problem: Young's modulus page 172, problem 16

### Discussion question

**A** A car is connected to its axles through big, stiff springs called shock absorbers, or "shocks." Although we've discussed Hooke's law above only in the case of stretching a spring, a car's shocks are continually going through both stretching and compression. In this situation, how would you interpret the positive and negative signs in Hooke's law?

## 5.6 Simple machines: the pulley

Even the most complex machines, such as cars or pianos, are built out of certain basic units called *simple machines*. The following are some of the main functions of simple machines:

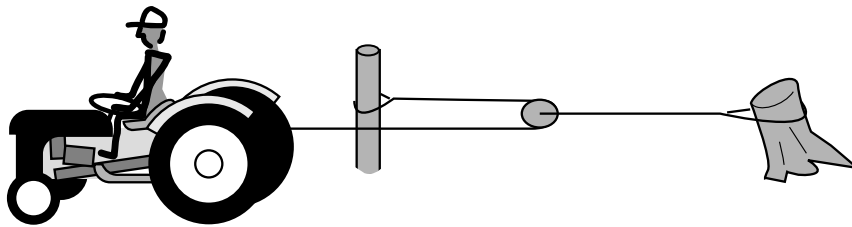
transmitting a force: The chain on a bicycle transmits a force from the crank set to the rear wheel.

changing the direction of a force: If you push down on a see-saw, the other end goes up.

changing the speed and precision of motion: When you make the “come here” motion, your biceps only moves a couple of centimeters where it attaches to your forearm, but your arm moves much farther and more rapidly.

changing the amount of force: A lever or pulley can be used to increase or decrease the amount of force.

You are now prepared to understand one-dimensional simple machines, of which the pulley is the main example.



w / Example 7.

#### *A pulley*

#### *example 7*

▷ Farmer Bill says this pulley arrangement doubles the force of his tractor. Is he just a dumb hayseed, or does he know what he’s doing?

▷ To use Newton’s first law, we need to pick an object and consider the sum of the forces on it. Since our goal is to relate the tension in the part of the cable attached to the stump to the tension in the part attached to the tractor, we should pick an object to which both those cables are attached, i.e., the pulley itself. The tension in a string or cable remains approximately constant as it passes around an idealized pulley.<sup>1</sup> There are therefore two leftward forces acting on the pulley, each equal to the force exerted by the tractor. Since the acceleration of the pulley is essentially zero, the forces on it must be canceling out, so the rightward force of the pulley-stump cable on the pulley must be double the force exerted by the tractor. Yes, Farmer Bill knows what he’s talking about.

<sup>1</sup>This was asserted in section 5.4 without proof. Essentially it holds because of symmetry. E.g., if the U-shaped piece of rope in figure w had unequal tension in its two legs, then this would have to be caused by some asymmetry between clockwise and counterclockwise rotation. But such an asymmetry can only be caused by friction or inertia, which we assume don’t exist.

## Summary

### Selected vocabulary

repulsive . . . . .	describes a force that tends to push the two participating objects apart
attractive . . . . .	describes a force that tends to pull the two participating objects together
oblique . . . . .	describes a force that acts at some other angle, one that is not a direct repulsion or attraction
normal force . . .	the force that keeps two objects from occupying the same space
static friction . .	a friction force between surfaces that are not slipping past each other
kinetic friction .	a friction force between surfaces that are slipping past each other
fluid . . . . .	a gas or a liquid
fluid friction . . .	a friction force in which at least one of the object is is a fluid
spring constant .	the constant of proportionality between force and elongation of a spring or other object under strain

### Notation

$F_N$ . . . . .	a normal force
$F_s$ . . . . .	a static frictional force
$F_k$ . . . . .	a kinetic frictional force
$\mu_s$ . . . . .	the coefficient of static friction; the constant of proportionality between the maximum static frictional force and the normal force; depends on what types of surfaces are involved
$\mu_k$ . . . . .	the coefficient of kinetic friction; the constant of proportionality between the kinetic frictional force and the normal force; depends on what types of surfaces are involved
$k$ . . . . .	the spring constant; the constant of proportionality between the force exerted on an object and the amount by which the object is lengthened or compressed

### Summary

Newton's third law states that forces occur in equal and opposite pairs. If object A exerts a force on object B, then object B must simultaneously be exerting an equal and opposite force on object A. Each instance of Newton's third law involves exactly two objects, and exactly two forces, which are of the same type.

There are two systems for classifying forces. We are presently using the more practical but less fundamental one. In this system, forces are classified by whether they are repulsive, attractive, or oblique; whether they are contact or noncontact forces; and whether

the two objects involved are solids or fluids.

Static friction adjusts itself to match the force that is trying to make the surfaces slide past each other, until the maximum value is reached,

$$F_{s,max} = \mu_s F_N \quad .$$

Once this force is exceeded, the surfaces slip past one another, and kinetic friction applies,

$$F_k = \mu_k F_N \quad .$$

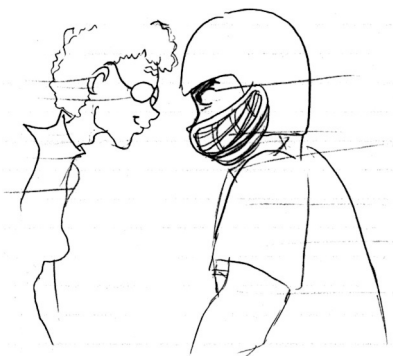
Both types of frictional force are nearly independent of surface area, and kinetic friction is usually approximately independent of the speed at which the surfaces are slipping. The direction of the force is in the direction that would tend to stop or prevent slipping.

A good first step in applying Newton's laws of motion to any physical situation is to pick an object of interest, and then to list all the forces acting on that object. We classify each force by its type, and find its Newton's-third-law partner, which is exerted by the object on some other object.

When two objects are connected by a third low-mass object, their forces are transmitted to each other nearly unchanged.

Objects under strain always obey Hooke's law to a good approximation, as long as the force is small. Hooke's law states that the stretching or compression of the object is proportional to the force exerted on it,

$$F \approx k(x - x_o) \quad .$$



Problem 1.



Problem 6.



Problem 7.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** A little old lady and a pro football player collide head-on. Compare their forces on each other, and compare their accelerations. Explain.

**2** The earth is attracted to an object with a force equal and opposite to the force of the earth on the object. If this is true, why is it that when you drop an object, the earth does not have an acceleration equal and opposite to that of the object?

**3** When you stand still, there are two forces acting on you, the force of gravity (your weight) and the normal force of the floor pushing up on your feet. Are these forces equal and opposite? Does Newton's third law relate them to each other? Explain.

*In problems 4-8, analyze the forces using a table in the format shown in section 5.3. Analyze the forces in which the italicized object participates.*

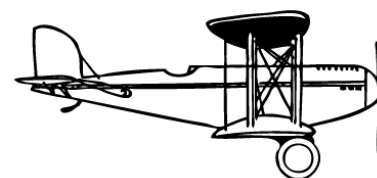
**4** Some people put a spare car key in a little magnetic *box* that they stick under the chassis of their car. Let's say that the box is stuck directly underneath a horizontal surface, and the car is parked. (See instructions above.)

**5** Analyze two examples of *objects* at rest relative to the earth that are being kept from falling by forces other than the normal force. Do not use objects in outer space, and do not duplicate problem 4 or 8. (See instructions above.)

**6** A *person* is rowing a boat, with her feet braced. She is doing the part of the stroke that propels the boat, with the ends of the oars in the water (not the part where the oars are out of the water). (See instructions above.)

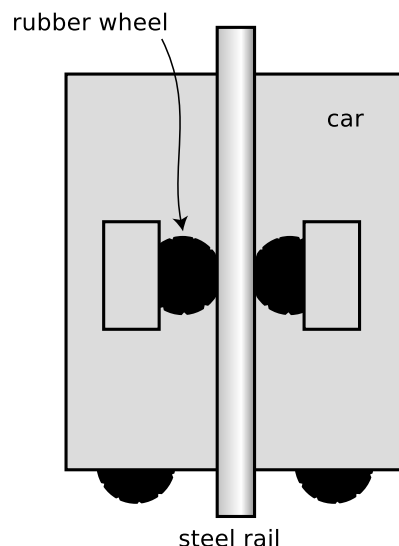
**7** A *farmer* is in a stall with a cow when the cow decides to press him against the wall, pinning him with his feet off the ground. Analyze the forces in which the farmer participates. (See instructions above.)

**8** A propeller *plane* is cruising east at constant speed and altitude. (See instructions above.)



Problem 8.

**9** Today's tallest buildings are really not that much taller than the tallest buildings of the 1940's. One big problem with making an even taller skyscraper is that every elevator needs its own shaft running the whole height of the building. So many elevators are needed to serve the building's thousands of occupants that the elevator shafts start taking up too much of the space within the building. An alternative is to have elevators that can move both horizontally and vertically: with such a design, many elevator cars can share a few shafts, and they don't get in each other's way too much because they can detour around each other. In this design, it becomes impossible to hang the cars from cables, so they would instead have to ride on rails which they grab onto with wheels. Friction would keep them from slipping. The figure shows such a frictional elevator in its vertical travel mode. (The wheels on the bottom are for when it needs to switch to horizontal motion.)

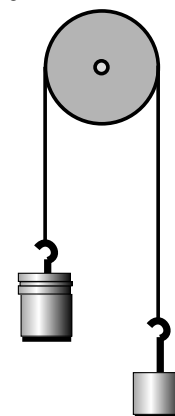


Problem 9.

(a) If the coefficient of static friction between rubber and steel is  $\mu_s$ , and the maximum mass of the car plus its passengers is  $M$ , how much force must there be pressing each wheel against the rail in order to keep the car from slipping? (Assume the car is not accelerating.) ✓

(b) Show that your result has physically reasonable behavior with respect to  $\mu_s$ . In other words, if there was less friction, would the wheels need to be pressed more firmly or less firmly? Does your equation behave that way?

**10** Unequal masses  $M$  and  $m$  are suspended from a pulley as shown in the figure.



Problem 10.

(a) Analyze the forces in which mass  $m$  participates, using a table the format shown in section 5.3. [The forces in which the other mass participates will of course be similar, but not numerically the same.]

(b) Find the magnitude of the accelerations of the two masses. [Hints: (1) Pick a coordinate system, and use positive and negative signs consistently to indicate the directions of the forces and accelerations. (2) The two accelerations of the two masses have to be equal in magnitude but of opposite signs, since one side eats up rope at the same rate at which the other side pays it out. (3) You need to apply Newton's second law twice, once to each mass, and then solve the two equations for the unknowns: the acceleration,  $a$ , and the tension in the rope,  $T$ .] ✓

(c) Many people expect that in the special case of  $M = m$ , the two masses will naturally settle down to an equilibrium position side by side. Based on your answer from part b, is this correct?

(d) Find the tension in the rope,  $T$ . ✓

(e) Interpret your equation from part d in the special case where one of the masses is zero. Here "interpret" means to figure out what happens mathematically, figure out what should happen physically, and connect the two.

**11** A tugboat of mass  $m$  pulls a ship of mass  $M$ , accelerating it. The speeds are low enough that you can ignore fluid friction acting on their hulls, although there will of course need to be fluid friction acting on the tug's propellers.

(a) Analyze the forces in which the tugboat participates, using a table in the format shown in section 5.3. Don't worry about vertical forces.

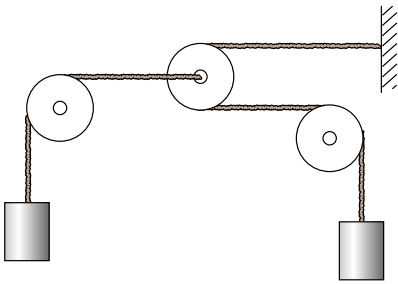
(b) Do the same for the ship.

(c) If the force acting on the tug's propeller is  $F$ , what is the tension,  $T$ , in the cable connecting the two ships? [Hint: Write down two equations, one for Newton's second law applied to each object. Solve these for the two unknowns  $T$  and  $a$ .]  $\checkmark$

(d) Interpret your answer in the special cases of  $M = 0$  and  $M = \infty$ .

**12** Someone tells you she knows of a certain type of Central American earthworm whose skin, when rubbed on polished diamond, has  $\mu_k > \mu_s$ . Why is this not just empirically unlikely but logically suspect?

**13** In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two masses are identical. Show that the mass on the left will have an upward acceleration equal to  $g/5$ . Assume all the ropes and pulleys are massless and frictionless.

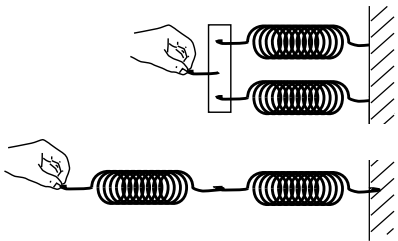


Problem 13.

**14** The figure shows two different ways of combining a pair of identical springs, each with spring constant  $k$ . We refer to the top setup as parallel, and the bottom one as a series arrangement.

(a) For the parallel arrangement, analyze the forces acting on the connector piece on the left, and then use this analysis to determine the equivalent spring constant of the whole setup. Explain whether the combined spring constant should be interpreted as being stiffer or less stiff.

(b) For the series arrangement, analyze the forces acting on each spring and figure out the same things.  $\triangleright$  Solution, p. 520



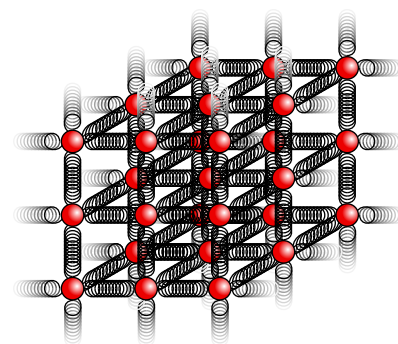
Problem 14.

**15** Generalize the results of problem 14 to the case where the two spring constants are unequal.

**16** (a) Using the solution of problem 14, which is given in the back of the book, predict how the spring constant of a fiber will depend on its length and cross-sectional area.

(b) The constant of proportionality is called the Young's modulus,  $E$ , and typical values of the Young's modulus are about  $10^{10}$  to  $10^{11}$ . What units would the Young's modulus have in the SI (meter-kilogram-second) system?  $\triangleright$  Solution, p. 521

**17** This problem depends on the results of problems 14 and 16, whose solutions are in the back of the book. When atoms form chemical bonds, it makes sense to talk about the spring constant of the bond as a measure of how “stiff” it is. Of course, there aren’t really little springs — this is just a mechanical model. The purpose of this problem is to estimate the spring constant,  $k$ , for a single bond in a typical piece of solid matter. Suppose we have a fiber, like a hair or a piece of fishing line, and imagine for simplicity that it is made of atoms of a single element stacked in a cubical manner, as shown in the figure, with a center-to-center spacing  $b$ . A typical value for  $b$  would be about  $10^{-10}$  m.



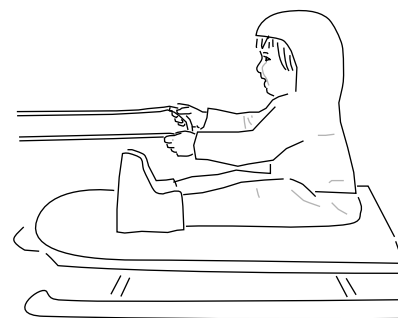
Problem 17.

- (a) Find an equation for  $k$  in terms of  $b$ , and in terms of the Young’s modulus,  $E$ , defined in problem 16 and its solution.
- (b) Estimate  $k$  using the numerical data given in problem 16.
- (c) Suppose you could grab one of the atoms in a diatomic molecule like  $\text{H}_2$  or  $\text{O}_2$ , and let the other atom hang vertically below it. Does the bond stretch by any appreciable fraction due to gravity?

**18** In each case, identify the force that causes the acceleration, and give its Newton’s-third-law partner. Describe the effect of the partner force. (a) A swimmer speeds up. (b) A golfer hits the ball off of the tee. (c) An archer fires an arrow. (d) A locomotive slows down.

▷ Solution, p. 521

**19** Ginny has a plan. She is going to ride her sled while her dog Foo pulls her, and she holds on to his leash. However, Ginny hasn’t taken physics, so there may be a problem: she may slide right off the sled when Foo starts pulling.



Problem 19.

- (a) Analyze all the forces in which Ginny participates, making a table as in section 5.3.

- (b) Analyze all the forces in which the sled participates.

(c) The sled has mass  $m$ , and Ginny has mass  $M$ . The coefficient of static friction between the sled and the snow is  $\mu_1$ , and  $\mu_2$  is the corresponding quantity for static friction between the sled and her snow pants. Ginny must have a certain minimum mass so that she will not slip off the sled. Find this in terms of the other three variables. ✓

- (d) Interpreting your equation from part c, under what conditions will there be no physically realistic solution for  $M$ ? Discuss what this means physically.

**20** Example 2 on page 151 involves a person pushing a box up a hill. The incorrect answer describes three forces. For each of these three forces, give the force that it is related to by Newton’s third law, and state the type of force.

▷ Solution, p. 521

**21** Example 7 on page 167 describes a force-doubling setup involving a pulley. Make up a more complicated arrangement, using two pulleys, that would multiply the force by four. The basic idea is to take the output of one force doubler and feed it into the input



of a second one.

**22** Pick up a heavy object such as a backpack or a chair, and stand on a bathroom scale. Shake the object up and down. What do you observe? Interpret your observations in terms of Newton's third law.

**23** A cop investigating the scene of an accident measures the length  $L$  of a car's skid marks in order to find out its speed  $v$  at the beginning of the skid. Express  $v$  in terms of  $L$  and any other relevant variables. ✓

**24** The following reasoning leads to an apparent paradox; explain what's wrong with the logic. A baseball player hits a ball. The ball and the bat spend a fraction of a second in contact. During that time they're moving together, so their accelerations must be equal. Newton's third law says that their forces on each other are also equal. But  $a = F/m$ , so how can this be, since their masses are unequal? (Note that the paradox isn't resolved by considering the force of the batter's hands on the bat. Not only is this force very small compared to the ball-bat force, but the batter could have just thrown the bat at the ball.)

**25** This problem has been deleted.

**26** (a) Compare the mass of a one-liter water bottle on earth, on the moon, and in interstellar space. ▷ Solution, p. 522  
(b) Do the same for its weight.

**27** An ice skater builds up some speed, and then coasts across the ice passively in a straight line. (a) Analyze the forces, using a table the format shown in section 5.3.

(b) If his initial speed is  $v$ , and the coefficient of kinetic friction is  $\mu_k$ , find the maximum theoretical distance he can glide before coming to a stop. Ignore air resistance. ✓

(c) Show that your answer to part b has the right units.

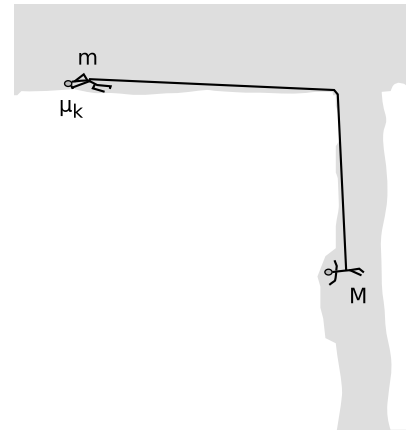
(d) Show that your answer to part b depends on the variables in a way that makes sense physically.

(e) Evaluate your answer numerically for  $\mu_k = 0.0046$ , and a world-record speed of 14.58 m/s. (The coefficient of friction was measured by De Koning et al., using special skates worn by real speed skaters.) ✓

(f) Comment on whether your answer in part e seems realistic. If it doesn't, suggest possible reasons why.

**28** Mountain climbers with masses  $m$  and  $M$  are roped together while crossing a horizontal glacier when a vertical crevasse opens up under the climber with mass  $M$ . The climber with mass  $m$  drops down on the snow and tries to stop by digging into the snow with the pick of an ice ax. Alas, this story does not have a happy ending, because this doesn't provide enough friction to stop. Both  $m$  and  $M$  continue accelerating, with  $M$  dropping down into the crevasse and  $m$  being dragged across the snow, slowed only by the kinetic friction with coefficient  $\mu_k$  acting between the ax and the snow. There is no significant friction between the rope and the lip of the crevasse.

- (a) Find the acceleration  $a$ . ✓
- (b) Check the units of your result.
- (c) Check the dependence of your equation on the variables. That means that for each variable, you should determine what its effect on  $a$  should be physically, and then what your answer from part a says its effect would be mathematically.



Problem 28.

## **Motion in three dimensions**







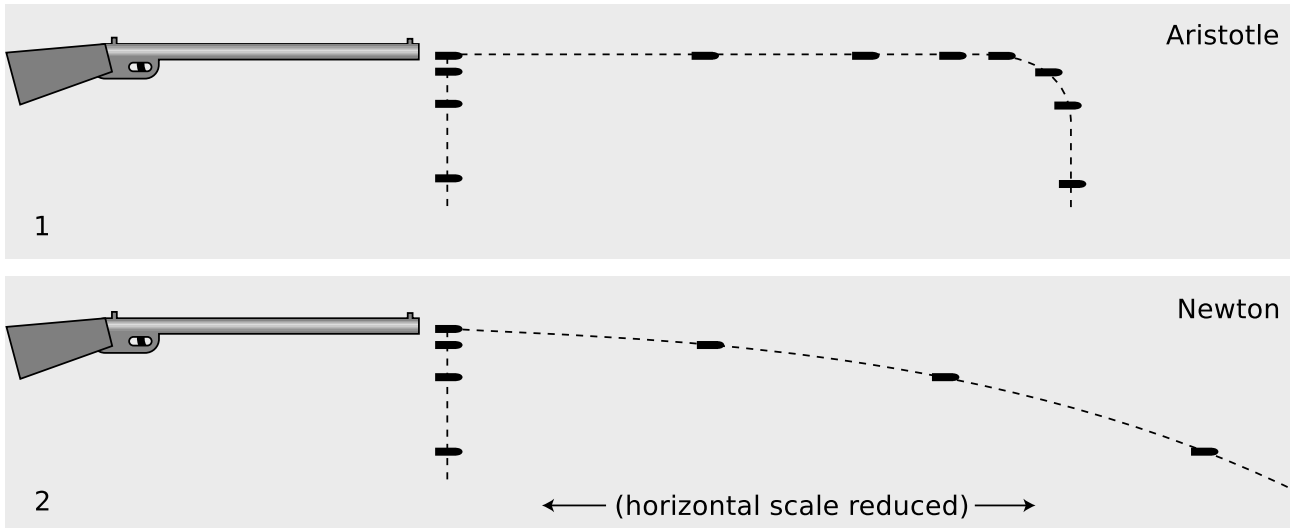
## Chapter 6

# Newton's laws in three dimensions

### 6.1 Forces have no perpendicular effects

Suppose you could shoot a rifle and arrange for a second bullet to be dropped from the same height at the exact moment when the first left the barrel. Which would hit the ground first? Nearly everyone expects that the dropped bullet will reach the dirt first,

and Aristotle would have agreed. Aristotle would have described it like this. The shot bullet receives some forced motion from the gun. It travels forward for a split second, slowing down rapidly because there is no longer any force to make it continue in motion. Once it is done with its forced motion, it changes to natural motion, i.e. falling straight down. While the shot bullet is slowing down, the dropped bullet gets on with the business of falling, so according to Aristotle it will hit the ground first.



a / A bullet is shot from a gun, and another bullet is simultaneously dropped from the same height. 1. Aristotelian physics says that the horizontal motion of the shot bullet delays the onset of falling, so the dropped bullet hits the ground first. 2. Newtonian physics says the two bullets have the same vertical motion, regardless of their different horizontal motions.

Luckily, nature isn't as complicated as Aristotle thought! To convince yourself that Aristotle's ideas were wrong and needlessly complex, stand up now and try this experiment. Take your keys out of your pocket, and begin walking briskly forward. Without speeding up or slowing down, release your keys and let them fall while you continue walking at the same pace.

You have found that your keys hit the ground right next to your feet. Their horizontal motion never slowed down at all, and the whole time they were dropping, they were right next to you. The horizontal motion and the vertical motion happen at the same time, and they are independent of each other. Your experiment proves that the horizontal motion is unaffected by the vertical motion, but it's also true that the vertical motion is not changed in any way by the horizontal motion. The keys take exactly the same amount of time to get to the ground as they would have if you simply dropped them, and the same is true of the bullets: both bullets hit the ground

simultaneously.

These have been our first examples of motion in more than one dimension, and they illustrate the most important new idea that is required to understand the three-dimensional generalization of Newtonian physics:

**Forces have no perpendicular effects.**

When a force acts on an object, it has no effect on the part of the object's motion that is perpendicular to the force.

In the examples above, the vertical force of gravity had no effect on the horizontal motions of the objects. These were examples of projectile motion, which interested people like Galileo because of its military applications. The principle is more general than that, however. For instance, if a rolling ball is initially heading straight for a wall, but a steady wind begins blowing from the side, the ball does not take any longer to get to the wall. In the case of projectile motion, the force involved is gravity, so we can say more specifically that the vertical acceleration is  $9.8 \text{ m/s}^2$ , regardless of the horizontal motion.

*self-check A*

In the example of the ball being blown sideways, why doesn't the ball take longer to get there, since it has to travel a greater distance?   ▷

Answer, p. 532

**Relationship to relative motion**

These concepts are directly related to the idea that motion is relative. Galileo's opponents argued that the earth could not possibly be rotating as he claimed, because then if you jumped straight up in the air you wouldn't be able to come down in the same place. Their argument was based on their incorrect Aristotelian assumption that once the force of gravity began to act on you and bring you back down, your horizontal motion would stop. In the correct Newtonian theory, the earth's downward gravitational force is acting before, during, and after your jump, but has no effect on your motion in the perpendicular (horizontal) direction.

If Aristotle had been correct, then we would have a handy way to determine absolute motion and absolute rest: jump straight up in the air, and if you land back where you started, the surface from which you jumped must have been in a state of rest. In reality, this test gives the same result as long as the surface under you is an inertial frame. If you try this in a jet plane, you land back on the same spot on the deck from which you started, regardless of whether the plane is flying at 500 miles per hour or parked on the runway. The method would in fact only be good for detecting whether the



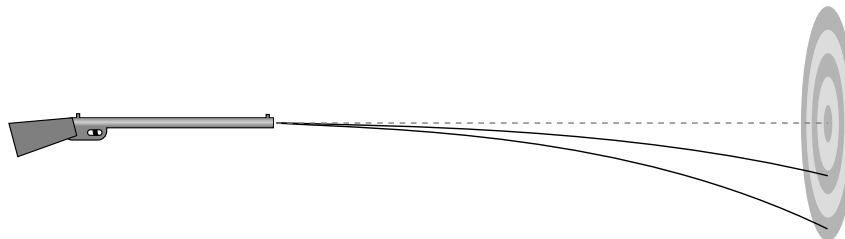
plane was accelerating.

## Discussion questions

**A** The following is an incorrect explanation of a fact about target shooting:

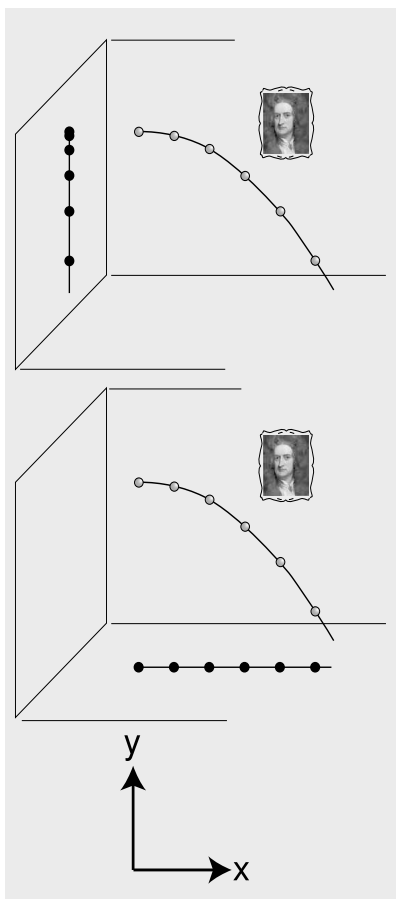
“Shooting a high-powered rifle with a high muzzle velocity is different from shooting a less powerful gun. With a less powerful gun, you have to aim quite a bit above your target, but with a more powerful one you don’t have to aim so high because the bullet doesn’t drop as fast.”

Explain why it’s incorrect. What is the correct explanation?



**B** You have thrown a rock, and it is flying through the air in an arc. If the earth’s gravitational force on it is always straight down, why doesn’t it just go straight down once it leaves your hand?

**C** Consider the example of the bullet that is dropped at the same moment another bullet is fired from a gun. What would the motion of the two bullets look like to a jet pilot flying alongside in the same direction as the shot bullet and at the same horizontal speed?



c / The shadow on the wall shows the ball’s  $y$  motion, the shadow on the floor its  $x$  motion.

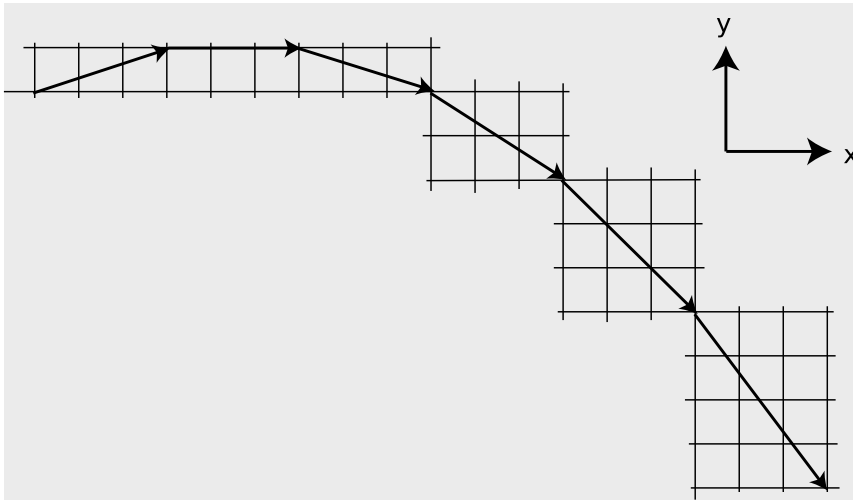
## 6.2 Coordinates and components

’Cause we’re all  
Bold as love,  
Just ask the axis.

*Jimi Hendrix*

How do we convert these ideas into mathematics? Figure b shows a good way of connecting the intuitive ideas to the numbers. In one dimension, we impose a number line with an  $x$  coordinate on a certain stretch of space. In two dimensions, we imagine a grid of squares which we label with  $x$  and  $y$  values, as shown in figure b.

But of course motion doesn’t really occur in a series of discrete hops like in chess or checkers. Figure c shows a way of conceptualizing the smooth variation of the  $x$  and  $y$  coordinates. The ball’s shadow on the wall moves along a line, and we describe its position with a single coordinate,  $y$ , its height above the floor. The wall shadow has a constant acceleration of  $-9.8 \text{ m/s}^2$ . A shadow on the floor, made by a second light source, also moves along a line, and we describe its motion with an  $x$  coordinate, measured from the wall.



b / This object experiences a force that pulls it down toward the bottom of the page. In each equal time interval, it moves three units to the right. At the same time, its vertical motion is making a simple pattern of  $+1, 0, -1, -2, -3, -4, \dots$  units. Its motion can be described by an  $x$  coordinate that has zero acceleration and a  $y$  coordinate with constant acceleration. The arrows labeled  $x$  and  $y$  serve to explain that we are defining increasing  $x$  to the right and increasing  $y$  as upward.

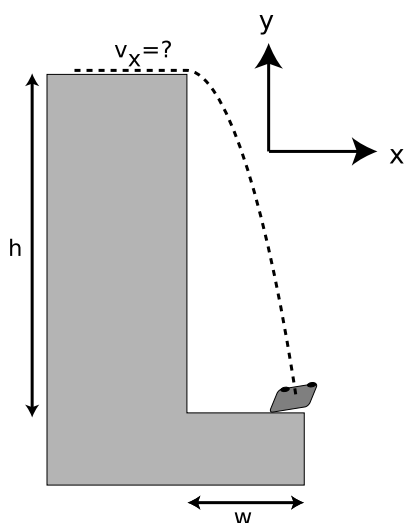
The velocity of the floor shadow is referred to as the  $x$  component of the velocity, written  $v_x$ . Similarly we can notate the acceleration of the floor shadow as  $a_x$ . Since  $v_x$  is constant,  $a_x$  is zero.

Similarly, the velocity of the wall shadow is called  $v_y$ , its acceleration  $a_y$ . This example has  $a_y = -9.8 \text{ m/s}^2$ .

Because the earth's gravitational force on the ball is acting along the  $y$  axis, we say that the force has a negative  $y$  component,  $F_y$ , but  $F_x = F_z = 0$ .

The general idea is that we imagine two observers, each of whom perceives the entire universe as if it was flattened down to a single line. The  $y$ -observer, for instance, perceives  $y$ ,  $v_y$ , and  $a_y$ , and will infer that there is a force,  $F_y$ , acting downward on the ball. That is, a  $y$  component means the aspect of a physical phenomenon, such as velocity, acceleration, or force, that is observable to someone who can only see motion along the  $y$  axis.

All of this can easily be generalized to three dimensions. In the example above, there could be a  $z$ -observer who only sees motion toward or away from the back wall of the room.



d / Example 1.

### A car going over a cliff

example 1

▷ The police find a car at a distance  $w = 20$  m from the base of a cliff of height  $h = 100$  m. How fast was the car going when it went over the edge? Solve the problem symbolically first, then plug in the numbers.

▷ Let's choose  $y$  pointing up and  $x$  pointing away from the cliff. The car's vertical motion was independent of its horizontal motion, so we know it had a constant vertical acceleration of  $a = -g = -9.8 \text{ m/s}^2$ . The time it spent in the air is therefore related to the vertical distance it fell by the constant-acceleration equation

$$\Delta y = \frac{1}{2}a_y\Delta t^2 \quad ,$$

or

$$-h = \frac{1}{2}(-g)\Delta t^2 \quad .$$

Solving for  $\Delta t$  gives

$$\Delta t = \sqrt{\frac{2h}{g}} \quad .$$

Since the vertical force had no effect on the car's horizontal motion, it had  $a_x = 0$ , i.e., constant horizontal velocity. We can apply the constant-velocity equation

$$v_x = \frac{\Delta x}{\Delta t} \quad ,$$

i.e.,

$$v_x = \frac{w}{\Delta t} \quad .$$

We now substitute for  $\Delta t$  to find

$$v_x = w / \sqrt{\frac{2h}{g}} \quad ,$$

which simplifies to

$$v_x = w \sqrt{\frac{g}{2h}} \quad .$$

Plugging in numbers, we find that the car's speed when it went over the edge was 4 m/s, or about 10 mi/hr.

## Projectiles move along parabolas.

What type of mathematical curve does a projectile follow through space? To find out, we must relate  $x$  to  $y$ , eliminating  $t$ . The reasoning is very similar to that used in the example above. Arbitrarily choosing  $x = y = t = 0$  to be at the top of the arc, we conveniently have  $x = \Delta x$ ,  $y = \Delta y$ , and  $t = \Delta t$ , so

$$y = \frac{1}{2}a_y t^2 \quad (a_y < 0)$$
$$x = v_x t$$

We solve the second equation for  $t = x/v_x$  and eliminate  $t$  in the first equation:

$$y = \frac{1}{2}a_y \left( \frac{x}{v_x} \right)^2.$$

Since everything in this equation is a constant except for  $x$  and  $y$ , we conclude that  $y$  is proportional to the square of  $x$ . As you may or may not recall from a math class,  $y \propto x^2$  describes a parabola.

▷ *Solved problem: A cannon*

*page 188, problem 5*

### Discussion question

**A** At the beginning of this section I represented the motion of a projectile on graph paper, breaking its motion into equal time intervals. Suppose instead that there is no force on the object at all. It obeys Newton's first law and continues without changing its state of motion. What would the corresponding graph-paper diagram look like? If the time interval represented by each arrow was 1 second, how would you relate the graph-paper diagram to the velocity components  $v_x$  and  $v_y$ ?

**B** Make up several different coordinate systems oriented in different ways, and describe the  $a_x$  and  $a_y$  of a falling object in each one.

## 6.3 Newton's laws in three dimensions

It is now fairly straightforward to extend Newton's laws to three dimensions:

### Newton's first law

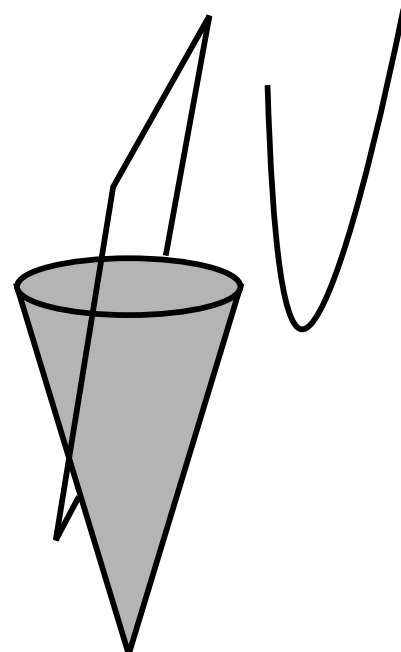
If all three components of the total force on an object are zero, then it will continue in the same state of motion.

### Newton's second law

The components of an object's acceleration are predicted by the equations

$$a_x = F_{x,\text{total}}/m \quad ,$$
$$a_y = F_{y,\text{total}}/m \quad , \quad \text{and}$$
$$a_z = F_{z,\text{total}}/m \quad .$$

### Newton's third law



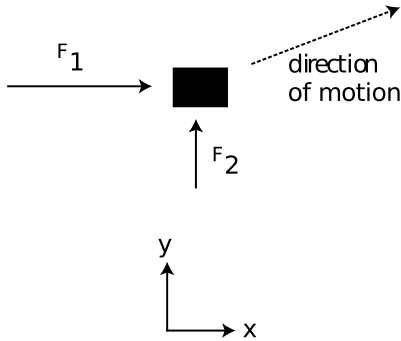
e / A parabola can be defined as the shape made by cutting a cone parallel to its side. A parabola is also the graph of an equation of the form  $y \propto x^2$ .



f / Each water droplet follows a parabola. The faster drops' parabolas are bigger.

If two objects A and B interact via forces, then the components of their forces on each other are equal and opposite:

$$\begin{aligned} F_{A \text{ on } B, x} &= -F_{B \text{ on } A, x} & , \\ F_{A \text{ on } B, y} &= -F_{B \text{ on } A, y} & , \quad \text{and} \\ F_{A \text{ on } B, z} &= -F_{B \text{ on } A, z} & . \end{aligned}$$



g / Example 2.

### Forces in perpendicular directions on the same object example 2

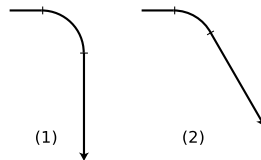
▷ An object is initially at rest. Two constant forces begin acting on it, and continue acting on it for a while. As suggested by the two arrows, the forces are perpendicular, and the rightward force is stronger. What happens?

▷ Aristotle believed, and many students still do, that only one force can “give orders” to an object at one time. They therefore think that the object will begin speeding up and moving in the direction of the stronger force. In fact the object will move along a diagonal. In the example shown in the figure, the object will respond to the large rightward force with a large acceleration component to the right, and the small upward force will give it a small acceleration component upward. The stronger force does not overwhelm the weaker force, or have any effect on the upward motion at all. The force components simply add together:

$$\begin{aligned} F_{x, \text{total}} &= F_{1, x} + F_{2, x} \\ F_{y, \text{total}} &= F_{1, y} + F_{2, y} \end{aligned}$$

### Discussion question

**A** The figure shows two trajectories, made by splicing together lines and circular arcs, which are unphysical for an object that is only being acted on by gravity. Prove that they are impossible based on Newton’s laws.



## Summary

### Selected vocabulary

component . . . .	the part of a velocity, acceleration, or force that would be perceptible to an observer who could only see the universe projected along a certain one-dimensional axis
parabola . . . . .	the mathematical curve whose graph has $y$ proportional to $x^2$

### Notation

$x, y, z$ . . . . .	an object's positions along the $x$ , $y$ , and $z$ axes
$v_x, v_y, v_z$ . . . . .	the $x$ , $y$ , and $z$ components of an object's velocity; the rates of change of the object's $x$ , $y$ , and $z$ coordinates
$a_x, a_y, a_z$ . . . . .	the $x$ , $y$ , and $z$ components of an object's acceleration; the rates of change of $v_x$ , $v_y$ , and $v_z$

## Summary

A force does not produce any effect on the motion of an object in a perpendicular direction. The most important application of this principle is that the horizontal motion of a projectile has zero acceleration, while the vertical motion has an acceleration equal to  $g$ . That is, an object's horizontal and vertical motions are independent. The arc of a projectile is a parabola.

Motion in three dimensions is measured using three coordinates,  $x$ ,  $y$ , and  $z$ . Each of these coordinates has its own corresponding velocity and acceleration. We say that the velocity and acceleration both have  $x$ ,  $y$ , and  $z$  components

Newton's second law is readily extended to three dimensions by rewriting it as three equations predicting the three components of the acceleration,

$$\begin{aligned}a_x &= F_{x,total}/m & , \\a_y &= F_{y,total}/m & , \\a_z &= F_{z,total}/m & ,\end{aligned}$$

and likewise for the first and third laws.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** (a) A ball is thrown straight up with velocity  $v$ . Find an equation for the height to which it rises. ✓  
 (b) Generalize your equation for a ball thrown at an angle  $\theta$  above horizontal, in which case its initial velocity components are  $v_x = v \cos \theta$  and  $v_y = v \sin \theta$ . ✓

**2** At the 2010 Salinas Lettuce Festival Parade, the Lettuce Queen drops her bouquet while riding on a float moving toward the right. Sketch the shape of its trajectory in her frame of reference, and compare with the shape seen by one of her admirers standing on the sidewalk.

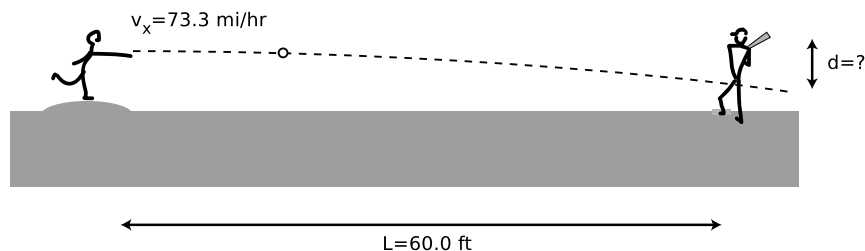
**3** Two daredevils, Wendy and Bill, go over Niagara Falls. Wendy sits in an inner tube, and lets the 30 km/hr velocity of the river throw her out horizontally over the falls. Bill paddles a kayak, adding an extra 10 km/hr to his velocity. They go over the edge of the falls at the same moment, side by side. Ignore air friction. Explain your reasoning.

- (a) Who hits the bottom first?
- (b) What is the horizontal component of Wendy's velocity on impact?
- (c) What is the horizontal component of Bill's velocity on impact?
- (d) Who is going faster on impact?

**4** A baseball pitcher throws a pitch clocked at  $v_x = 73.3$  mi/h. He throws horizontally. By what amount,  $d$ , does the ball drop by the time it reaches home plate,  $L = 60.0$  ft away?

- (a) First find a symbolic answer in terms of  $L$ ,  $v_x$ , and  $g$ . ✓
- (b) Plug in and find a numerical answer. Express your answer in units of ft. (Note: 1 ft = 12 in, 1 mi = 5280 ft, and 1 in = 2.54 cm) ✓

Problem 4.



**5** A cannon standing on a flat field fires a cannonball with a muzzle velocity  $v$ , at an angle  $\theta$  above horizontal. The cannonball

thus initially has velocity components  $v_x = v \cos \theta$  and  $v_y = v \sin \theta$ .

(a) Show that the cannon's range (horizontal distance to where the cannonball falls) is given by the equation  $R = (2v^2/g) \sin \theta \cos \theta$ .

(b) Interpret your equation in the cases of  $\theta = 0$  and  $\theta = 90^\circ$ .

▷ Solution, p. 522

**6** Assuming the result of problem 5 for the range of a projectile,  $R = (2v^2/g) \sin \theta \cos \theta$ , show that the maximum range is for  $\theta = 45^\circ$ .

∫

**7** Two cars go over the same speed bump in a parking lot, Maria's Maserati at 25 miles per hour and Park's Porsche at 37. How many times greater is the vertical acceleration of the Porsche? Hint: Remember that acceleration depends both on how much the velocity changes and on how much time it takes to change. ✓







a / Vectors are used in aerial navigation.

## Chapter 7

# Vectors

### 7.1 Vector notation

The idea of components freed us from the confines of one-dimensional physics, but the component notation can be unwieldy, since every one-dimensional equation has to be written as a set of three separate equations in the three-dimensional case. Newton was stuck with the component notation until the day he died, but eventually someone sufficiently lazy and clever figured out a way of abbreviating three equations as one.

(a)	$\vec{F}_{A \text{ on } B} = -\vec{F}_{B \text{ on } A}$	stands for	$F_{A \text{ on } B, x} = -F_{B \text{ on } A, x}$ $F_{A \text{ on } B, y} = -F_{B \text{ on } A, y}$ $F_{A \text{ on } B, z} = -F_{B \text{ on } A, z}$
(b)	$\vec{F}_{\text{total}} = \vec{F}_1 + \vec{F}_2 + \dots$	stands for	$F_{\text{total}, x} = F_{1, x} + F_{2, x} + \dots$ $F_{\text{total}, y} = F_{1, y} + F_{2, y} + \dots$ $F_{\text{total}, z} = F_{1, z} + F_{2, z} + \dots$
(c)	$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$	stands for	$a_x = \Delta v_x / \Delta t$ $a_y = \Delta v_y / \Delta t$ $a_z = \Delta v_z / \Delta t$

Example (a) shows both ways of writing Newton's third law. Which would you rather write?

The idea is that each of the algebra symbols with an arrow writ-

ten on top, called a vector, is actually an abbreviation for three different numbers, the  $x$ ,  $y$ , and  $z$  components. The three components are referred to as the components of the vector, e.g.,  $F_x$  is the  $x$  component of the vector  $\vec{F}$ . The notation with an arrow on top is good for handwritten equations, but is unattractive in a printed book, so books use boldface,  $\mathbf{F}$ , to represent vectors. After this point, I'll use boldface for vectors throughout this book.

In general, the vector notation is useful for any quantity that has both an amount and a direction in space. Even when you are not going to write any actual vector notation, the concept itself is a useful one. We say that force and velocity, for example, are vectors. A quantity that has no direction in space, such as mass or time, is called a scalar. The amount of a vector quantity is called its magnitude. The notation for the magnitude of a vector  $\mathbf{A}$  is  $|\mathbf{A}|$ , like the absolute value sign used with scalars.

Often, as in example (b), we wish to use the vector notation to represent adding up all the  $x$  components to get a total  $x$  component, etc. The plus sign is used between two vectors to indicate this type of component-by-component addition. Of course, vectors are really triplets of numbers, not numbers, so this is not the same as the use of the plus sign with individual numbers. But since we don't want to have to invent new words and symbols for this operation on vectors, we use the same old plus sign, and the same old addition-related words like "add," "sum," and "total." Combining vectors this way is called vector addition.

Similarly, the minus sign in example (a) was used to indicate negating each of the vector's three components individually. The equals sign is used to mean that all three components of the vector on the left side of an equation are the same as the corresponding components on the right.

Example (c) shows how we abuse the division symbol in a similar manner. When we write the vector  $\Delta\mathbf{v}$  divided by the scalar  $\Delta t$ , we mean the new vector formed by dividing each one of the velocity components by  $\Delta t$ .

It's not hard to imagine a variety of operations that would combine vectors with vectors or vectors with scalars, but only four of them are required in order to express Newton's laws:

operation	definition
<b>vector</b> + <b>vector</b>	Add component by component to make a new set of three numbers.
<b>vector</b> – <b>vector</b>	Subtract component by component to make a new set of three numbers.
<b>vector</b> · scalar	Multiply each component of the vector by the scalar.
<b>vector</b> /scalar	Divide each component of the vector by the scalar.

As an example of an operation that is not useful for physics, there just aren't any useful physics applications for dividing a vector by another vector component by component. In optional section 7.5, we discuss in more detail the fundamental reasons why some vector operations are useful and others useless.

We can do algebra with vectors, or with a mixture of vectors and scalars in the same equation. Basically all the normal rules of algebra apply, but if you're not sure if a certain step is valid, you should simply translate it into three component-based equations and see if it works.

#### Order of addition

#### example 1

▷ If we are adding two force vectors,  $\mathbf{F} + \mathbf{G}$ , is it valid to assume as in ordinary algebra that  $\mathbf{F} + \mathbf{G}$  is the same as  $\mathbf{G} + \mathbf{F}$ ?

▷ To tell if this algebra rule also applies to vectors, we simply translate the vector notation into ordinary algebra notation. In terms of ordinary numbers, the components of the vector  $\mathbf{F} + \mathbf{G}$  would be  $F_x + G_x$ ,  $F_y + G_y$ , and  $F_z + G_z$ , which are certainly the same three numbers as  $G_x + F_x$ ,  $G_y + F_y$ , and  $G_z + F_z$ . Yes,  $\mathbf{F} + \mathbf{G}$  is the same as  $\mathbf{G} + \mathbf{F}$ .

It is useful to define a symbol  $\mathbf{r}$  for the vector whose components are  $x$ ,  $y$ , and  $z$ , and a symbol  $\Delta\mathbf{r}$  made out of  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ .

Although this may all seem a little formidable, keep in mind that it amounts to nothing more than a way of abbreviating equations! Also, to keep things from getting too confusing the remainder of this chapter focuses mainly on the  $\Delta\mathbf{r}$  vector, which is relatively easy to visualize.

#### self-check A

Translate the equations  $v_x = \Delta x / \Delta t$ ,  $v_y = \Delta y / \Delta t$ , and  $v_z = \Delta z / \Delta t$  for motion with constant velocity into a single equation in vector notation.

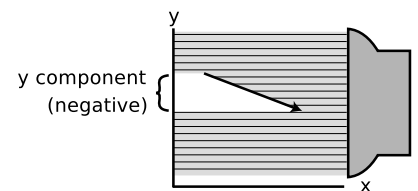
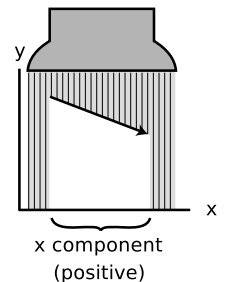
▷ Answer, p. 532

### Drawing vectors as arrows

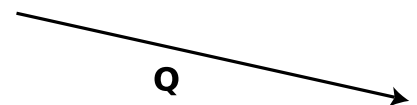
A vector in two dimensions can be easily visualized by drawing an arrow whose length represents its magnitude and whose direction represents its direction. The  $x$  component of a vector can then be visualized as the length of the shadow it would cast in a beam of light projected onto the  $x$  axis, and similarly for the  $y$  component. Shadows with arrowheads pointing back against the direction of the positive axis correspond to negative components.

In this type of diagram, the negative of a vector is the vector with the same magnitude but in the opposite direction. Multiplying a vector by a scalar is represented by lengthening the arrow by that factor, and similarly for division.

#### self-check B



b / The  $x$  and  $y$  components of a vector can be thought of as the shadows it casts onto the  $x$  and  $y$  axes.



c / Self-check B.

Given vector  $\mathbf{Q}$  represented by an arrow in figure c, draw arrows representing the vectors  $1.5\mathbf{Q}$  and  $-\mathbf{Q}$ .

▷ Answer, p.

532

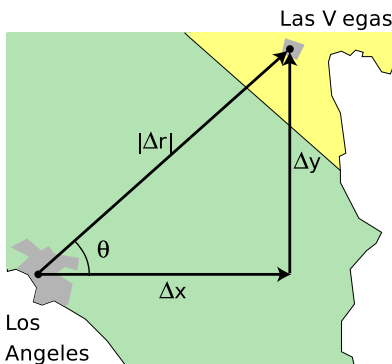
### Discussion questions

**A** Would it make sense to define a zero vector? Discuss what the zero vector's components, magnitude, and direction would be; are there any issues here? If you wanted to disqualify such a thing from being a vector, consider whether the system of vectors would be complete. For comparison, can you think of a simple arithmetic problem with ordinary numbers where you need zero as the result? Does the same reasoning apply to vectors, or not? From your group, choose one person to act as zero's advocate and one to argue against letting zero in the club. The rest of the group should act as jurors.

**B** You drive to your friend's house. How does the magnitude of your  $\Delta\mathbf{r}$  vector compare with the distance you've added to the car's odometer?

## 7.2 Calculations with magnitude and direction

If you ask someone where Las Vegas is compared to Los Angeles, they are unlikely to say that the  $\Delta x$  is 290 km and the  $\Delta y$  is 230 km, in a coordinate system where the positive  $x$  axis is east and the  $y$  axis points north. They will probably say instead that it's 370 km to the northeast. If they were being precise, they might specify the direction as  $38^\circ$  counterclockwise from east. In two dimensions, we can always specify a vector's direction like this, using a single angle. A magnitude plus an angle suffice to specify everything about the vector. The following two examples show how we use trigonometry and the Pythagorean theorem to go back and forth between the  $x$ - $y$  and magnitude-angle descriptions of vectors.



d / Example 2.

### Finding magnitude and angle from components example 2

▷ Given that the  $\Delta\mathbf{r}$  vector from LA to Las Vegas has  $\Delta x = 290$  km and  $\Delta y = 230$  km, how would we find the magnitude and direction of  $\Delta\mathbf{r}$ ?

▷ We find the magnitude of  $\Delta\mathbf{r}$  from the Pythagorean theorem:

$$|\Delta\mathbf{r}| = \sqrt{\Delta x^2 + \Delta y^2} \\ = 370 \text{ km}$$

We know all three sides of the triangle, so the angle  $\theta$  can be found using any of the inverse trig functions. For example, we know the opposite and adjacent sides, so

$$\theta = \tan^{-1} \frac{\Delta y}{\Delta x} \\ = 38^\circ$$

*Finding components from magnitude and angle* *example 3*

▷ Given that the straight-line distance from Los Angeles to Las Vegas is 370 km, and that the angle  $\theta$  in the figure is  $38^\circ$ , how can the  $x$  and  $y$  components of the  $\Delta \mathbf{r}$  vector be found?

▷ The sine and cosine of  $\theta$  relate the given information to the information we wish to find:

$$\cos \theta = \frac{\Delta x}{|\Delta \mathbf{r}|}$$

$$\sin \theta = \frac{\Delta y}{|\Delta \mathbf{r}|}$$

Solving for the unknowns gives

$$\begin{aligned}\Delta x &= |\Delta \mathbf{r}| \cos \theta \\ &= 290 \text{ km} \quad \text{and} \\ \Delta y &= |\Delta \mathbf{r}| \sin \theta \\ &= 230 \text{ km} \quad .\end{aligned}$$

The following example shows the correct handling of the plus and minus signs, which is usually the main cause of mistakes.

*Negative components* *example 4*

▷ San Diego is 120 km east and 150 km south of Los Angeles. An airplane pilot is setting course from San Diego to Los Angeles. At what angle should she set her course, measured counterclockwise from east, as shown in the figure?

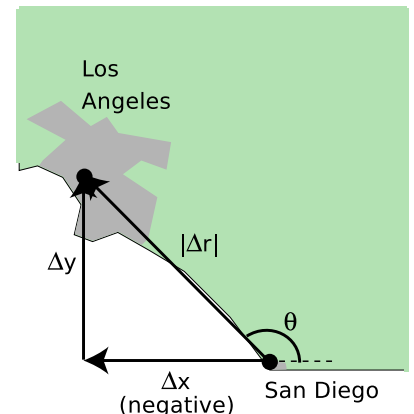
▷ If we make the traditional choice of coordinate axes, with  $x$  pointing to the right and  $y$  pointing up on the map, then her  $\Delta x$  is negative, because her final  $x$  value is less than her initial  $x$  value. Her  $\Delta y$  is positive, so we have

$$\begin{aligned}\Delta x &= -120 \text{ km} \\ \Delta y &= 150 \text{ km} \quad .\end{aligned}$$

If we work by analogy with example 2, we get

$$\begin{aligned}\theta &= \tan^{-1} \frac{\Delta y}{\Delta x} \\ &= \tan^{-1}(-1.25) \\ &= -51^\circ \quad .\end{aligned}$$

According to the usual way of defining angles in trigonometry, a negative result means an angle that lies clockwise from the  $x$  axis, which would have her heading for the Baja California. What went wrong? The answer is that when you ask your calculator to



e / Example 4.

take the arctangent of a number, there are always two valid possibilities differing by  $180^\circ$ . That is, there are two possible angles whose tangents equal  $-1.25$ :

$$\tan 129^\circ = -1.25$$

$$\tan -51^\circ = -1.25$$

Your calculator doesn't know which is the correct one, so it just picks one. In this case, the one it picked was the wrong one, and it was up to you to add  $180^\circ$  to it to find the right answer.

### Discussion question

**A** In example 4, we dealt with *components* that were negative. Does it make sense to classify *vectors* as positive and negative?

## 7.3 Techniques for adding vectors

Vector addition is one of the three essential mathematical skills, summarized on pp.514-515, that you need for success in this course.

### Addition of vectors given their components

The easiest type of vector addition is when you are in possession of the components, and want to find the components of their sum.

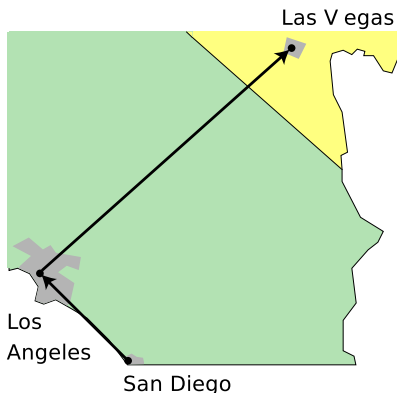
#### Adding components

example 5

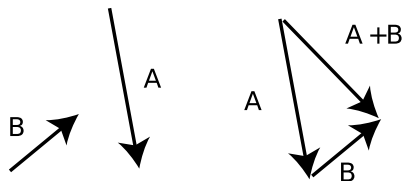
▷ Given the  $\Delta x$  and  $\Delta y$  values from the previous examples, find the  $\Delta x$  and  $\Delta y$  from San Diego to Las Vegas.

▷

$$\begin{aligned}\Delta x_{total} &= \Delta x_1 + \Delta x_2 \\ &= -120 \text{ km} + 290 \text{ km} \\ &= 170 \text{ km} \\ \Delta y_{total} &= \Delta y_1 + \Delta y_2 \\ &= 150 \text{ km} + 230 \text{ km} \\ &= 380\end{aligned}$$



f / Example 5.



g / Vectors can be added graphically by placing them tip to tail, and then drawing a vector from the tail of the first vector to the tip of the second vector.

Note how the signs of the  $x$  components take care of the westward and eastward motions, which partially cancel.

### Addition of vectors given their magnitudes and directions

In this case, you must first translate the magnitudes and directions into components, and then add the components. In our San Diego-Los Angeles-Las Vegas example, we can simply string together the preceding examples; this is done on p. 515.

### Graphical addition of vectors

Often the easiest way to add vectors is by making a scale drawing on a piece of paper. This is known as graphical addition, as opposed

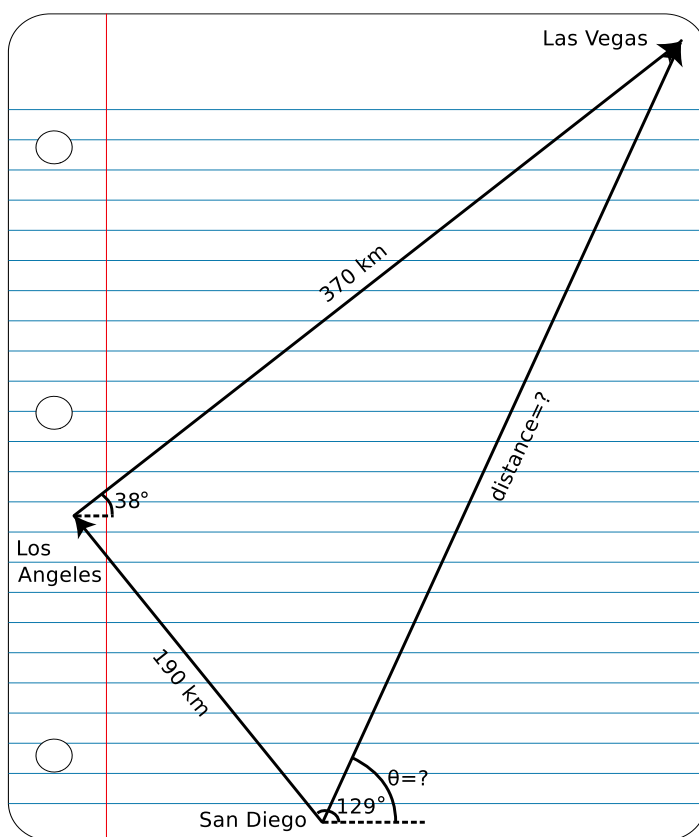
to the analytic techniques discussed previously. (It has nothing to do with  $x - y$  graphs or graph paper. “Graphical” here simply means drawing. It comes from the Greek verb “*grapho*,” to write, like related English words including “graphic.”)

**LA to Vegas, graphically**

**example 6**

▷ Given the magnitudes and angles of the  $\Delta \mathbf{r}$  vectors from San Diego to Los Angeles and from Los Angeles to Las Vegas, find the magnitude and angle of the  $\Delta \mathbf{r}$  vector from San Diego to Las Vegas.

▷ Using a protractor and a ruler, we make a careful scale drawing, as shown in figure h. The protractor can be conveniently aligned with the blue rules on the notebook paper. A scale of  $1 \text{ mm} \rightarrow 2 \text{ km}$  was chosen for this solution because it was as big as possible (for accuracy) without being so big that the drawing wouldn’t fit on the page. With a ruler, we measure the distance from San Diego to Las Vegas to be 206 mm, which corresponds to 412 km. With a protractor, we measure the angle  $\theta$  to be  $65^\circ$ .



h / Example 6.

Even when we don’t intend to do an actual graphical calculation with a ruler and protractor, it can be convenient to diagram the addition of vectors in this way. With  $\Delta \mathbf{r}$  vectors, it intuitively makes sense to lay the vectors tip-to-tail and draw the sum vector from the



tail of the first vector to the tip of the second vector. We can do the same when adding other vectors such as force vectors.

*self-check C*

How would you subtract vectors graphically?

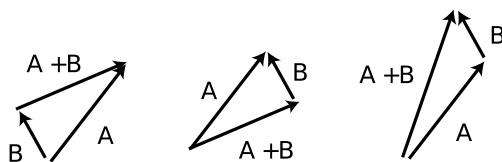
▷ Answer, p. 532

### Discussion questions

**A** If you're doing *graphical* addition of vectors, does it matter which vector you start with and which vector you start from the other vector's tip?

**B** If you add a vector with magnitude 1 to a vector of magnitude 2, what magnitudes are possible for the vector sum?

**C** Which of these examples of vector addition are correct, and which are incorrect?



## 7.4 ★ Unit vector notation

When we want to specify a vector by its components, it can be cumbersome to have to write the algebra symbol for each component:

$$\Delta x = 290 \text{ km}, \Delta y = 230 \text{ km}$$

A more compact notation is to write

$$\Delta \mathbf{r} = (290 \text{ km})\hat{\mathbf{x}} + (230 \text{ km})\hat{\mathbf{y}},$$

where the vectors  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$ , called the unit vectors, are defined as the vectors that have magnitude equal to 1 and directions lying along the  $x$ ,  $y$ , and  $z$  axes. In speech, they are referred to as “x-hat” and so on.

A slightly different, and harder to remember, version of this notation is unfortunately more prevalent. In this version, the unit vectors are called  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , and  $\hat{\mathbf{k}}$ :

$$\Delta \mathbf{r} = (290 \text{ km})\hat{\mathbf{i}} + (230 \text{ km})\hat{\mathbf{j}}.$$

## 7.5 ★ Rotational invariance

Let's take a closer look at why certain vector operations are useful and others are not. Consider the operation of multiplying two vectors component by component to produce a third vector:

$$R_x = P_x Q_x$$

$$R_y = P_y Q_y$$

$$R_z = P_z Q_z$$

As a simple example, we choose vectors **P** and **Q** to have length 1, and make them perpendicular to each other, as shown in figure i/1. If we compute the result of our new vector operation using the coordinate system in i/2, we find:

$$R_x = 0$$

$$R_y = 0$$

$$R_z = 0$$

The  $x$  component is zero because  $P_x = 0$ , the  $y$  component is zero because  $Q_y = 0$ , and the  $z$  component is of course zero because both vectors are in the  $x - y$  plane. However, if we carry out the same operations in coordinate system i/3, rotated 45 degrees with respect to the previous one, we find

$$R_x = 1/2$$

$$R_y = -1/2$$

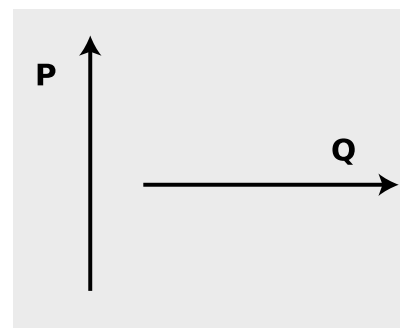
$$R_z = 0$$

The operation's result depends on what coordinate system we use, and since the two versions of **R** have different lengths (one being zero and the other nonzero), they don't just represent the same answer expressed in two different coordinate systems. Such an operation will never be useful in physics, because experiments show physics works the same regardless of which way we orient the laboratory building! The *useful* vector operations, such as addition and scalar multiplication, are rotationally invariant, i.e., come out the same regardless of the orientation of the coordinate system.

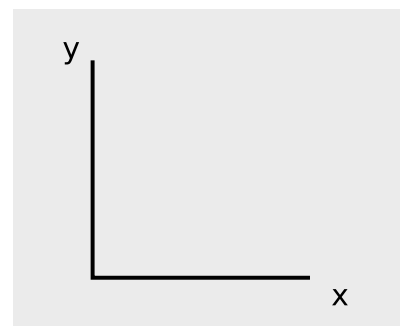
### Calibrating an electronic compass

### example 7

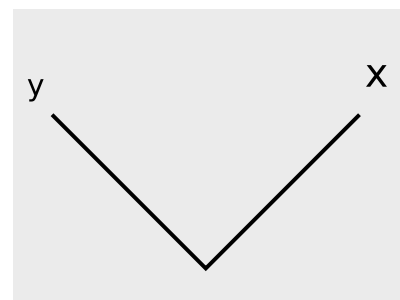
Some smart phones and GPS units contain electronic compasses that can sense the direction of the earth's magnetic field vector, notated **B**. Because all vectors work according to the same rules, you don't need to know anything special about magnetism in order to understand this example. Unlike a traditional compass that uses a magnetized needle on a bearing, an electronic compass has no moving parts. It contains two sensors oriented perpendicular to one another, and each sensor is only sensitive to the component of the earth's field that lies along its own axis. Because a



1



2



3

i / Component-by-component multiplication of the vectors in 1 would produce different vectors in coordinate systems 2 and 3.

choice of coordinates is arbitrary, we can take one of these sensors as defining the  $x$  axis and the other the  $y$ . Given the two components  $B_x$  and  $B_y$ , the device's computer chip can compute the angle of magnetic north relative to its sensors,  $\tan^{-1}(B_y/B_x)$ .

All compasses are vulnerable to errors because of nearby magnetic materials, and in particular it may happen that some part of the compass's own housing becomes magnetized. In an electronic compass, rotational invariance provides a convenient way of calibrating away such effects by having the user rotate the device in a horizontal circle.

Suppose that when the compass is oriented in a certain way, it measures  $B_x = 1.00$  and  $B_y = 0.00$  (in certain units). We then expect that when it is rotated 90 degrees clockwise, the sensors will detect  $B_x = 0.00$  and  $B_y = 1.00$ .

But imagine instead that we get  $B_x = 0.20$  and  $B_y = 0.80$ . This would violate rotational invariance, since rotating the coordinate system is supposed to give a different description of the *same* vector. The magnitude appears to have changed from 1.00 to  $\sqrt{0.20^2 + 0.80^2} = 0.82$ , and a vector can't change its magnitude just because you rotate it. The compass's computer chip figures out that some effect, possibly a slight magnetization of its housing, must be adding an erroneous 0.2 units to all the  $B_x$  readings, because subtracting this amount from all the  $B_x$  values gives vectors that have the same magnitude, satisfying rotational invariance.

## Summary

### Selected vocabulary

vector . . . . .	a quantity that has both an amount (magnitude) and a direction in space
magnitude . . . .	the “amount” associated with a vector
scalar . . . . .	a quantity that has no direction in space, only an amount

### Notation

$\mathbf{A}$ . . . . .	a vector with components $A_x$ , $A_y$ , and $A_z$
$\vec{A}$ . . . . .	handwritten notation for a vector
$ \mathbf{A} $ . . . . .	the magnitude of vector $\mathbf{A}$
$\mathbf{r}$ . . . . .	the vector whose components are $x$ , $y$ , and $z$
$\Delta\mathbf{r}$ . . . . .	the vector whose components are $\Delta x$ , $\Delta y$ , and $\Delta z$
$\hat{x}$ , $\hat{y}$ , $\hat{z}$ . . . . .	(optional topic) unit vectors; the vectors with magnitude 1 lying along the $x$ , $y$ , and $z$ axes
$\hat{i}$ , $\hat{j}$ , $\hat{k}$ . . . . .	a harder to remember notation for the unit vectors

### Other terminology and notation

displacement vector . . . . .	a name for the symbol $\Delta\mathbf{r}$
speed . . . . .	the magnitude of the velocity vector, i.e., the velocity stripped of any information about its direction

## Summary

A vector is a quantity that has both a magnitude (amount) and a direction in space, as opposed to a scalar, which has no direction. The vector notation amounts simply to an abbreviation for writing the vector’s three components.

In two dimensions, a vector can be represented either by its two components or by its magnitude and direction. The two ways of describing a vector can be related by trigonometry.

The two main operations on vectors are addition of a vector to a vector, and multiplication of a vector by a scalar.

Vector addition means adding the components of two vectors to form the components of a new vector. In graphical terms, this corresponds to drawing the vectors as two arrows laid tip-to-tail and drawing the sum vector from the tail of the first vector to the tip of the second one. Vector subtraction is performed by negating the vector to be subtracted and then adding.

Multiplying a vector by a scalar means multiplying each of its components by the scalar to create a new vector. Division by a scalar is defined similarly.

## Problems

### Key

✓ A computerized answer check is available online.

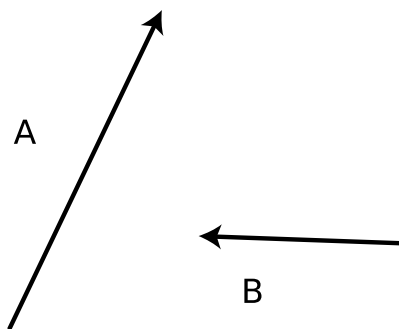
∫ A problem that requires calculus.

★ A difficult problem.

**1** The figure shows vectors **A** and **B**. As in figure g on p. 196, graphically calculate the following:

**A + B, A - B, B - A, -2B, A - 2B**

No numbers are involved.



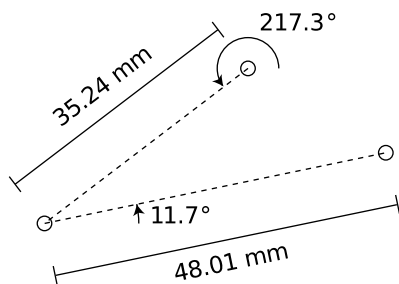
Problem 1.

**2** Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.

(a) Choose a coordinate system, and translate these data into  $\Delta x$  and  $\Delta y$  values with the proper plus and minus signs.

(b) Find the components of the  $\Delta \mathbf{r}$  vector pointing from Bangkok to Hanoi. ✓

**3** If you walk 35 km at an angle  $25^\circ$  counterclockwise from east, and then 22 km at  $230^\circ$  counterclockwise from east, find the distance and direction from your starting point to your destination. ✓



Problem 4.

**4** A machinist is drilling holes in a piece of aluminum according to the plan shown in the figure. She starts with the top hole, then moves to the one on the left, and then to the one on the right. Since this is a high-precision job, she finishes by moving in the direction and at the angle that should take her back to the top hole, and checks that she ends up in the same place. What are the distance and direction from the right-hand hole to the top one? ✓

**5** Suppose someone proposes a new operation in which a vector **A** and a scalar  $B$  are added together to make a new vector **C** like this:

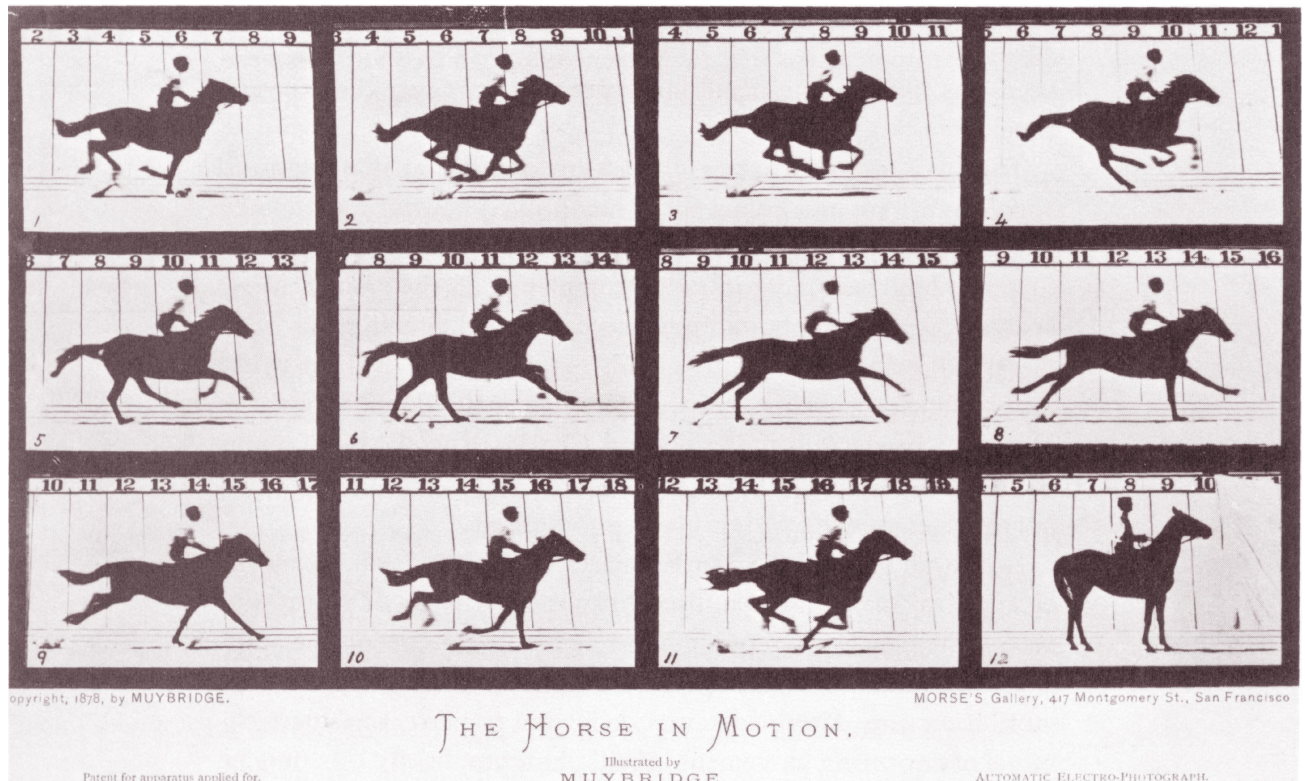
$$C_x = A_x + B$$

$$C_y = A_y + B$$

$$C_z = A_z + B$$

Prove that this operation won't be useful in physics, because it's not rotationally invariant.





## Chapter 8

# Vectors and motion

In 1872, capitalist and former California governor Leland Stanford asked photographer Eadweard Muybridge if he would work for him on a project to settle a \$25,000 bet (a princely sum at that time). Stanford's friends were convinced that a galloping horse always had at least one foot on the ground, but Stanford claimed that there was a moment during each cycle of the motion when all four feet were in the air. The human eye was simply not fast enough to settle the question. In 1878, Muybridge finally succeeded in producing what amounted to a motion picture of the horse, showing conclusively that all four feet did leave the ground at one point. (Muybridge was a colorful figure in San Francisco history, and his acquittal for the murder of his wife's lover was considered the trial of the century in California.)

The losers of the bet had probably been influenced by Aristotelian reasoning, for instance the expectation that a leaping horse would lose horizontal velocity while in the air with no force to push it forward, so that it would be more efficient for the horse to run without leaping. But even for students who have converted whole-



heartedly to Newtonianism, the relationship between force and acceleration leads to some conceptual difficulties, the main one being a problem with the true but seemingly absurd statement that an object can have an acceleration vector whose direction is not the same as the direction of motion. The horse, for instance, has nearly constant horizontal velocity, so its  $a_x$  is zero. But as anyone can tell you who has ridden a galloping horse, the horse accelerates up and down. The horse's acceleration vector therefore changes back and forth between the up and down directions, but is never in the same direction as the horse's motion. In this chapter, we will examine more carefully the properties of the velocity, acceleration, and force vectors. No new principles are introduced, but an attempt is made to tie things together and show examples of the power of the vector formulation of Newton's laws.

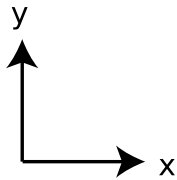
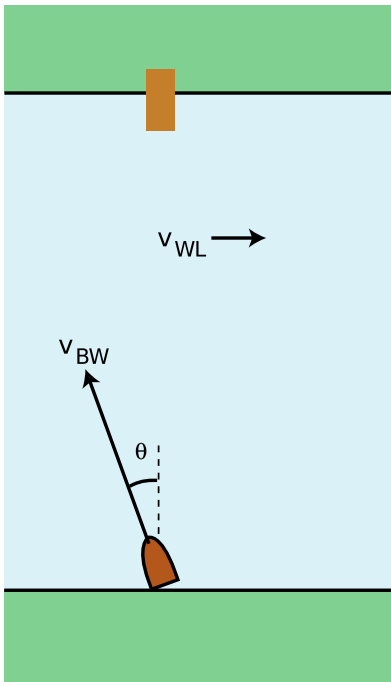
## 8.1 The velocity vector

For motion with constant velocity, the velocity vector is

$$\mathbf{v} = \Delta \mathbf{r} / \Delta t \quad . \quad [\text{only for constant velocity}]$$

The  $\Delta \mathbf{r}$  vector points in the direction of the motion, and dividing it by the scalar  $\Delta t$  only changes its length, not its direction, so the velocity vector points in the same direction as the motion. When the velocity is not constant, i.e., when the  $x-t$ ,  $y-t$ , and  $z-t$  graphs are not all linear, we use the slope-of-the-tangent-line approach to define the components  $v_x$ ,  $v_y$ , and  $v_z$ , from which we assemble the velocity vector. Even when the velocity vector is not constant, it still points along the direction of motion.

Vector addition is the correct way to generalize the one-dimensional concept of adding velocities in relative motion, as shown in the following example:



a / Example 1.

### *Velocity vectors in relative motion*

### *example 1*

▷ You wish to cross a river and arrive at a dock that is directly across from you, but the river's current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle  $\theta$ , given the magnitude,  $|\mathbf{v}_{WL}|$ , of the water's velocity relative to the land, and the maximum speed,  $|\mathbf{v}_{BW}|$ , of which the boat is capable relative to the water.

▷ The boat's velocity relative to the land equals the vector sum of its velocity with respect to the water and the water's velocity with respect to the land,

$$\mathbf{v}_{BL} = \mathbf{v}_{BW} + \mathbf{v}_{WL} \quad .$$

If the boat is to travel straight across the river, i.e., along the  $y$  axis, then we need to have  $\mathbf{v}_{BL,x} = 0$ . This  $x$  component equals the sum of the  $x$  components of the other two vectors,

$$\mathbf{v}_{BL,x} = \mathbf{v}_{BW,x} + \mathbf{v}_{WL,x} \quad ,$$

or

$$0 = -|\mathbf{v}_{BW}| \sin \theta + |\mathbf{v}_{WL}| \quad .$$

Solving for  $\theta$ , we find

$$\sin \theta = |\mathbf{v}_{WL}| / |\mathbf{v}_{BW}| \quad ,$$

so

$$\theta = \sin^{-1} \frac{|\mathbf{v}_{WL}|}{|\mathbf{v}_{BW}|} \quad .$$

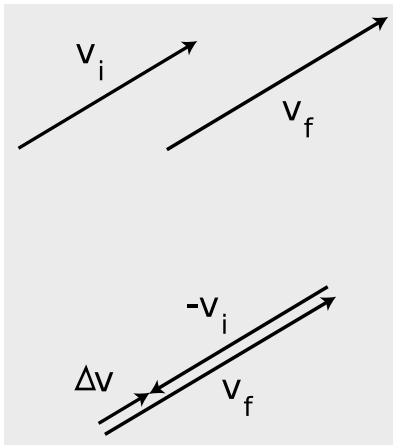
▷ *Solved problem: Annie Oakley*

*page 218, problem 8*

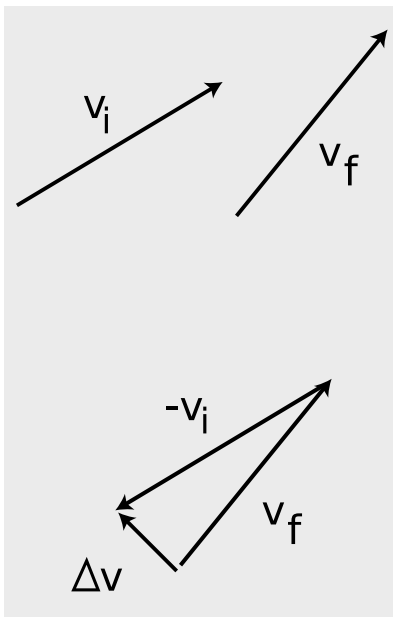
### Discussion questions

**A** Is it possible for an airplane to maintain a constant velocity vector but not a constant  $|\mathbf{v}|$ ? How about the opposite – a constant  $|\mathbf{v}|$  but not a constant velocity vector? Explain.

**B** New York and Rome are at about the same latitude, so the earth's rotation carries them both around nearly the same circle. Do the two cities have the same velocity vector (relative to the center of the earth)? If not, is there any way for two cities to have the same velocity vector?



b / A change in the magnitude of the velocity vector implies an acceleration.



c / A change in the direction of the velocity vector also produces a nonzero  $\Delta \mathbf{v}$  vector, and thus a nonzero acceleration vector,  $\Delta \mathbf{v} / \Delta t$ .

## 8.2 The acceleration vector

When all three acceleration components are constant, i.e., when the  $v_x - t$ ,  $v_y - t$ , and  $v_z - t$  graphs are all linear, we can define the acceleration vector as

$$\mathbf{a} = \Delta \mathbf{v} / \Delta t \quad , \quad [\text{only for constant acceleration}]$$

which can be written in terms of initial and final velocities as

$$\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i) / \Delta t \quad . \quad [\text{only for constant acceleration}]$$

If the acceleration is not constant, we define it as the vector made out of the  $a_x$ ,  $a_y$ , and  $a_z$  components found by applying the slope-of-the-tangent-line technique to the  $v_x - t$ ,  $v_y - t$ , and  $v_z - t$  graphs.

Now there are two ways in which we could have a nonzero acceleration. Either the magnitude or the direction of the velocity vector could change. This can be visualized with arrow diagrams as shown in figures b and c. Both the magnitude and direction can change simultaneously, as when a car accelerates while turning. Only when the magnitude of the velocity changes while its direction stays constant do we have a  $\Delta \mathbf{v}$  vector and an acceleration vector along the same line as the motion.

### self-check A

(1) In figure b, is the object speeding up, or slowing down? (2) What would the diagram look like if  $\mathbf{v}_i$  was the same as  $\mathbf{v}_f$ ? (3) Describe how the  $\Delta \mathbf{v}$  vector is different depending on whether an object is speeding up or slowing down. ▷ Answer, p. 532

If this all seems a little strange and abstract to you, you're not alone. It doesn't mean much to most physics students the first time someone tells them that acceleration is a vector, and that the acceleration vector does not have to be in the same direction as the velocity vector. One way to understand those statements better is to imagine an object such as an air freshener or a pair of fuzzy dice hanging from the rear-view mirror of a car. Such a hanging object, called a bob, constitutes an accelerometer. If you watch the bob as you accelerate from a stop light, you'll see it swing backward. The horizontal direction in which the bob tilts is opposite to the direction of the acceleration. If you apply the brakes and the car's acceleration vector points backward, the bob tilts forward.

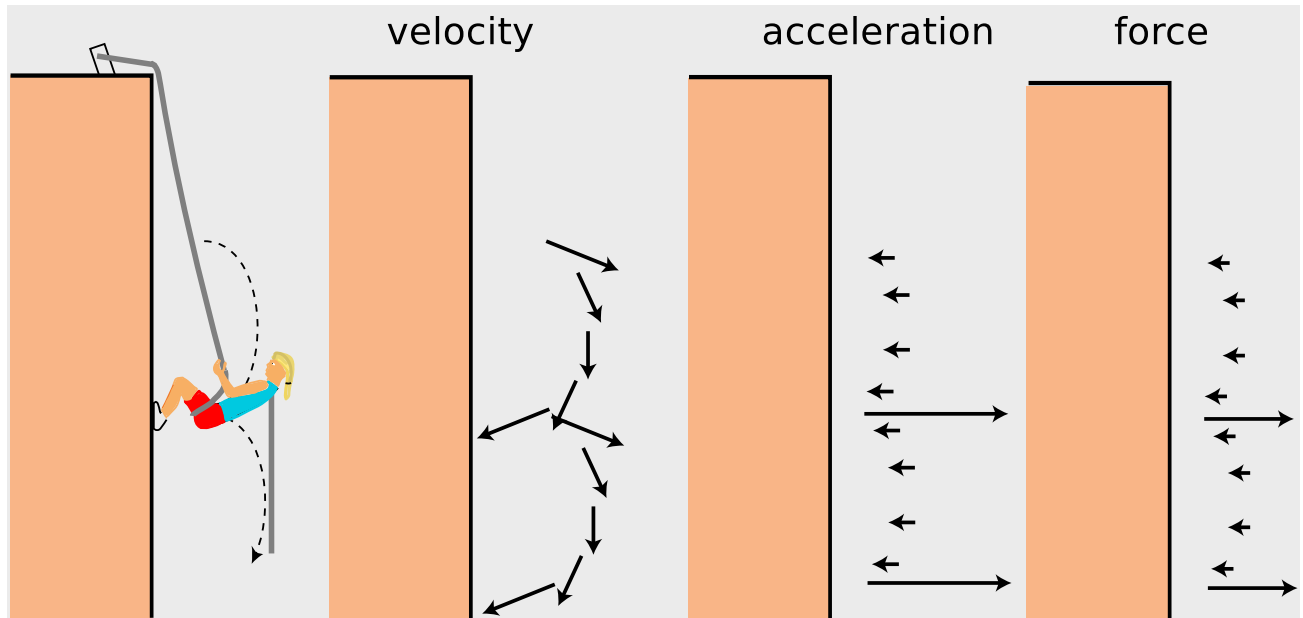
After accelerating and slowing down a few times, you think you've put your accelerometer through its paces, but then you make a right turn. Surprise! Acceleration is a vector, and needn't point in the same direction as the velocity vector. As you make a right turn, the bob swings outward, to your left. That means the car's acceleration vector is to your right, perpendicular to your velocity vector. A useful definition of an acceleration vector should relate in a systematic way to the actual physical effects produced by the

acceleration, so a physically reasonable definition of the acceleration vector must allow for cases where it is not in the same direction as the motion.

*self-check B*

In projectile motion, what direction does the acceleration vector have?

▷ Answer, p. 532



d / Example 2.

*Rappelling*

*example 2*

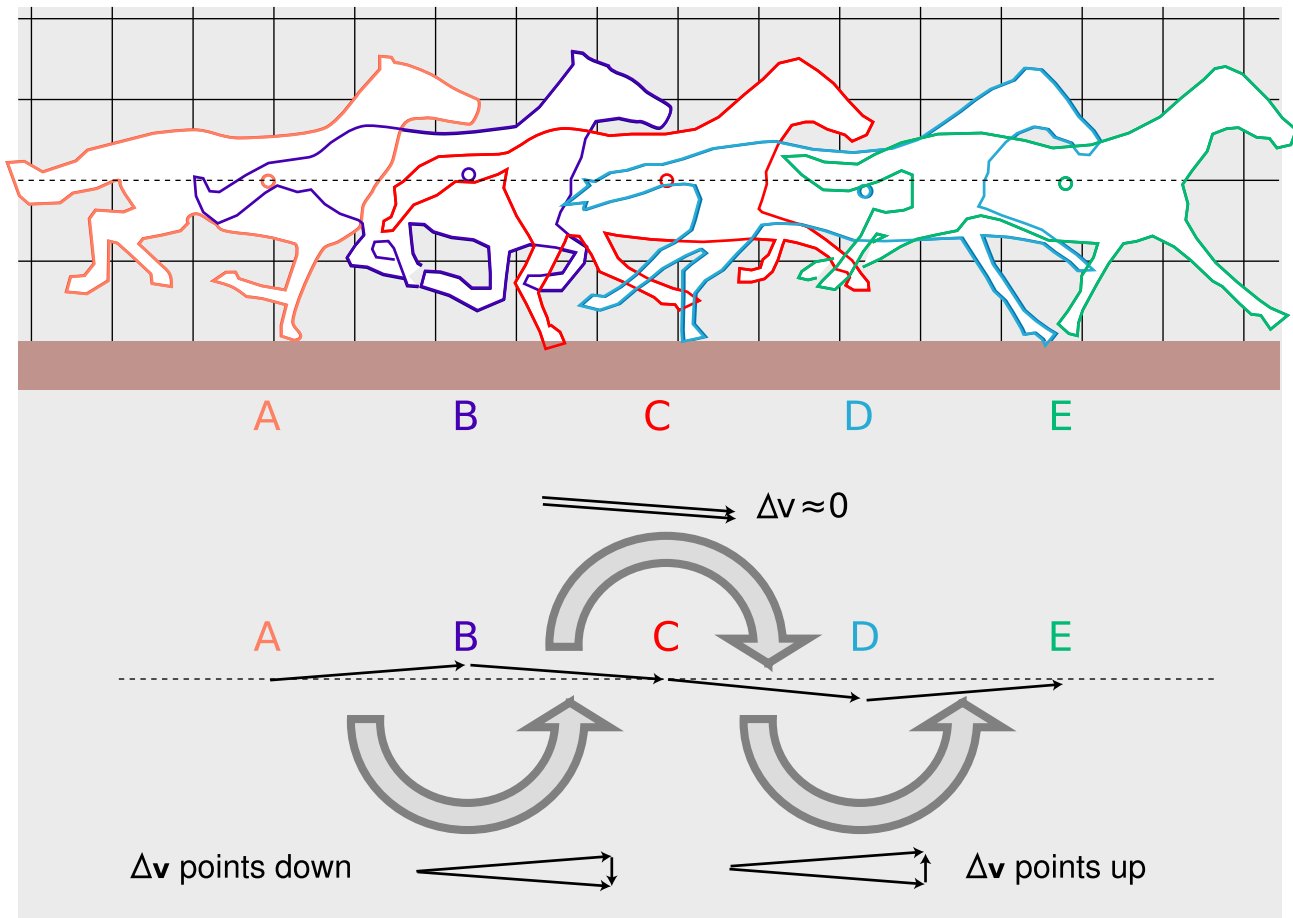
In figure d, the rappeller's velocity has long periods of gradual change interspersed with short periods of rapid change. These correspond to periods of small acceleration and force, and periods of large acceleration and force.

*The galloping horse*

*example 3*

Figure e on page 210 shows outlines traced from the first, third, fifth, seventh, and ninth frames in Muybridge's series of photographs of the galloping horse. The estimated location of the horse's center of mass is shown with a circle, which bobs above and below the horizontal dashed line.

If we don't care about calculating velocities and accelerations in any particular system of units, then we can pretend that the time between frames is one unit. The horse's velocity vector as it moves from one point to the next can then be found simply by drawing an arrow to connect one position of the center of mass to the next. This produces a series of velocity vectors which alter-



e / Example 3.

nate between pointing above and below horizontal.

The  $\Delta \mathbf{v}$  vector is the vector which we would have to add onto one velocity vector in order to get the next velocity vector in the series. The  $\Delta \mathbf{v}$  vector alternates between pointing down (around the time when the horse is in the air, B) and up (around the time when the horse has two feet on the ground, D).

### Discussion questions

**A** When a car accelerates, why does a bob hanging from the rearview mirror swing toward the back of the car? Is it because a force throws it backward? If so, what force? Similarly, describe what happens in the other cases described above.

**B** Superman is guiding a crippled spaceship into port. The ship's engines are not working. If Superman suddenly changes the direction of his force on the ship, does the ship's velocity vector change suddenly? Its acceleration vector? Its direction of motion?

## 8.3 The force vector and simple machines

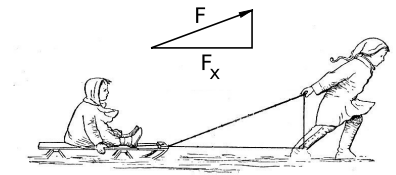
Force is relatively easy to intuit as a vector. The force vector points in the direction in which it is trying to accelerate the object it is acting on.

Since force vectors are so much easier to visualize than acceleration vectors, it is often helpful to first find the direction of the (total) force vector acting on an object, and then use that information to determine the direction of the acceleration vector. Newton's second law,  $\mathbf{F}_{total} = m\mathbf{a}$ , tells us that the two must be in the same direction.

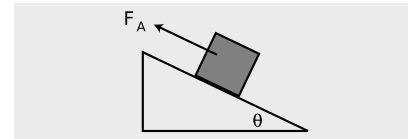
### A component of a force vector

### example 4

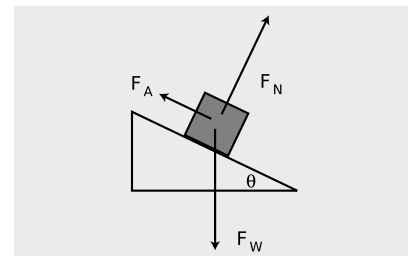
Figure f, redrawn from a classic 1920 textbook, shows a boy pulling another child on a sled. His force has both a horizontal component and a vertical one, but only the horizontal one accelerates the sled. (The vertical component just partially cancels the force of gravity, causing a decrease in the normal force between the runners and the snow.) There are two triangles in the figure. One triangle's hypotenuse is the rope, and the other's is the magnitude of the force. These triangles are similar, so their internal angles are all the same, but they are not the same triangle. One is a distance triangle, with sides measured in meters, the other a force triangle, with sides in newtons. In both cases, the horizontal leg is 93% as long as the hypotenuse. It does not make sense, however, to compare the sizes of the triangles — the force triangle is not smaller in any meaningful sense.



f / Example 4.



g / The applied force  $\mathbf{F}_A$  pushes the block up the frictionless ramp.



h / Three forces act on the block. Their vector sum is zero.

### Pushing a block up a ramp

### example 5

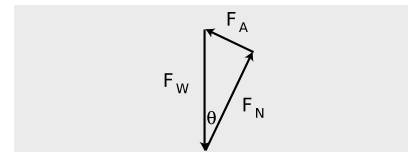
▷ Figure g shows a block being pushed up a frictionless ramp at constant speed by an applied force  $\mathbf{F}_A$ . How much force is required, in terms of the block's mass,  $m$ , and the angle of the ramp,  $\theta$ ?

▷ Figure h shows the other two forces acting on the block: a normal force,  $\mathbf{F}_N$ , created by the ramp, and the weight force,  $\mathbf{F}_W$ , created by the earth's gravity. Because the block is being pushed up at constant speed, it has zero acceleration, and the total force on it must be zero. From figure i, we find

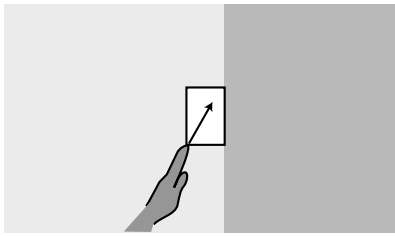
$$\begin{aligned} |\mathbf{F}_A| &= |\mathbf{F}_W| \sin \theta \\ &= mg \sin \theta \end{aligned}$$

Since the sine is always less than one, the applied force is always less than  $mg$ , i.e., pushing the block up the ramp is easier than lifting it straight up. This is presumably the principle on which the pyramids were constructed: the ancient Egyptians would have had a hard time applying the forces of enough slaves to equal the full weight of the huge blocks of stone.

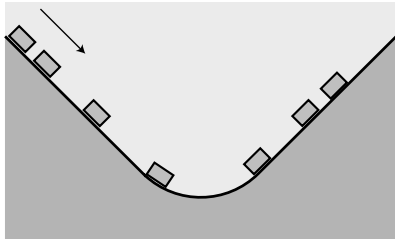
Essentially the same analysis applies to several other simple ma-



i / If the block is to move at constant velocity, Newton's first law says that the three force vectors acting on it must add up to zero. To perform vector addition, we put the vectors tip to tail, and in this case we are adding three vectors, so each one's tail goes against the tip of the previous one. Since they are supposed to add up to zero, the third vector's tip must come back to touch the tail of the first vector. They form a triangle, and since the applied force is perpendicular to the normal force, it is a right triangle.



Discussion question A.



j / Discussion question A.

chines, such as the wedge and the screw.

▷ Solved problem: A cargo plane page 218, problem 9

▷ Solved problem: The angle of repose page 219, problem 11

▷ Solved problem: A wagon page 218, problem 10

### Discussion questions

**A** The figure shows a block being pressed diagonally upward against a wall, causing it to slide up the wall. Analyze the forces involved, including their directions.

**B** The figure shows a roller coaster car rolling down and then up under the influence of gravity. Sketch the car's velocity vectors and acceleration vectors. Pick an interesting point in the motion and sketch a set of force vectors acting on the car whose vector sum could have resulted in the right acceleration vector.

## 8.4 ∫ Calculus with vectors

Using the unit vector notation introduced in section 7.4, the definitions of the velocity and acceleration components given in chapter 6 can be translated into calculus notation as

$$\mathbf{v} = \frac{dx}{dt} \hat{\mathbf{x}} + \frac{dy}{dt} \hat{\mathbf{y}} + \frac{dz}{dt} \hat{\mathbf{z}}$$

and

$$\mathbf{a} = \frac{dv_x}{dt} \hat{\mathbf{x}} + \frac{dv_y}{dt} \hat{\mathbf{y}} + \frac{dv_z}{dt} \hat{\mathbf{z}} \quad .$$

To make the notation less cumbersome, we generalize the concept of the derivative to include derivatives of vectors, so that we can abbreviate the above equations as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \quad .$$

In words, to take the derivative of a vector, you take the derivatives of its components and make a new vector out of those. This definition means that the derivative of a vector function has the familiar properties

$$\frac{d(c\mathbf{f})}{dt} = c \frac{d(\mathbf{f})}{dt} \quad [c \text{ is a constant}]$$

and

$$\frac{d(\mathbf{f} + \mathbf{g})}{dt} = \frac{d(\mathbf{f})}{dt} + \frac{d(\mathbf{g})}{dt} \quad .$$

The integral of a vector is likewise defined as integrating component by component.

▷ Two objects have positions as functions of time given by the equations

$$\mathbf{r}_1 = 3t^2\hat{\mathbf{x}} + t\hat{\mathbf{y}}$$

and

$$\mathbf{r}_2 = 3t^4\hat{\mathbf{x}} + t\hat{\mathbf{y}} \quad .$$

Find both objects' accelerations using calculus. Could either answer have been found without calculus?

▷ Taking the first derivative of each component, we find

$$\begin{aligned}\mathbf{v}_1 &= 6t\hat{\mathbf{x}} + \hat{\mathbf{y}} \\ \mathbf{v}_2 &= 12t^3\hat{\mathbf{x}} + \hat{\mathbf{y}} \quad ,\end{aligned}$$

and taking the derivatives again gives acceleration,

$$\begin{aligned}\mathbf{a}_1 &= 6\hat{\mathbf{x}} \\ \mathbf{a}_2 &= 36t^2\hat{\mathbf{x}} \quad .\end{aligned}$$

The first object's acceleration could have been found without calculus, simply by comparing the  $x$  and  $y$  coordinates with the constant-acceleration equation  $\Delta x = v_0\Delta t + \frac{1}{2}a\Delta t^2$ . The second equation, however, isn't just a second-order polynomial in  $t$ , so the acceleration isn't constant, and we really did need calculus to find the corresponding acceleration.

▷ Starting from rest, a flying saucer of mass  $m$  is observed to vary its propulsion with mathematical precision according to the equation

$$\mathbf{F} = bt^{42}\hat{\mathbf{x}} + ct^{137}\hat{\mathbf{y}} \quad .$$

(The aliens inform us that the numbers 42 and 137 have a special religious significance for them.) Find the saucer's velocity as a function of time.

▷ From the given force, we can easily find the acceleration

$$\begin{aligned}\mathbf{a} &= \frac{\mathbf{F}}{m} \\ &= \frac{b}{m}t^{42}\hat{\mathbf{x}} + \frac{c}{m}t^{137}\hat{\mathbf{y}} \quad .\end{aligned}$$

The velocity vector  $\mathbf{v}$  is the integral with respect to time of the acceleration,

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} \, dt \\ &= \int \left( \frac{b}{m}t^{42}\hat{\mathbf{x}} + \frac{c}{m}t^{137}\hat{\mathbf{y}} \right) dt \quad ,\end{aligned}$$



and integrating component by component gives

$$\begin{aligned}
 &= \left( \int \frac{b}{m} t^{42} dt \right) \hat{\mathbf{x}} + \left( \int \frac{c}{m} t^{137} dt \right) \hat{\mathbf{y}} \\
 &= \frac{b}{43m} t^{43} \hat{\mathbf{x}} + \frac{c}{138m} t^{138} \hat{\mathbf{y}} \quad ,
 \end{aligned}$$

where we have omitted the constants of integration, since the saucer was starting from rest.

*A fire-extinguisher stunt on ice* example 8

▷ Prof. Puerile smuggles a fire extinguisher into a skating rink. Climbing out onto the ice without any skates on, he sits down and pushes off from the wall with his feet, acquiring an initial velocity  $v_0 \hat{\mathbf{y}}$ . At  $t = 0$ , he then discharges the fire extinguisher at a 45-degree angle so that it applies a force to him that is backward and to the left, i.e., along the negative  $y$  axis and the positive  $x$  axis. The fire extinguisher's force is strong at first, but then dies down according to the equation  $|\mathbf{F}| = b - ct$ , where  $b$  and  $c$  are constants. Find the professor's velocity as a function of time.

▷ Measured counterclockwise from the  $x$  axis, the angle of the force vector becomes  $315^\circ$ . Breaking the force down into  $x$  and  $y$  components, we have

$$\begin{aligned}
 F_x &= |\mathbf{F}| \cos 315^\circ \\
 &= (b - ct) \\
 F_y &= |\mathbf{F}| \sin 315^\circ \\
 &= (-b + ct) \quad .
 \end{aligned}$$

In unit vector notation, this is

$$\mathbf{F} = (b - ct) \hat{\mathbf{x}} + (-b + ct) \hat{\mathbf{y}} \quad .$$

Newton's second law gives

$$\begin{aligned}
 \mathbf{a} &= \mathbf{F}/m \\
 &= \frac{b - ct}{\sqrt{2}m} \hat{\mathbf{x}} + \frac{-b + ct}{\sqrt{2}m} \hat{\mathbf{y}} \quad .
 \end{aligned}$$

To find the velocity vector as a function of time, we need to integrate the acceleration vector with respect to time,

$$\begin{aligned}
 \mathbf{v} &= \int \mathbf{a} dt \\
 &= \int \left( \frac{b - ct}{\sqrt{2}m} \hat{\mathbf{x}} + \frac{-b + ct}{\sqrt{2}m} \hat{\mathbf{y}} \right) dt \\
 &= \frac{1}{\sqrt{2}m} \int [(b - ct) \hat{\mathbf{x}} + (-b + ct) \hat{\mathbf{y}}] dt
 \end{aligned}$$

A vector function can be integrated component by component, so this can be broken down into two integrals,

$$\begin{aligned}\mathbf{v} &= \frac{\hat{\mathbf{x}}}{\sqrt{2m}} \int (b - ct) dt + \frac{\hat{\mathbf{y}}}{\sqrt{2m}} \int (-b + ct) dt \\ &= \left( \frac{bt - \frac{1}{2}ct^2}{\sqrt{2m}} + \text{constant \#1} \right) \hat{\mathbf{x}} + \left( \frac{-bt + \frac{1}{2}ct^2}{\sqrt{2m}} + \text{constant \#2} \right) \hat{\mathbf{y}}\end{aligned}$$

Here the physical significance of the two constants of integration is that they give the initial velocity. Constant #1 is therefore zero, and constant #2 must equal  $v_0$ . The final result is

$$\mathbf{v} = \left( \frac{bt - \frac{1}{2}ct^2}{\sqrt{2m}} \right) \hat{\mathbf{x}} + \left( \frac{-bt + \frac{1}{2}ct^2}{\sqrt{2m}} + v_0 \right) \hat{\mathbf{y}} \quad .$$

## Summary

The velocity vector points in the direction of the object's motion. Relative motion can be described by vector addition of velocities.

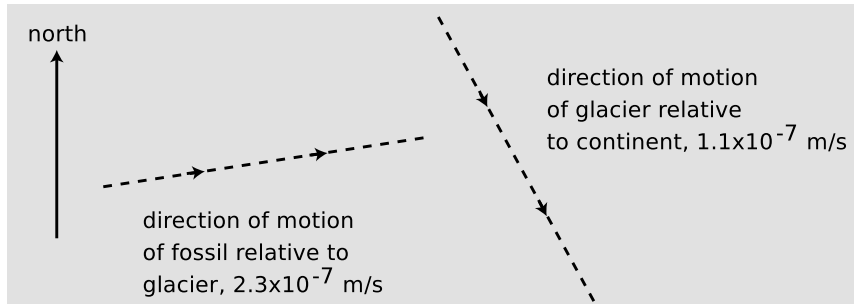
The acceleration vector need not point in the same direction as the object's motion. We use the word “acceleration” to describe any change in an object's velocity vector, which can be either a change in its magnitude or a change in its direction.

An important application of the vector addition of forces is the use of Newton's first law to analyze mechanical systems.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

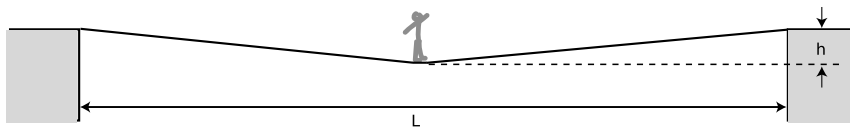


### Problem 1.

**1** A dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil's velocity relative to the continent. You will need a ruler and protractor. ✓

**2** Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

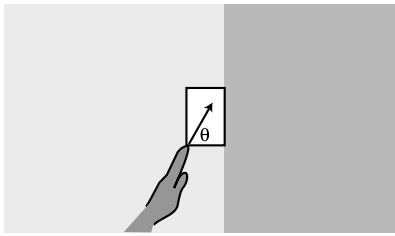
**3** A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and  $7^\circ$  north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.) ✓



### Problem 4.

**4** A person of mass  $M$  stands in the middle of a tightrope, which is fixed at the ends to two buildings separated by a horizontal distance  $L$ . The rope sags in the middle, stretching and lengthening the rope slightly.

(a) If the tightrope walker wants the rope to sag vertically by no



Problem 5.

more than a height  $h$ , find the minimum tension,  $T$ , that the rope must be able to withstand without breaking, in terms of  $h$ ,  $g$ ,  $M$ , and  $L$ .

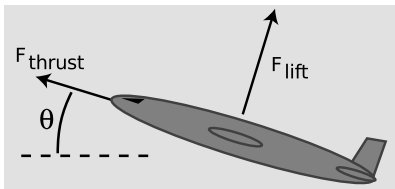
(b) Based on your equation, explain why it is not possible to get  $h = 0$ , and give a physical interpretation.

**5** Your hand presses a block of mass  $m$  against a wall with a force  $\mathbf{F}_H$  acting at an angle  $\theta$ , as shown in the figure. Find the minimum and maximum possible values of  $|\mathbf{F}_H|$  that can keep the block stationary, in terms of  $m$ ,  $g$ ,  $\theta$ , and  $\mu_s$ , the coefficient of static friction between the block and the wall. Check both your answers in the case of  $\theta = 90^\circ$ , and interpret the case where the maximum force is infinite. ✓ ★

**6** A skier of mass  $m$  is coasting down a slope inclined at an angle  $\theta$  compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is  $\mu_k$ , and in addition the skier experiences an air friction force of magnitude  $bv^2$ , where  $b$  is a constant.

(a) Find the maximum speed that the skier will attain, in terms of the variables  $m$ ,  $g$ ,  $\theta$ ,  $\mu_k$ , and  $b$ . ✓

(b) For angles below a certain minimum angle  $\theta_{min}$ , the equation gives a result that is not mathematically meaningful. Find an equation for  $\theta_{min}$ , and give a physical explanation of what is happening for  $\theta < \theta_{min}$ . ✓



Problem 9.

**7** A gun is aimed horizontally to the west, and fired at  $t = 0$ . The bullet's position vector as a function of time is  $\mathbf{r} = b\hat{\mathbf{x}} + ct\hat{\mathbf{y}} + dt^2\hat{\mathbf{z}}$ , where  $b$ ,  $c$ , and  $d$  are positive constants.

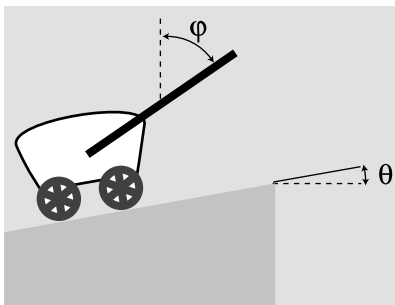
(a) What units would  $b$ ,  $c$ , and  $d$  need to have for the equation to make sense?

(b) Find the bullet's velocity and acceleration as functions of time.

(c) Give physical interpretations of  $b$ ,  $c$ ,  $d$ ,  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$ .  $\int$

**8** Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet. ▷ Solution, p. 522

**9** A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of  $\theta = 17^\circ$  with respect to horizontal. Its engines supply a thrust of  $F_{thrust} = 200$  kN, and the lift from its wings is  $F_{lift} = 654$  kN. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg? ▷ Solution, p. 523



Problem 10.

**10** A wagon is being pulled at constant speed up a slope  $\theta$  by a

rope that makes an angle  $\phi$  with the vertical.

(a) Assuming negligible friction, show that the tension in the rope is given by the equation

$$F_T = \frac{\sin \theta}{\sin(\theta + \phi)} F_W \quad ,$$

where  $F_W$  is the weight force acting on the wagon.

(b) Interpret this equation in the special cases of  $\phi = 0$  and  $\phi = 180^\circ - \theta$ .  
 ▷ Solution, p. 523

**11** The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose.

(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.

(b) On an asteroid, where  $g$  can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?

▷ Solution, p. 524

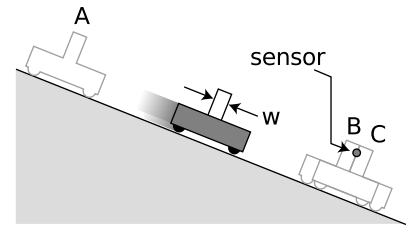
**12** The figure shows an experiment in which a cart is released from rest at A, and accelerates down the slope through a distance  $x$  until it passes through a sensor's light beam. The point of the experiment is to determine the cart's acceleration. At B, a cardboard vane mounted on the cart enters the light beam, blocking the light beam, and starts an electronic timer running. At C, the vane emerges from the beam, and the timer stops.

(a) Find the final velocity of the cart in terms of the width  $w$  of the vane and the time  $t_b$  for which the sensor's light beam was blocked. ✓

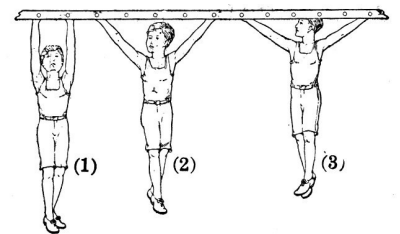
(b) Find the magnitude of the cart's acceleration in terms of the measurable quantities  $x$ ,  $t_b$ , and  $w$ . ✓

(c) Analyze the forces in which the cart participates, using a table in the format introduced in section 5.3. Assume friction is negligible.

(d) Find a theoretical value for the acceleration of the cart, which could be compared with the experimentally observed value extracted in part b. Express the theoretical value in terms of the angle  $\theta$  of the slope, and the strength  $g$  of the gravitational field. ✓



Problem 12.

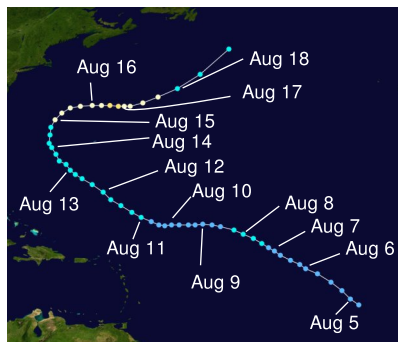


Problem 13 (Millikan and Gale, 1920).

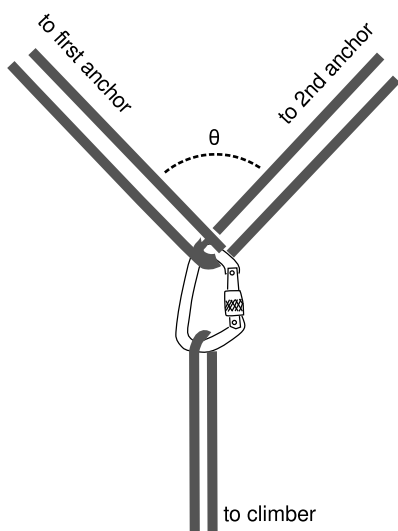
**13** The figure shows a boy hanging in three positions: (1) with his arms straight up, (2) with his arms at 45 degrees, and (3) with his arms at 60 degrees with respect to the vertical. Compare the tension in his arms in the three cases.

**14** Driving down a hill inclined at an angle  $\theta$  with respect to horizontal, you slam on the brakes to keep from hitting a deer. Your antilock brakes kick in, and you don't skid.

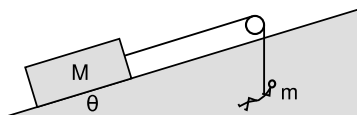
- Analyze the forces. (Ignore rolling resistance and air friction.)
- Find the car's maximum possible deceleration,  $a$  (expressed as a positive number), in terms of  $g$ ,  $\theta$ , and the relevant coefficient of friction. ✓
- Explain physically why the car's mass has no effect on your answer.
- Discuss the mathematical behavior and physical interpretation of your result for negative values of  $\theta$ .
- Do the same for very large positive values of  $\theta$ .



Problem 15.



Problem 16.



Problem 17.

**15** The figure shows the path followed by Hurricane Irene in 2005 as it moved north. The dots show the location of the center of the storm at six-hour intervals, with lighter dots at the time when the storm reached its greatest intensity. Find the time when the storm's center had a velocity vector to the northeast and an acceleration vector to the southeast. Explain.

**16** For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a “Y” at an angle  $\theta$ . The usual advice is to make  $\theta < 90^\circ$ ; for large values of  $\theta$ , the stress placed on the anchors can be many times greater than the actual load  $L$ , so that two anchors are actually *less* safe than one.

- Find the force  $S$  at each anchor in terms of  $L$  and  $\theta$ . ✓
- Verify that your answer makes sense in the case of  $\theta = 0$ .
- Interpret your answer in the case of  $\theta = 180^\circ$ .
- What is the smallest value of  $\theta$  for which  $S$  equals or exceeds  $L$ , so that for larger angles a failure of at least one anchor is *more* likely than it would have been with a single anchor? ✓

**17** (a) The person with mass  $m$  hangs from the rope, hauling the box of mass  $M$  up a slope inclined at an angle  $\theta$ . There is friction between the box and the slope, described by the usual coefficients of friction. The pulley, however, is frictionless. Find the magnitude of the box's acceleration. ✓

- Show that the units of your answer make sense.
- Check the physical behavior of your answer in the special cases of  $M = 0$  and  $\theta = -90^\circ$ .





## Exercise 8: Vectors and motion

Each diagram on page 223 shows the motion of an object in an  $x - y$  plane. Each dot is one location of the object at one moment in time. The time interval from one dot to the next is always the same, so you can think of the vector that connects one dot to the next as a  $\mathbf{v}$  vector, and subtract to find  $\Delta\mathbf{v}$  vectors.

1. Suppose the object in diagram 1 is moving from the top left to the bottom right. Deduce whatever you can about the force acting on it. Does the force always have the same magnitude? The same direction?

Invent a physical situation that this diagram could represent.

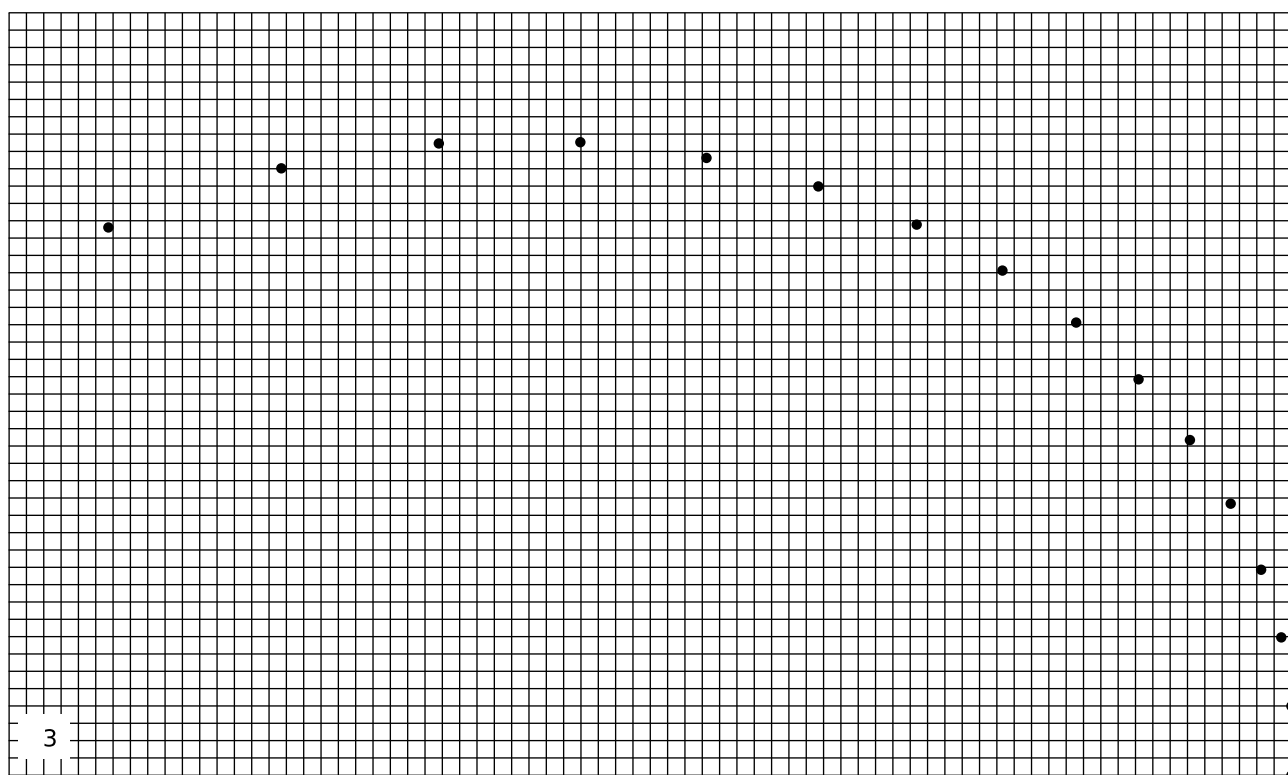
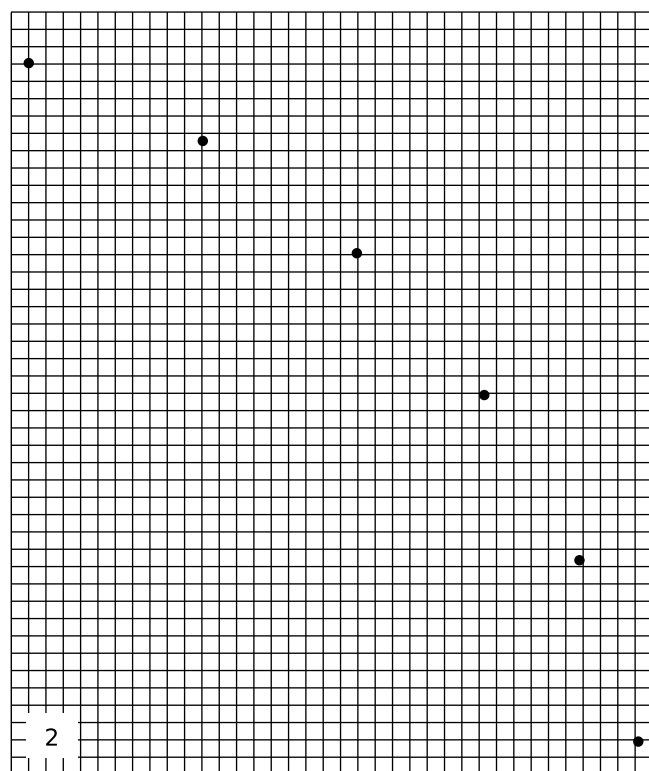
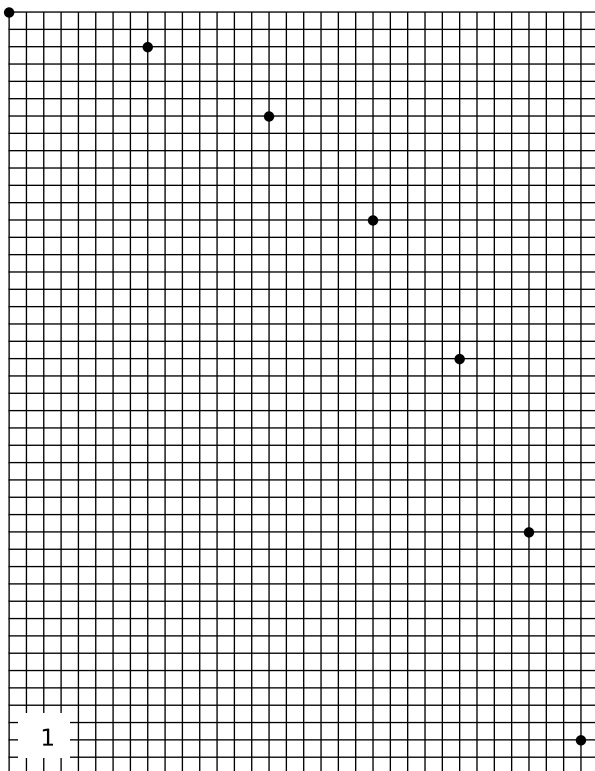
What if you reinterpret the diagram by reversing the object's direction of motion? Redo the construction of one of the  $\Delta\mathbf{v}$  vectors and see what happens.

2. What can you deduce about the force that is acting in diagram 2?

Invent a physical situation that diagram 2 could represent.

3. What can you deduce about the force that is acting in diagram 3?

Invent a physical situation.







## Chapter 9

# Circular motion

### 9.1 Conceptual framework

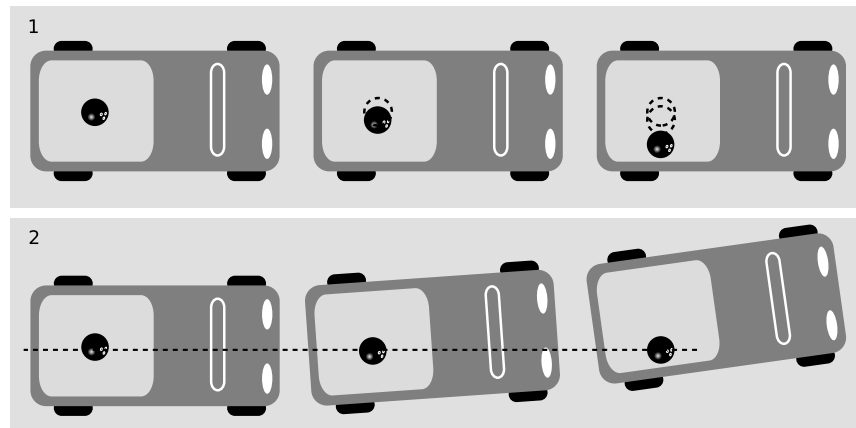
I now live fifteen minutes from Disneyland, so my friends and family in my native Northern California think it's a little strange that I've never visited the Magic Kingdom again since a childhood trip to the south. The truth is that for me as a preschooler, Disneyland was not the Happiest Place on Earth. My mother took me on a ride in which little cars shaped like rocket ships circled rapidly around a central pillar. I knew I was going to die. There was a force trying to throw me outward, and the safety features of the ride would surely have been inadequate if I hadn't screamed the whole time to make sure Mom would hold on to me. Afterward, she seemed surprisingly indifferent to the extreme danger we had experienced.

#### **Circular motion does not produce an outward force**

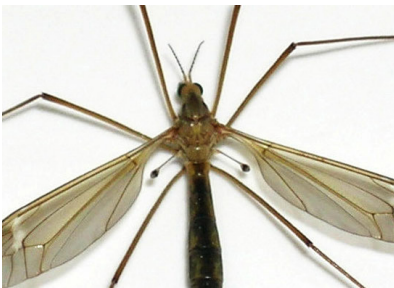
My younger self's understanding of circular motion was partly right and partly wrong. I was wrong in believing that there was a force pulling me outward, away from the center of the circle. The easiest way to understand this is to bring back the parable of the bowling ball in the pickup truck from chapter 4. As the truck makes a left turn, the driver looks in the rearview mirror and thinks that some mysterious force is pulling the ball outward, but the truck is accelerating, so the driver's frame of reference is not an inertial frame. Newton's laws are violated in a noninertial frame, so the ball appears to accelerate without any actual force acting on it. Because we are used to inertial frames, in which accelerations are caused by

forces, the ball's acceleration creates a vivid illusion that there must be an outward force.

a / 1. In the turning truck's frame of reference, the ball appears to violate Newton's laws, displaying a sideways acceleration that is not the result of a force-interaction with any other object. 2. In an inertial frame of reference, such as the frame fixed to the earth's surface, the ball obeys Newton's first law. No forces are acting on it, and it continues moving in a straight line. It is the truck that is participating in an interaction with the asphalt, the truck that accelerates as it should according to Newton's second law.



In an inertial frame everything makes more sense. The ball has no force on it, and goes straight as required by Newton's first law. The truck has a force on it from the asphalt, and responds to it by accelerating (changing the direction of its velocity vector) as Newton's second law says it should.



b / This crane fly's halteres help it to maintain its orientation in flight.

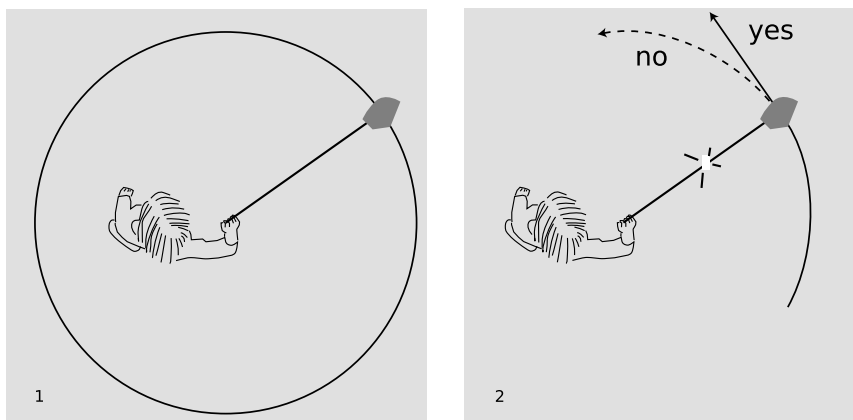
#### The halteres

#### example 1

Another interesting example is an insect organ called the halteres, a pair of small knobbed limbs behind the wings, which vibrate up and down and help the insect to maintain its orientation in flight. The halteres evolved from a second pair of wings possessed by earlier insects. Suppose, for example, that the halteres are on their upward stroke, and at that moment an air current causes the fly to pitch its nose down. The halteres follow Newton's first law, continuing to rise vertically, but in the fly's rotating frame of reference, it seems as though they have been subjected to a backward force. The fly has special sensory organs that perceive this twist, and help it to correct itself by raising its nose.

### Circular motion does not persist without a force

I was correct, however, on a different point about the Disneyland ride. To make me curve around with the car, I really did need some force such as a force from my mother, friction from the seat, or a normal force from the side of the car. (In fact, all three forces were probably adding together.) One of the reasons why Galileo failed to



c / 1. An overhead view of a person swinging a rock on a rope. A force from the string is required to make the rock's velocity vector keep changing direction. 2. If the string breaks, the rock will follow Newton's first law and go straight instead of continuing around the circle.

refine the principle of inertia into a quantitative statement like Newton's first law is that he was not sure whether motion without a force would naturally be circular or linear. In fact, the most impressive examples he knew of the persistence of motion were mostly circular: the spinning of a top or the rotation of the earth, for example. Newton realized that in examples such as these, there really were forces at work. Atoms on the surface of the top are prevented from flying off straight by the ordinary force that keeps atoms stuck together in solid matter. The earth is nearly all liquid, but gravitational forces pull all its parts inward.

### Uniform and nonuniform circular motion

Circular motion always involves a change in the direction of the velocity vector, but it is also possible for the magnitude of the velocity to change at the same time. Circular motion is referred to as *uniform* if  $|\mathbf{v}|$  is constant, and *nonuniform* if it is changing.

Your speedometer tells you the magnitude of your car's velocity vector, so when you go around a curve while keeping your speedometer needle steady, you are executing uniform circular motion. If your speedometer reading is changing as you turn, your circular motion is nonuniform. Uniform circular motion is simpler to analyze mathematically, so we will attack it first and then pass to the nonuniform case.

#### self-check A

Which of these are examples of uniform circular motion and which are nonuniform?

(1) the clothes in a clothes dryer (assuming they remain against the inside of the drum, even at the top)

(2) a rock on the end of a string being whirled in a vertical circle ▶

Answer, p. 533

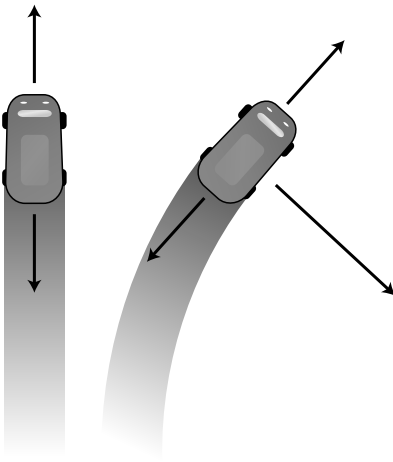


d / To make the brick go in a circle, I had to exert an inward force on the rope.

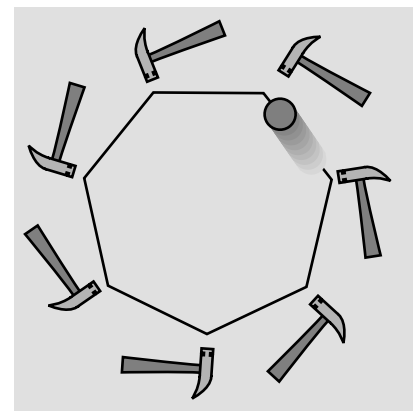
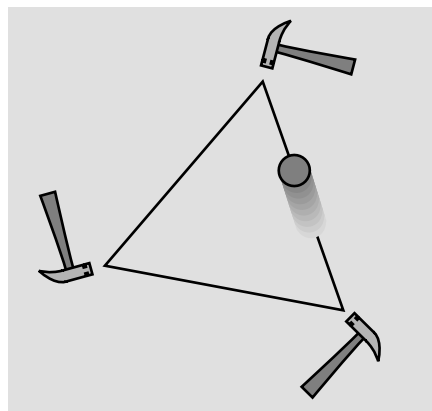
### Only an inward force is required for uniform circular motion.

Figure c showed the string pulling in straight along a radius of the circle, but many people believe that when they are doing this they must be “leading” the rock a little to keep it moving along. That is, they believe that the force required to produce uniform circular motion is not directly inward but at a slight angle to the radius of the circle. This intuition is incorrect, which you can easily verify for yourself now if you have some string handy. It is only while you are getting the object going that your force needs to be at an angle to the radius. During this initial period of speeding up, the motion is not uniform. Once you settle down into uniform circular motion, you only apply an inward force.

If you have not done the experiment for yourself, here is a theoretical argument to convince you of this fact. We have discussed in chapter 6 the principle that forces have no perpendicular effects. To keep the rock from speeding up or slowing down, we only need to make sure that our force is perpendicular to its direction of motion. We are then guaranteed that its forward motion will remain unaffected: our force can have no perpendicular effect, and there is no other force acting on the rock which could slow it down. The rock requires no forward force to maintain its forward motion, any more than a projectile needs a horizontal force to “help it over the top” of its arc.



f / When a car is going straight at constant speed, the forward and backward forces on it are canceling out, producing a total force of zero. When it moves in a circle at constant speed, there are three forces on it, but the forward and backward forces cancel out, so the vector sum is an inward force.



e / A series of three hammer taps makes the rolling ball trace a triangle, seven hammers a heptagon. If the number of hammers was large enough, the ball would essentially be experiencing a steady inward force, and it would go in a circle. In no case is any forward force necessary.

Why, then, does a car driving in circles in a parking lot stop executing uniform circular motion if you take your foot off the gas? The source of confusion here is that Newton's laws predict an object's motion based on the *total* force acting on it. A car driving in circles has three forces on it

- (1) an inward force from the asphalt, controlled with the steering wheel;
- (2) a forward force from the asphalt, controlled with the gas pedal; and
- (3) backward forces from air resistance and rolling resistance.

You need to make sure there is a forward force on the car so that the backward forces will be exactly canceled out, creating a vector sum that points directly inward.

#### *A motorcycle making a turn*

#### *example 2*

The motorcyclist in figure g is moving along an arc of a circle. It looks like he's chosen to ride the slanted surface of the dirt at a place where it makes just the angle he wants, allowing him to get the force he needs on the tires as a normal force, without needing any frictional force. The dirt's normal force on the tires points up and to our left. The vertical component of that force is canceled by gravity, while its horizontal component causes him to curve.

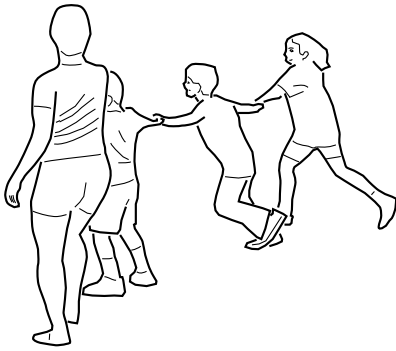
### **In uniform circular motion, the acceleration vector is inward**

Since experiments show that the force vector points directly inward, Newton's second law implies that the acceleration vector points inward as well. This fact can also be proven on purely kinematical grounds, and we will do so in the next section.

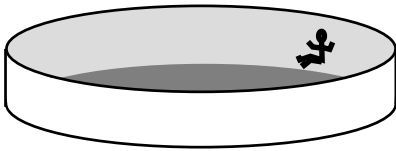


g / Example 2.





Discussion questions A-D



Discussion question E.

## Discussion questions

**A** In the game of crack the whip, a line of people stand holding hands, and then they start sweeping out a circle. One person is at the center, and rotates without changing location. At the opposite end is the person who is running the fastest, in a wide circle. In this game, someone always ends up losing their grip and flying off. Suppose the person on the end loses her grip. What path does she follow as she goes flying off? (Assume she is going so fast that she is really just trying to put one foot in front of the other fast enough to keep from falling; she is not able to get any significant horizontal force between her feet and the ground.)

**B** Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What force or forces are acting on her, and in what directions are they? (We are not interested in the vertical forces, which are the earth's gravitational force pulling down, and the ground's normal force pushing up.)

**C** Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What is wrong with the following analysis of the situation? "The person whose hand she's holding exerts an inward force on her, and because of Newton's third law, there's an equal and opposite force acting outward. That outward force is the one she feels throwing her outward, and the outward force is what might make her go flying off, if it's strong enough."

**D** If the only force felt by the person on the outside is an inward force, why doesn't she go straight in?

**E** In the amusement park ride shown in the figure, the cylinder spins faster and faster until the customer can pick her feet up off the floor without falling. In the old Coney Island version of the ride, the floor actually dropped out like a trap door, showing the ocean below. (There is also a version in which the whole thing tilts up diagonally, but we're discussing the version that stays flat.) If there is no outward force acting on her, why does she stick to the wall? Analyze all the forces on her.

**F** What is an example of circular motion where the inward force is a normal force? What is an example of circular motion where the inward force is friction? What is an example of circular motion where the inward force is the sum of more than one force?

**G** Does the acceleration vector always change continuously in circular motion? The velocity vector?

## 9.2 Uniform circular motion

In this section I derive a simple and very useful equation for the magnitude of the acceleration of an object undergoing constant acceleration. The law of sines is involved, so I've recapped it in figure h.

The derivation is brief, but the method requires some explanation and justification. The idea is to calculate a  $\Delta \mathbf{v}$  vector describing the change in the velocity vector as the object passes through an angle  $\theta$ . We then calculate the acceleration,  $\mathbf{a} = \Delta \mathbf{v} / \Delta t$ . The astute reader will recall, however, that this equation is only valid for motion with constant acceleration. Although the magnitude of the acceleration is constant for uniform circular motion, the acceleration vector changes its direction, so it is not a constant vector, and the equation  $\mathbf{a} = \Delta \mathbf{v} / \Delta t$  does not apply. The justification for using it is that we will then examine its behavior when we make the time interval very short, which means making the angle  $\theta$  very small. For smaller and smaller time intervals, the  $\Delta \mathbf{v} / \Delta t$  expression becomes a better and better approximation, so that the final result of the derivation is exact.

In figure i/1, the object sweeps out an angle  $\theta$ . Its direction of motion also twists around by an angle  $\theta$ , from the vertical dashed line to the tilted one. Figure i/2 shows the initial and final velocity vectors, which have equal magnitude, but directions differing by  $\theta$ . In i/3, I've reassembled the vectors in the proper positions for vector subtraction. They form an isosceles triangle with interior angles  $\theta$ ,  $\eta$ , and  $\eta$ . (Eta,  $\eta$ , is my favorite Greek letter.) The law of sines gives

$$\frac{|\Delta \mathbf{v}|}{\sin \theta} = \frac{|\mathbf{v}|}{\sin \eta} \quad .$$

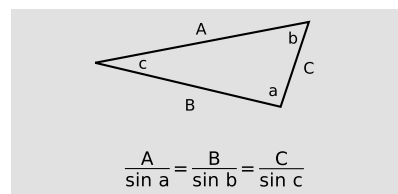
This tells us the magnitude of  $\Delta \mathbf{v}$ , which is one of the two ingredients we need for calculating the magnitude of  $\mathbf{a} = \Delta \mathbf{v} / \Delta t$ . The other ingredient is  $\Delta t$ . The time required for the object to move through the angle  $\theta$  is

$$\Delta t = \frac{\text{length of arc}}{|\mathbf{v}|} \quad .$$

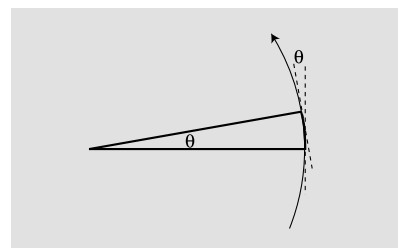
Now if we measure our angles in radians we can use the definition of radian measure, which is (angle) = (length of arc)/(radius), giving  $\Delta t = \theta r / |\mathbf{v}|$ . Combining this with the first expression involving  $|\Delta \mathbf{v}|$  gives

$$\begin{aligned} |\mathbf{a}| &= |\Delta \mathbf{v}| / \Delta t \\ &= \frac{|\mathbf{v}|^2}{r} \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\sin \eta} \quad . \end{aligned}$$

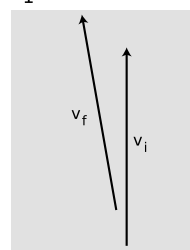
When  $\theta$  becomes very small, the small-angle approximation  $\sin \theta \approx \theta$  applies, and also  $\eta$  becomes close to  $90^\circ$ , so  $\sin \eta \approx 1$ , and we have



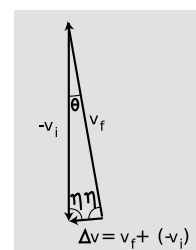
h / The law of sines.



1



2



3

i / Deriving  $|\mathbf{a}| = |\mathbf{v}|^2 / r$  for uniform circular motion.

an equation for  $|\mathbf{a}|$ :

$$|\mathbf{a}| = \frac{|\mathbf{v}|^2}{r} \quad . \quad [\text{uniform circular motion}]$$

---

*Force required to turn on a bike* *example 3*

▷ A bicyclist is making a turn along an arc of a circle with radius 20 m, at a speed of 5 m/s. If the combined mass of the cyclist plus the bike is 60 kg, how great a static friction force must the road be able to exert on the tires?

▷ Taking the magnitudes of both sides of Newton's second law gives

$$\begin{aligned} |\mathbf{F}| &= m|\mathbf{a}| \\ &= m|\mathbf{a}| \quad . \end{aligned}$$

Substituting  $|\mathbf{a}| = |\mathbf{v}|^2/r$  gives

$$\begin{aligned} |\mathbf{F}| &= m|\mathbf{v}|^2/r \\ &\approx 80 \text{ N} \end{aligned}$$

(rounded off to one sig fig).

---

*Don't hug the center line on a curve!* *example 4*

▷ You're driving on a mountain road with a steep drop on your right. When making a left turn, is it safer to hug the center line or to stay closer to the outside of the road?

▷ You want whichever choice involves the least acceleration, because that will require the least force and entail the least risk of exceeding the maximum force of static friction. Assuming the curve is an arc of a circle and your speed is constant, your car is performing uniform circular motion, with  $|\mathbf{a}| = |\mathbf{v}|^2/r$ . The dependence on the square of the speed shows that driving slowly is the main safety measure you can take, but for any given speed you also want to have the largest possible value of  $r$ . Even though your instinct is to keep away from that scary precipice, you are actually less likely to skid if you keep toward the outside, because then you are describing a larger circle.

---

*Acceleration related to radius and period of rotation* *example 5*

▷ How can the equation for the acceleration in uniform circular motion be rewritten in terms of the radius of the circle and the period,  $T$ , of the motion, i.e., the time required to go around once?

▷ The period can be related to the speed as follows:

$$\begin{aligned} |\mathbf{v}| &= \frac{\text{circumference}}{T} \\ &= 2\pi r/T \quad . \end{aligned}$$

Substituting into the equation  $|\mathbf{a}| = |\mathbf{v}|^2/r$  gives

$$|\mathbf{a}| = \frac{4\pi^2 r}{T^2} \quad .$$



j / Example 6.

---

**A clothes dryer***example 6*

▷ My clothes dryer has a drum with an inside radius of 35 cm, and it spins at 48 revolutions per minute. What is the acceleration of the clothes inside?

▷ We can solve this by finding the period and plugging in to the result of the previous example. If it makes 48 revolutions in one minute, then the period is  $1/48$  of a minute, or 1.25 s. To get an acceleration in mks units, we must convert the radius to 0.35 m. Plugging in, the result is  $8.8 \text{ m/s}^2$ .

---

**More about clothes dryers!***example 7*

▷ In a discussion question in the previous section, we made the assumption that the clothes remain against the inside of the drum as they go over the top. In light of the previous example, is this a correct assumption?

▷ No. We know that there must be some minimum speed at which the motor can run that will result in the clothes just barely staying against the inside of the drum as they go over the top. If the clothes dryer ran at just this minimum speed, then there would be no normal force on the clothes at the top: they would be on the verge of losing contact. The only force acting on them at the top would be the force of gravity, which would give them an acceleration of  $g = 9.8 \text{ m/s}^2$ . The actual dryer must be running slower than this minimum speed, because it produces an acceleration of only  $8.8 \text{ m/s}^2$ . My theory is that this is done intentionally, to make the clothes mix and tumble.

▷ *Solved problem: The tilt-a-whirl*

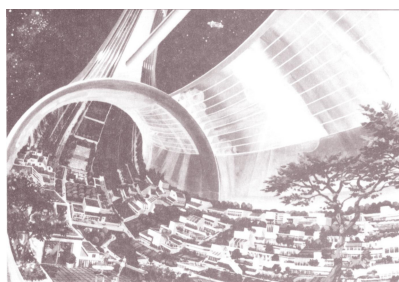
*page 238, problem 6*

▷ *Solved problem: An off-ramp*

*page 238, problem 7***Discussion questions**

**A** A certain amount of force is needed to provide the acceleration of circular motion. What if we are exerting a force perpendicular to the direction of motion in an attempt to make an object trace a circle of radius  $r$ , but the force isn't as big as  $m|\mathbf{v}|^2/r$ ?

**B** Suppose a rotating space station, as in figure k on page 234, is built. It gives its occupants the illusion of ordinary gravity. What happens when a person in the station lets go of a ball? What happens when she throws a ball straight "up" in the air (i.e., towards the center)?



k / Discussion question B. An artist's conception of a rotating space colony in the form of a giant wheel. A person living in this noninertial frame of reference has an illusion of a force pulling her outward, toward the deck, for the same reason that a person in the pickup truck has the illusion of a force pulling the bowling ball. By adjusting the speed of rotation, the designers can make an acceleration  $|\mathbf{v}|^2/r$  equal to the usual acceleration of gravity on earth. On earth, your acceleration standing on the ground is zero, and a falling rock heads for your feet with an acceleration of  $9.8 \text{ m/s}^2$ . A person standing on the deck of the space colony has an *upward* acceleration of  $9.8 \text{ m/s}^2$ , and when she lets go of a rock, her feet head *up* at the nonaccelerating rock. To her, it seems the same as true gravity.

## 9.3 Nonuniform circular motion

What about nonuniform circular motion? Although so far we have been discussing components of vectors along fixed  $x$  and  $y$  axes, it now becomes convenient to discuss components of the acceleration vector along the radial line (in-out) and the tangential line (along the direction of motion). For nonuniform circular motion, the radial component of the acceleration obeys the same equation as for uniform circular motion,

$$a_r = v^2/r \quad ,$$

where  $v = |\mathbf{v}|$ , but the acceleration vector also has a tangential component,

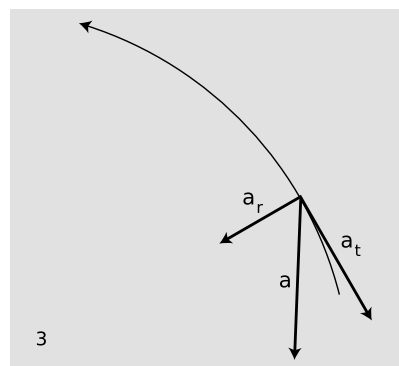
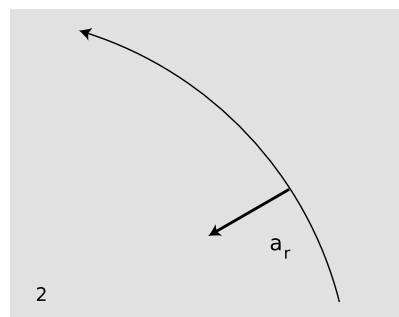
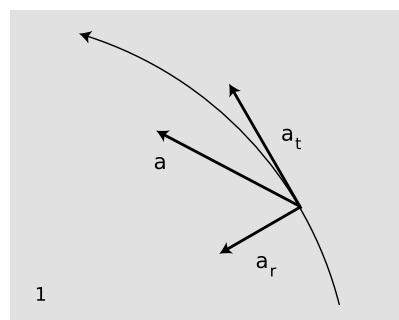
$$a_t = \text{slope of the graph of } v \text{ versus } t \quad .$$

The latter quantity has a simple interpretation. If you are going around a curve in your car, and the speedometer needle is moving, the tangential component of the acceleration vector is simply what you would have thought the acceleration was if you saw the speedometer and didn't know you were going around a curve.

*Slow down before a turn, not during it.* example 8

▷ When you're making a turn in your car and you're afraid you may skid, isn't it a good idea to slow down?

▷ If the turn is an arc of a circle, and you've already completed part of the turn at constant speed without skidding, then the road and tires are apparently capable of enough static friction to supply an acceleration of  $|\mathbf{v}|^2/r$ . There is no reason why you would skid out now if you haven't already. If you get nervous and brake, however, then you need to have a tangential acceleration component in addition to the radial component you were already able to produce successfully. This would require an acceleration vector with a greater magnitude, which in turn would require a larger force. Static friction might not be able to supply that much force, and you might skid out. As in the previous example on a similar



1 / 1. Moving in a circle while speeding up. 2. Uniform circular motion. 3. Slowing down.

topic, the safe thing to do is to approach the turn at a comfortably low speed.

▷ *Solved problem: A bike race*

*page 237, problem 5*

# Summary

## Selected vocabulary

uniform circular motion . . . . .	circular motion in which the magnitude of the velocity vector remains constant
nonuniform circular motion . . . . .	circular motion in which the magnitude of the velocity vector changes
radial . . . . .	parallel to the radius of a circle; the in-out direction
tangential . . . . .	tangent to the circle, perpendicular to the radial direction

## Notation

$a_r$ . . . . .	radial acceleration; the component of the acceleration vector along the in-out direction
$a_t$ . . . . .	tangential acceleration; the component of the acceleration vector tangent to the circle

## Summary

If an object is to have circular motion, a force must be exerted on it toward the center of the circle. There is no outward force on the object; the illusion of an outward force comes from our experiences in which our point of view was rotating, so that we were viewing things in a noninertial frame.

An object undergoing uniform circular motion has an inward acceleration vector of magnitude

$$|\mathbf{a}| = v^2/r \quad ,$$

where  $v = |\mathbf{v}|$ . In nonuniform circular motion, the radial and tangential components of the acceleration vector are

$$\begin{aligned} a_r &= v^2/r \\ a_t &= \text{slope of the graph of } v \text{ versus } t \quad . \end{aligned}$$

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** When you're done using an electric mixer, you can get most of the batter off of the beaters by lifting them out of the batter with the motor running at a high enough speed. Let's imagine, to make things easier to visualize, that we instead have a piece of tape stuck to one of the beaters.

(a) Explain why static friction has no effect on whether or not the tape flies off.

(b) Analyze the forces in which the tape participates, using a table the format shown in section 5.3.

(c) Suppose you find that the tape doesn't fly off when the motor is on a low speed, but at a greater speed, the tape won't stay on. Why would the greater speed change things? [Hint: If you don't invoke any law of physics, you haven't explained it.]

**2** Show that the expression  $|\mathbf{v}|^2/r$  has the units of acceleration.

**3** A plane is flown in a loop-the-loop of radius 1.00 km. The plane starts out flying upside-down, straight and level, then begins curving up along the circular loop, and is right-side up when it reaches the top. (The plane may slow down somewhat on the way up.) How fast must the plane be going at the top if the pilot is to experience no force from the seat or the seatbelt while at the top of the loop? ✓

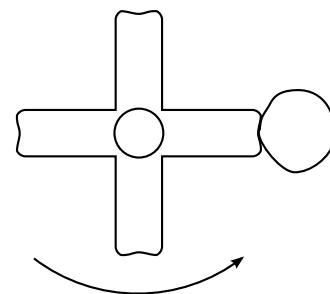
**4** In this problem, you'll derive the equation  $|\mathbf{a}| = |\mathbf{v}|^2/r$  using calculus. Instead of comparing velocities at two points in the particle's motion and then taking a limit where the points are close together, you'll just take derivatives. The particle's position vector is  $\mathbf{r} = (r \cos \theta)\hat{\mathbf{x}} + (r \sin \theta)\hat{\mathbf{y}}$ , where  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are the unit vectors along the  $x$  and  $y$  axes. By the definition of radians, the distance traveled since  $t = 0$  is  $r\theta$ , so if the particle is traveling at constant speed  $v = |\mathbf{v}|$ , we have  $v = r\theta/t$ .

(a) Eliminate  $\theta$  to get the particle's position vector as a function of time.

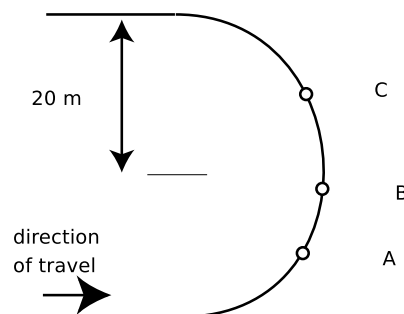
(b) Find the particle's acceleration vector.

(c) Show that the magnitude of the acceleration vector equals  $v^2/r$ . ∫

**5** Three cyclists in a race are rounding a semicircular curve. At the moment depicted, cyclist A is using her brakes to apply a force of 375 N to her bike. Cyclist B is coasting. Cyclist C is pedaling, resulting in a force of 375 N on her bike. Each cyclist, with her bike, has a mass of 75 kg. At the instant shown, the



Problem 1.

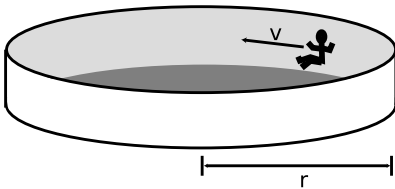


Problem 5.



instantaneous speed of all three cyclists is 10 m/s. On the diagram, draw each cyclist's acceleration vector with its tail on top of her present position, indicating the directions and lengths reasonably accurately. Indicate approximately the consistent scale you are using for all three acceleration vectors. Extreme precision is not necessary as long as the directions are approximately right, and lengths of vectors that should be equal appear roughly equal, etc. Assume all three cyclists are traveling along the road all the time, not wandering across their lane or wiping out and going off the road.

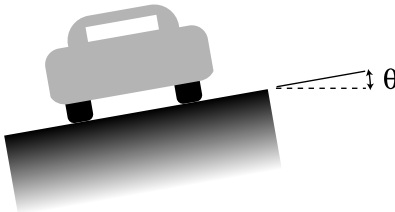
▷ Solution, p. 524



Problem 6.

**6** The amusement park ride shown in the figure consists of a cylindrical room that rotates about its vertical axis. When the rotation is fast enough, a person against the wall can pick his or her feet up off the floor and remain “stuck” to the wall without falling. (a) Suppose the rotation results in the person having a speed  $v$ . The radius of the cylinder is  $r$ , the person's mass is  $m$ , the downward acceleration of gravity is  $g$ , and the coefficient of static friction between the person and the wall is  $\mu_s$ . Find an equation for the speed,  $v$ , required, in terms of the other variables. (You will find that one of the variables cancels out.)

(b) Now suppose two people are riding the ride. Huy is wearing denim, and Gina is wearing polyester, so Huy's coefficient of static friction is three times greater. The ride starts from rest, and as it begins rotating faster and faster, Gina must wait longer before being able to lift her feet without sliding to the floor. Based on your equation from part a, how many times greater must the speed be before Gina can lift her feet without sliding down? ▷ Solution, p. 524 ★



Problem 7.

**7** An engineer is designing a curved off-ramp for a freeway. Since the off-ramp is curved, she wants to bank it to make it less likely that motorists going too fast will wipe out. If the radius of the curve is  $r$ , how great should the banking angle,  $\theta$ , be so that for a car going at a speed  $v$ , no static friction force whatsoever is required to allow the car to make the curve? State your answer in terms of  $v$ ,  $r$ , and  $g$ , and show that the mass of the car is irrelevant.

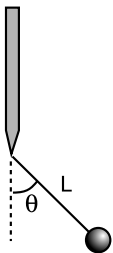
▷ Solution, p. 525

**8** Lionel brand toy trains come with sections of track in standard lengths and shapes. For circular arcs, the most commonly used sections have diameters of 662 and 1067 mm at the inside of the outer rail. The maximum speed at which a train can take the broader curve without flying off the tracks is 0.95 m/s. At what speed must the train be operated to avoid derailing on the tighter curve? ✓

**9** The figure shows a ball on the end of a string of length  $L$  attached to a vertical rod which is spun about its vertical axis by a motor. The period (time for one rotation) is  $P$ .

(a) Analyze the forces in which the ball participates.

(b) Find how the angle  $\theta$  depends on  $P$ ,  $g$ , and  $L$ . [Hints: (1)



Problem 9.

Write down Newton's second law for the vertical and horizontal components of force and acceleration. This gives two equations, which can be solved for the two unknowns,  $\theta$  and the tension in the string. (2) If you introduce variables like  $v$  and  $r$ , relate them to the variables your solution is supposed to contain, and eliminate them.]  $\checkmark$

(c) What happens mathematically to your solution if the motor is run very slowly (very large values of  $P$ )? Physically, what do you think would actually happen in this case?

**10** Psychology professor R.O. Dent requests funding for an experiment on compulsive thrill-seeking behavior in guinea pigs, in which the subject is to be attached to the end of a spring and whirled around in a horizontal circle. The spring has relaxed length  $b$ , and obeys Hooke's law with spring constant  $k$ . It is stiff enough to keep from bending significantly under the guinea pig's weight.

(a) Calculate the length of the spring when it is undergoing steady circular motion in which one rotation takes a time  $T$ . Express your result in terms of  $k$ ,  $b$ ,  $T$ , and the guinea pig's mass  $m$ .  $\checkmark$

(b) The ethics committee somehow fails to veto the experiment, but the safety committee expresses concern. Why? Does your equation do anything unusual, or even spectacular, for any particular value of  $T$ ? What do you think is the physical significance of this mathematical behavior?

**11** The figure shows an old-fashioned device called a flyball governor, used for keeping an engine running at the correct speed. The whole thing rotates about the vertical shaft, and the mass  $M$  is free to slide up and down. This mass would have a connection (not shown) to a valve that controlled the engine. If, for instance, the engine ran too fast, the mass would rise, causing the engine to slow back down.

(a) Show that in the special case of  $a = 0$ , the angle  $\theta$  is given by

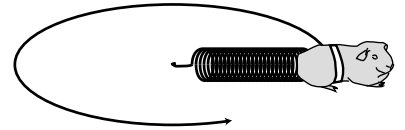
$$\theta = \cos^{-1} \left( \frac{g(m + M)P^2}{4\pi^2 mL} \right),$$

where  $P$  is the period of rotation (time required for one complete rotation).

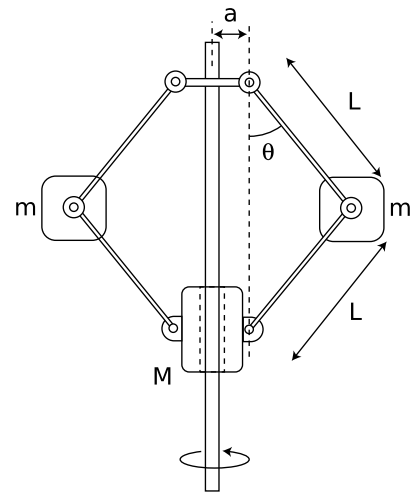
(b) There is no closed-form solution for  $\theta$  in the general case where  $a$  is not zero. However, explain how the undesirable low-speed behavior of the  $a = 0$  device would be improved by making  $a$  nonzero.

**12** The figure shows two blocks of masses  $m_1$  and  $m_2$  sliding in circles on a frictionless table. Find the tension in the strings if the period of rotation (time required for one rotation) is  $P$ .  $\checkmark$

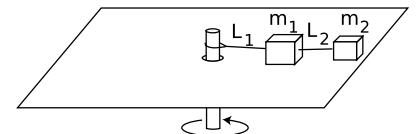
**13** The acceleration of an object in uniform circular motion can be given either by  $|\mathbf{a}| = |\mathbf{v}|^2/r$  or, equivalently, by  $|\mathbf{a}| = 4\pi^2 r/T^2$ , where  $T$  is the time required for one cycle (example 5 on page 232).



Problem 10.



Problem 11.



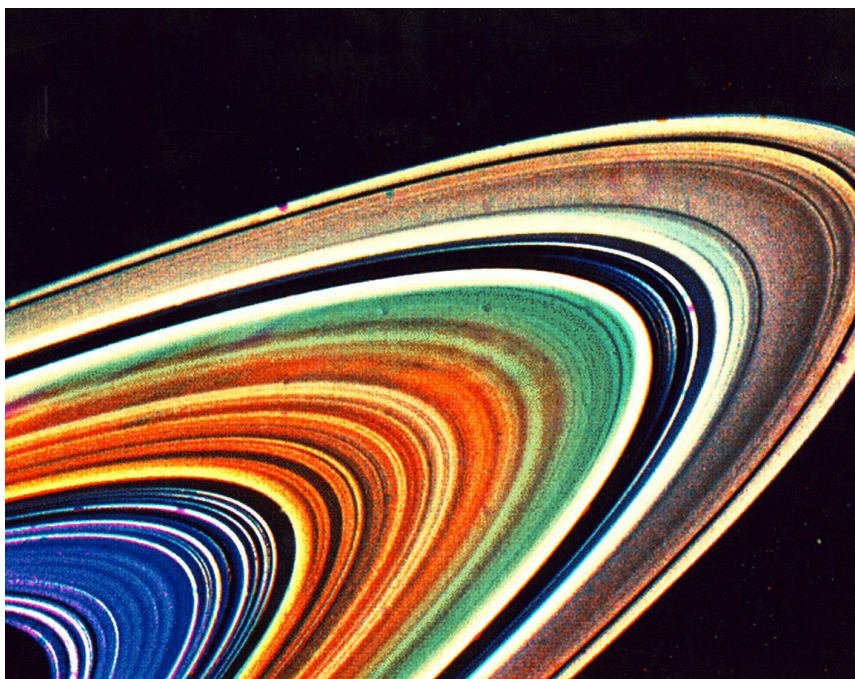
Problem 12.

Person A says based on the first equation that the acceleration in circular motion is greater when the circle is smaller. Person B, arguing from the second equation, says that the acceleration is smaller when the circle is smaller. Rewrite the two statements so that they are less misleading, eliminating the supposed paradox. [Based on a problem by Arnold Arons.]

**14** The bright star Sirius has a mass of  $4.02 \times 10^{30}$  kg and lies at a distance of  $8.1 \times 10^{16}$  m from our solar system. Suppose you're standing on a merry-go-round carousel rotating with a period of 10 seconds, and Sirius is on the horizon. You adopt a rotating, non-inertial frame of reference, in which the carousel is at rest, and the universe is spinning around it. If you drop a corndog, you see it accelerate horizontally away from the axis, and you interpret this is the result of some horizontal force. This force does not actually exist; it only seems to exist because you're insisting on using a noninertial frame. Similarly, calculate the force that seems to act on Sirius in this frame of reference. Comment on the physical plausibility of this force, and on what object could be exerting it. ✓

**15** In a well known stunt from circuses and carnivals, a motorcyclist rides around inside a big bowl, gradually speeding up and rising higher. Eventually the cyclist can get up to where the walls of the bowl are vertical. Let's estimate the conditions under which a running human could do the same thing.

- (a) If the runner can run at speed  $v$ , and her shoes have a coefficient of static friction  $\mu_s$ , what is the maximum radius of the circle? ✓
- (b) Show that the units of your answer make sense.
- (c) Check that its dependence on the variables makes sense.
- (d) Evaluate your result numerically for  $v = 10$  m/s (the speed of an olympic sprinter) and  $\mu_s = 5$ . (This is roughly the highest coefficient of static friction ever achieved for surfaces that are not sticky. The surface has an array of microscopic fibers like a hair brush, and is inspired by the hairs on the feet of a gecko. These assumptions are not necessarily realistic, since the person would have to run at an angle, which would be physically awkward.) ✓



Gravity is the only really important force on the cosmic scale. This false-color representation of saturn's rings was made from an image sent back by the Voyager 2 space probe. The rings are composed of innumerable tiny ice particles orbiting in circles under the influence of saturn's gravity.

## Chapter 10

# Gravity

Cruise your radio dial today and try to find any popular song that would have been imaginable without Louis Armstrong. By introducing solo improvisation into jazz, Armstrong took apart the jigsaw puzzle of popular music and fit the pieces back together in a different way. In the same way, Newton reassembled our view of the universe. Consider the titles of some recent physics books written for the general reader: *The God Particle*, *Dreams of a Final Theory*. Without Newton, such attempts at universal understanding would not merely have seemed a little pretentious, they simply would not have occurred to anyone.

This chapter is about Newton's theory of gravity, which he used to explain the motion of the planets as they orbited the sun. Whereas this book has concentrated on Newton's laws of motion, leaving gravity as a dessert, Newton tosses off the laws of motion in the first 20 pages of the *Principia Mathematica* and then spends the next 130 discussing the motion of the planets. Clearly he saw this as the crucial scientific focus of his work. Why? Because in it he



a / Johannes Kepler found a mathematical description of the motion of the planets, which led to Newton's theory of gravity.

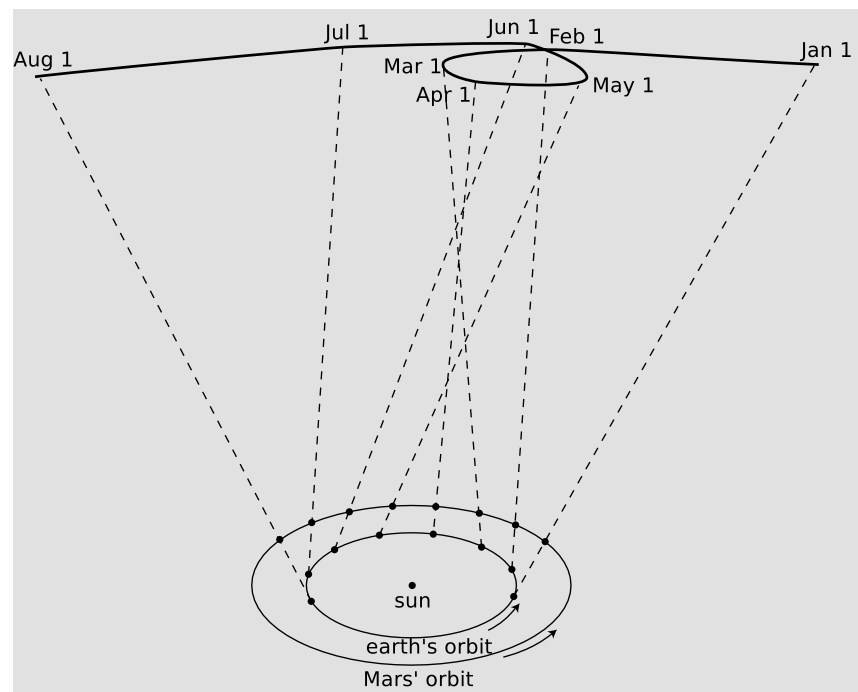


b / Tycho Brahe made his name as an astronomer by showing that the bright new star, today called a supernova, that appeared in the skies in 1572 was far beyond the Earth's atmosphere. This, along with Galileo's discovery of sunspots, showed that contrary to Aristotle, the heavens were not perfect and unchanging. Brahe's fame as an astronomer brought him patronage from King Frederick II, allowing him to carry out his historic high-precision measurements of the planets' motions. A contradictory character, Brahe enjoyed lecturing other nobles about the evils of dueling, but had lost his own nose in a youthful duel and had it replaced with a prosthesis made of an alloy of gold and silver. Willing to endure scandal in order to marry a peasant, he nevertheless used the feudal powers given to him by the king to impose harsh forced labor on the inhabitants of his parishes. The result of their work, an Italian-style palace with an observatory on top, surely ranks as one of the most luxurious science labs ever built. Kepler described Brahe as dying of a ruptured bladder after falling from a wagon on the way home from a party, but other contemporary accounts and modern medical analysis suggest mercury poisoning, possibly as a result of court intrigue.

showed that the same laws of motion applied to the heavens as to the earth, and that the gravitational force that made an apple fall was the same as the force that kept the earth's motion from carrying it away from the sun. What was radical about Newton was not his laws of motion but his concept of a universal science of physics.

## 10.1 Kepler's laws

Newton wouldn't have been able to figure out *why* the planets move the way they do if it hadn't been for the astronomer Tycho Brahe (1546-1601) and his protegee Johannes Kepler (1571-1630), who together came up with the first simple and accurate description of *how* the planets actually do move. The difficulty of their task is suggested by figure c, which shows how the relatively simple orbital motions of the earth and Mars combine so that as seen from earth Mars appears to be staggering in loops like a drunken sailor.



c / As the Earth and Mars revolve around the sun at different rates, the combined effect of their motions makes Mars appear to trace a strange, looped path across the background of the distant stars.

Brahe, the last of the great naked-eye astronomers, collected extensive data on the motions of the planets over a period of many years, taking the giant step from the previous observations' accuracy of about 10 minutes of arc (10/60 of a degree) to an unprecedented 1 minute. The quality of his work is all the more remarkable consid-

ering that his observatory consisted of four giant brass protractors mounted upright in his castle in Denmark. Four different observers would simultaneously measure the position of a planet in order to check for mistakes and reduce random errors.

With Brahe's death, it fell to his former assistant Kepler to try to make some sense out of the volumes of data. Kepler, in contradiction to his late boss, had formed a prejudice, a correct one as it turned out, in favor of the theory that the earth and planets revolved around the sun, rather than the earth staying fixed and everything rotating about it. Although motion is relative, it is not just a matter of opinion what circles what. The earth's rotation and revolution about the sun make it a noninertial reference frame, which causes detectable violations of Newton's laws when one attempts to describe sufficiently precise experiments in the earth-fixed frame. Although such direct experiments were not carried out until the 19th century, what convinced everyone of the sun-centered system in the 17th century was that Kepler was able to come up with a surprisingly simple set of mathematical and geometrical rules for describing the planets' motion using the sun-centered assumption. After 900 pages of calculations and many false starts and dead-end ideas, Kepler finally synthesized the data into the following three laws:

**Kepler's elliptical orbit law**

The planets orbit the sun in elliptical orbits with the sun at one focus.

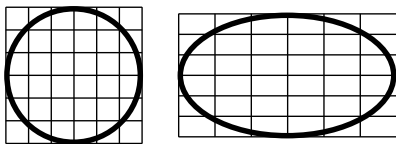
**Kepler's equal-area law**

The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

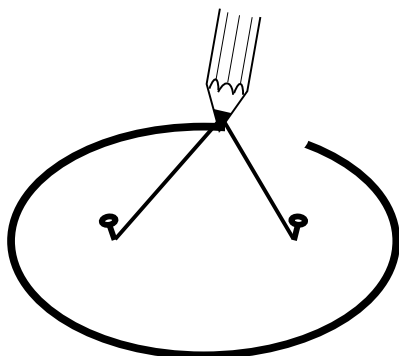
**Kepler's law of periods**

The time required for a planet to orbit the sun, called its period, is proportional to the long axis of the ellipse raised to the  $3/2$  power. The constant of proportionality is the same for all the planets.

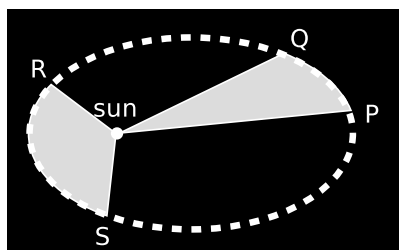
Although the planets' orbits are ellipses rather than circles, most are very close to being circular. The earth's orbit, for instance, is only flattened by 1.7% relative to a circle. In the special case of a planet in a circular orbit, the two foci (plural of "focus") coincide at the center of the circle, and Kepler's elliptical orbit law thus says that the circle is centered on the sun. The equal-area law implies that a planet in a circular orbit moves around the sun with constant speed. For a circular orbit, the law of periods then amounts to a statement that the time for one orbit is proportional to  $r^{3/2}$ , where  $r$  is the radius. If all the planets were moving in their orbits at the same speed, then the time for one orbit would simply depend on the circumference of the circle, so it would only be proportional to  $r$  to the first power. The more drastic dependence on  $r^{3/2}$  means



d / An ellipse is a circle that has been distorted by shrinking and stretching along perpendicular axes.



e / An ellipse can be constructed by tying a string to two pins and drawing like this with the pencil stretching the string taut. Each pin constitutes one focus of the ellipse.



f / If the time interval taken by the planet to move from P to Q is equal to the time interval from R to S, then according to Kepler's equal-area law, the two shaded areas are equal. The planet is moving faster during interval RS than it did during PQ, which Newton later determined was due to the sun's gravitational force accelerating it. The equal-area law predicts exactly how much it will speed up.

that the outer planets must be moving more slowly than the inner planets.

## 10.2 Newton's law of gravity

### The sun's force on the planets obeys an inverse square law.

Kepler's laws were a beautifully simple explanation of what the planets did, but they didn't address why they moved as they did. Did the sun exert a force that pulled a planet toward the center of its orbit, or, as suggested by Descartes, were the planets circulating in a whirlpool of some unknown liquid? Kepler, working in the Aristotelian tradition, hypothesized not just an inward force exerted by the sun on the planet, but also a second force in the direction of motion to keep the planet from slowing down. Some speculated that the sun attracted the planets magnetically.

Once Newton had formulated his laws of motion and taught them to some of his friends, they began trying to connect them to Kepler's laws. It was clear now that an inward force would be needed to bend the planets' paths. This force was presumably an attraction between the sun and each planet. (Although the sun does accelerate in response to the attractions of the planets, its mass is so great that the effect had never been detected by the prenewtonian astronomers.) Since the outer planets were moving slowly along more gently curving paths than the inner planets, their accelerations were apparently less. This could be explained if the sun's force was determined by distance, becoming weaker for the farther planets. Physicists were also familiar with the noncontact forces of electricity and magnetism, and knew that they fell off rapidly with distance, so this made sense.

In the approximation of a circular orbit, the magnitude of the sun's force on the planet would have to be

$$[1] \quad F = ma = mv^2/r \quad .$$

Now although this equation has the magnitude,  $v$ , of the velocity vector in it, what Newton expected was that there would be a more fundamental underlying equation for the force of the sun on a planet, and that that equation would involve the distance,  $r$ , from the sun to the object, but not the object's speed,  $v$  — motion doesn't make objects lighter or heavier.

#### self-check A

If eq. [1] really was generally applicable, what would happen to an object released at rest in some empty region of the solar system? ▶

Answer, p. 533

Equation [1] was thus a useful piece of information which could be related to the data on the planets simply because the planets happened to be going in nearly circular orbits, but Newton wanted



to combine it with other equations and eliminate  $v$  algebraically in order to find a deeper truth.

To eliminate  $v$ , Newton used the equation

$$[2] \quad v = \frac{\text{circumference}}{T} = \frac{2\pi r}{T} \quad .$$

Of course this equation would also only be valid for planets in nearly circular orbits. Plugging this into eq. [1] to eliminate  $v$  gives

$$[3] \quad F = \frac{4\pi^2 mr}{T^2} \quad .$$

This unfortunately has the side-effect of bringing in the period,  $T$ , which we expect on similar physical grounds will not occur in the final answer. That's where the circular-orbit case,  $T \propto r^{3/2}$ , of Kepler's law of periods comes in. Using it to eliminate  $T$  gives a result that depends only on the mass of the planet and its distance from the sun:

$$F \propto m/r^2 \quad . \quad \begin{array}{l} \text{[force of the sun on a planet of mass} \\ m \text{ at a distance } r \text{ from the sun; same} \\ \text{proportionality constant for all the planets]} \end{array}$$

(Since Kepler's law of periods is only a proportionality, the final result is a proportionality rather than an equation, so there is no point in hanging on to the factor of  $4\pi^2$ .)

As an example, the "twin planets" Uranus and Neptune have nearly the same mass, but Neptune is about twice as far from the sun as Uranus, so the sun's gravitational force on Neptune is about four times smaller.

*self-check B*

Fill in the steps leading from equation [3] to  $F \propto m/r^2$ .    ▷ Answer, p. 533

### The forces between heavenly bodies are the same type of force as terrestrial gravity.

OK, but what kind of force was it? It probably wasn't magnetic, since magnetic forces have nothing to do with mass. Then came Newton's great insight. Lying under an apple tree and looking up at the moon in the sky, he saw an apple fall. Might not the earth also attract the moon with the same kind of gravitational force? The moon orbits the earth in the same way that the planets orbit the sun, so maybe the earth's force on the falling apple, the earth's force on the moon, and the sun's force on a planet were all the same type of force.

There was an easy way to test this hypothesis numerically. If it was true, then we would expect the gravitational forces exerted by



60

1



$g$  / The moon's acceleration is  $60^2 = 3600$  times smaller than the apple's.



the earth to follow the same  $F \propto m/r^2$  rule as the forces exerted by the sun, but with a different constant of proportionality appropriate to the earth's gravitational strength. The issue arises now of how to define the distance,  $r$ , between the earth and the apple. An apple in England is closer to some parts of the earth than to others, but suppose we take  $r$  to be the distance from the center of the earth to the apple, i.e., the radius of the earth. (The issue of how to measure  $r$  did not arise in the analysis of the planets' motions because the sun and planets are so small compared to the distances separating them.) Calling the proportionality constant  $k$ , we have

$$\begin{aligned} F_{\text{earth on apple}} &= k m_{\text{apple}}/r_{\text{earth}}^2 \\ F_{\text{earth on moon}} &= k m_{\text{moon}}/d_{\text{earth-moon}}^2 \end{aligned} \quad .$$

Newton's second law says  $a = F/m$ , so

$$\begin{aligned} a_{\text{apple}} &= k / r_{\text{earth}}^2 \\ a_{\text{moon}} &= k / d_{\text{earth-moon}}^2 \end{aligned} \quad .$$

The Greek astronomer Hipparchus had already found 2000 years before that the distance from the earth to the moon was about 60 times the radius of the earth, so if Newton's hypothesis was right, the acceleration of the moon would have to be  $60^2 = 3600$  times less than the acceleration of the falling apple.

Applying  $a = v^2/r$  to the acceleration of the moon yielded an acceleration that was indeed 3600 times smaller than  $9.8 \text{ m/s}^2$ , and Newton was convinced he had unlocked the secret of the mysterious force that kept the moon and planets in their orbits.

### Newton's law of gravity

The proportionality  $F \propto m/r^2$  for the gravitational force on an object of mass  $m$  only has a consistent proportionality constant for various objects if they are being acted on by the gravity of the same object. Clearly the sun's gravitational strength is far greater than the earth's, since the planets all orbit the sun and do not exhibit any very large accelerations caused by the earth (or by one another). What property of the sun gives it its great gravitational strength? Its great volume? Its great mass? Its great temperature? Newton reasoned that if the force was proportional to the mass of the object being acted on, then it would also make sense if the determining factor in the gravitational strength of the object exerting the force was its own mass. Assuming there were no other factors affecting the gravitational force, then the only other thing needed to make quantitative predictions of gravitational forces would be a proportionality constant. Newton called that proportionality constant  $G$ , so here is the complete form of the law of gravity he hypothesized.

## Newton's law of gravity

$$F = \frac{Gm_1m_2}{r^2} \quad [\text{gravitational force between objects of mass } m_1 \text{ and } m_2, \text{ separated by a distance } r; r \text{ is not the radius of anything}]$$

Newton conceived of gravity as an attraction between any two masses in the universe. The constant  $G$  tells us how many newtons the attractive force is for two 1-kg masses separated by a distance of 1 m. The experimental determination of  $G$  in ordinary units (as opposed to the special, nonmetric, units used in astronomy) is described in section 10.5. This difficult measurement was not accomplished until long after Newton's death.

### The units of $G$

example 1

- ▷ What are the units of  $G$ ?
- ▷ Solving for  $G$  in Newton's law of gravity gives

$$G = \frac{Fr^2}{m_1m_2},$$

so the units of  $G$  must be  $\text{N}\cdot\text{m}^2/\text{kg}^2$ . Fully adorned with units, the value of  $G$  is  $6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$ .

### Newton's third law

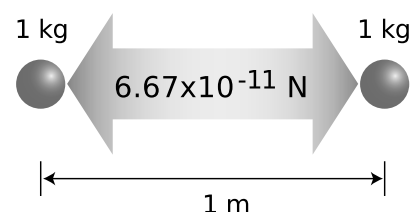
example 2

- ▷ Is Newton's law of gravity consistent with Newton's third law?
- ▷ The third law requires two things. First,  $m_1$ 's force on  $m_2$  should be the same as  $m_2$ 's force on  $m_1$ . This works out, because the product  $m_1m_2$  gives the same result if we interchange the labels 1 and 2. Second, the forces should be in opposite directions. This condition is also satisfied, because Newton's law of gravity refers to an attraction: each mass pulls the other toward itself.

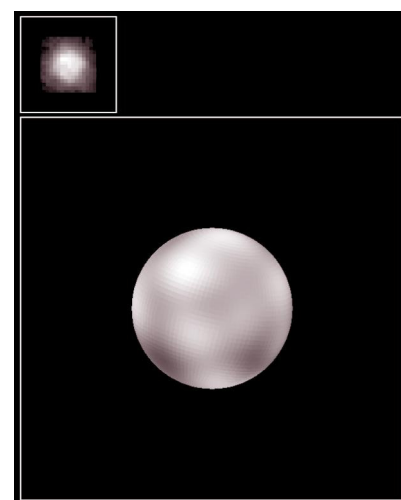
### Pluto and Charon

example 3

- ▷ Pluto's moon Charon is unusually large considering Pluto's size, giving them the character of a double planet. Their masses are  $1.25 \times 10^{22}$  and  $1.9 \times 10^{21}$  kg, and their average distance from one another is  $1.96 \times 10^4$  km. What is the gravitational force between them?
- ▷ If we want to use the value of  $G$  expressed in SI (meter-kilogram-second) units, we first have to convert the distance to  $1.96 \times$



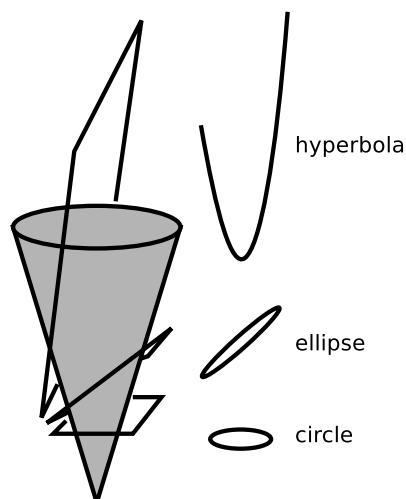
h / Students often have a hard time understanding the physical meaning of  $G$ . It's just a proportionality constant that tells you how strong gravitational forces are. If you could change it, all the gravitational forces all over the universe would get stronger or weaker. Numerically, the gravitational attraction between two 1-kg masses separated by a distance of 1 m is  $6.67 \times 10^{-11} \text{ N}$ , and this is what  $G$  is in SI units.



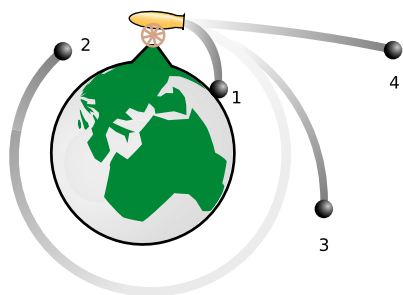
i / Example 3. Computer-enhanced images of Pluto and Charon, taken by the Hubble Space Telescope.

$10^7$  m. The force is

$$\frac{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2) (1.25 \times 10^{22} \text{ kg}) (1.9 \times 10^{21} \text{ kg})}{(1.96 \times 10^7 \text{ m})^2} = 4.1 \times 10^{18} \text{ N}$$



j / The conic sections are the curves made by cutting the surface of an infinite cone with a plane.



k / An imaginary cannon able to shoot cannonballs at very high speeds is placed on top of an imaginary, very tall mountain that reaches up above the atmosphere. Depending on the speed at which the ball is fired, it may end up in a tightly curved elliptical orbit, 1, a circular orbit, 2, a bigger elliptical orbit, 3, or a nearly straight hyperbolic orbit, 4.

The proportionality to  $1/r^2$  in Newton's law of gravity was not entirely unexpected. Proportionalities to  $1/r^2$  are found in many other phenomena in which some effect spreads out from a point. For instance, the intensity of the light from a candle is proportional to  $1/r^2$ , because at a distance  $r$  from the candle, the light has to be spread out over the surface of an imaginary sphere of area  $4\pi r^2$ . The same is true for the intensity of sound from a firecracker, or the intensity of gamma radiation emitted by the Chernobyl reactor. It's important, however, to realize that this is only an analogy. Force does not travel through space as sound or light does, and force is not a substance that can be spread thicker or thinner like butter on toast.

Although several of Newton's contemporaries had speculated that the force of gravity might be proportional to  $1/r^2$ , none of them, even the ones who had learned Newton's laws of motion, had had any luck proving that the resulting orbits would be ellipses, as Kepler had found empirically. Newton did succeed in proving that elliptical orbits would result from a  $1/r^2$  force, but we postpone the proof until the chapter 15 because it can be accomplished much more easily using the concepts of energy and angular momentum.

Newton also predicted that orbits in the shape of hyperbolas should be possible, and he was right. Some comets, for instance, orbit the sun in very elongated ellipses, but others pass through the solar system on hyperbolic paths, never to return. Just as the trajectory of a faster baseball pitch is flatter than that of a more slowly thrown ball, so the curvature of a planet's orbit depends on its speed. A spacecraft can be launched at relatively low speed, resulting in a circular orbit about the earth, or it can be launched at a higher speed, giving a more gently curved ellipse that reaches farther from the earth, or it can be launched at a very high speed which puts it in an even less curved hyperbolic orbit. As you go very far out on a hyperbola, it approaches a straight line, i.e., its curvature eventually becomes nearly zero.

Newton also was able to prove that Kepler's second law (sweeping out equal areas in equal time intervals) was a logical consequence of his law of gravity. Newton's version of the proof is moderately complicated, but the proof becomes trivial once you understand the concept of angular momentum, which will be covered later in the course. The proof will therefore be deferred until section section 15.7.

*self-check C*

Which of Kepler's laws would it make sense to apply to hyperbolic orbits?

▷ Answer, p.

533

- ▷ *Solved problem: Visiting Ceres*                      *page 260, problem 10*
- ▷ *Solved problem: Geosynchronous orbit*            *page 262, problem 16*
- ▷ *Solved problem: Why  $a$  equals  $g$*                       *page 261, problem 11*
- ▷ *Solved problem: Ida and Dactyl*                      *page 261, problem 12*
- ▷ *Solved problem: Another solar system*            *page 261, problem 15*
- ▷ *Solved problem: Weight loss*                          *page 263, problem 19*
- ▷ *Solved problem: The receding moon*                *page 262, problem 17*

**Discussion questions**

**A**     How could Newton find the speed of the moon to plug in to  $a = v^2/r$ ?

**B**     Two projectiles of different mass shot out of guns on the surface of the earth at the same speed and angle will follow the same trajectories, assuming that air friction is negligible. (You can verify this by throwing two objects together from your hand and seeing if they separate or stay side by side.) What corresponding fact would be true for satellites of the earth having different masses?

**C**     What is wrong with the following statement? "A comet in an elliptical orbit speeds up as it approaches the sun, because the sun's force on it is increasing."

**D**     Why would it not make sense to expect the earth's gravitational force on a bowling ball to be inversely proportional to the square of the distance between their surfaces rather than their centers?

**E**     Does the earth accelerate as a result of the moon's gravitational force on it? Suppose two planets were bound to each other gravitationally the way the earth and moon are, but the two planets had equal masses. What would their motion be like?

**F**     Spacecraft normally operate by firing their engines only for a few minutes at a time, and an interplanetary probe will spend months or years on its way to its destination without thrust. Suppose a spacecraft is in a circular orbit around Mars, and it then briefly fires its engines in reverse, causing a sudden decrease in speed. What will this do to its orbit? What about a forward thrust?

## 10.3 Apparent weightlessness

If you ask somebody at the bus stop why astronauts are weightless, you'll probably get one of the following two incorrect answers:

- (1) They're weightless because they're so far from the earth.
- (2) They're weightless because they're moving so fast.

The first answer is wrong, because the vast majority of astronauts never get more than a thousand miles from the earth's surface. The reduction in gravity caused by their altitude is significant, but not 100%. The second answer is wrong because Newton's law of gravity only depends on distance, not speed.

The correct answer is that astronauts in orbit around the earth are not really weightless at all. Their weightlessness is only apparent. If there was no gravitational force on the spaceship, it would obey Newton's first law and move off on a straight line, rather than orbiting the earth. Likewise, the astronauts inside the spaceship are in orbit just like the spaceship itself, with the earth's gravitational force continually twisting their velocity vectors around. The reason they appear to be weightless is that they are in the same orbit as the spaceship, so although the earth's gravity curves their trajectory down toward the deck, the deck drops out from under them at the same rate.

Apparent weightlessness can also be experienced on earth. Any time you jump up in the air, you experience the same kind of apparent weightlessness that the astronauts do. While in the air, you can lift your arms more easily than normal, because gravity does not make them fall any faster than the rest of your body, which is falling out from under them. The Russian air force now takes rich foreign tourists up in a big cargo plane and gives them the feeling of weightlessness for a short period of time while the plane is nose-down and dropping like a rock.

## 10.4 Vector addition of gravitational forces

Pick a flower on earth and you move the farthest star.

*Paul Dirac*

When you stand on the ground, which part of the earth is pulling down on you with its gravitational force? Most people are tempted to say that the effect only comes from the part directly under you, since gravity always pulls straight down. Here are three observations that might help to change your mind:

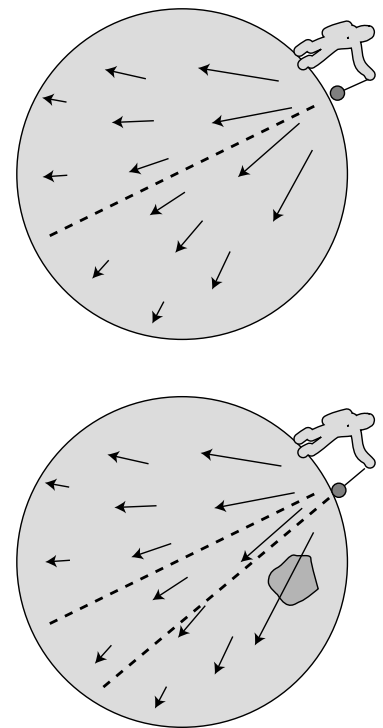
- If you jump up in the air, gravity does not stop affecting you just because you are not touching the earth: gravity is a non-contact force. That means you are not immune from the grav-

ity of distant parts of our planet just because you are not touching them.

- Gravitational effects are not blocked by intervening matter. For instance, in an eclipse of the moon, the earth is lined up directly between the sun and the moon, but only the sun's light is blocked from reaching the moon, not its gravitational force — if the sun's gravitational force on the moon was blocked in this situation, astronomers would be able to tell because the moon's acceleration would change suddenly. A more subtle but more easily observable example is that the tides are caused by the moon's gravity, and tidal effects can occur on the side of the earth facing away from the moon. Thus, far-off parts of the earth are not prevented from attracting you with their gravity just because there is other stuff between you and them.
- Prospectors sometimes search for underground deposits of dense minerals by measuring the direction of the local gravitational forces, i.e., the direction things fall or the direction a plumb bob hangs. For instance, the gravitational forces in the region to the west of such a deposit would point along a line slightly to the east of the earth's center. Just because the total gravitational force on you points down, that doesn't mean that only the parts of the earth directly below you are attracting you. It's just that the sideways components of all the force vectors acting on you come very close to canceling out.

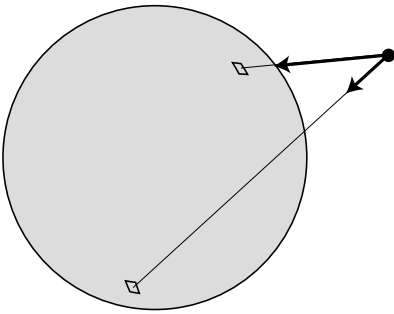
A cubic centimeter of lava in the earth's mantle, a grain of silica inside Mt. Kilimanjaro, and a flea on a cat in Paris are all attracting you with their gravity. What you feel is the vector sum of all the gravitational forces exerted by all the atoms of our planet, and for that matter by all the atoms in the universe.

When Newton tested his theory of gravity by comparing the orbital acceleration of the moon to the acceleration of a falling apple on earth, he assumed he could compute the earth's force on the apple using the distance from the apple to the earth's center. Was he wrong? After all, it isn't just the earth's center attracting the apple, it's the whole earth. A kilogram of dirt a few feet under his backyard in England would have a much greater force on the apple than a kilogram of molten rock deep under Australia, thousands of miles away. There's really no obvious reason why the force should come out right if you just pretend that the earth's whole mass is concentrated at its center. Also, we know that the earth has some parts that are more dense, and some parts that are less dense. The solid crust, on which we live, is considerably less dense than the molten rock on which it floats. By all rights, the computation of the vector sum of all the forces exerted by all the earth's parts should be a horrendous mess.



1 / Gravity only appears to pull straight down because the near perfect symmetry of the earth makes the sideways components of the total force on an object cancel almost exactly. If the symmetry is broken, e.g., by a dense mineral deposit, the total force is a little off to the side.

Actually, Newton had sound mathematical reasons for treating the earth's mass as if it was concentrated at its center. First, although Newton no doubt suspected the earth's density was nonuniform, he knew that the direction of its total gravitational force was very nearly toward the earth's center. That was strong evidence that the distribution of mass was very symmetric, so that we can think of the earth as being made of many layers, like an onion, with each layer having constant density throughout. (Today there is further evidence for symmetry based on measurements of how the vibrations from earthquakes and nuclear explosions travel through the earth.) Newton then concentrated on the gravitational forces exerted by a single such thin shell, and proved the following mathematical theorem, known as the shell theorem:



$m$  / An object outside a spherical shell of mass will feel gravitational forces from every part of the shell — stronger forces from the closer parts, and weaker ones from the parts farther away. The shell theorem states that the vector sum of all the forces is the same as if all the mass had been concentrated at the center of the shell.

If an object lies outside a thin, spherical shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if the shell's mass had been concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

For terrestrial gravity, each shell acts as though its mass was concentrated at the earth's center, so the final result is the same as if the earth's whole mass was concentrated at its center.

The second part of the shell theorem, about the gravitational forces canceling inside the shell, is a little surprising. Obviously the forces would all cancel out if you were at the exact center of a shell, but why should they still cancel out perfectly if you are inside the shell but off-center? The whole idea might seem academic, since we don't know of any hollow planets in our solar system that astronauts could hope to visit, but actually it's a useful result for understanding gravity within the earth, which is an important issue in geology. It doesn't matter that the earth is not actually hollow. In a mine shaft at a depth of, say, 2 km, we can use the shell theorem to tell us that the outermost 2 km of the earth has no net gravitational effect, and the gravitational force is the same as what would be produced if the remaining, deeper, parts of the earth were all concentrated at its center.

*self-check D*

Suppose you're at the bottom of a deep mineshaft, which means you're still quite far from the center of the earth. The shell theorem says that the shell of mass you've gone inside exerts zero total force on you. Discuss which parts of the shell are attracting you in which directions, and how strong these forces are. Explain why it's at least plausible that they cancel.

▷ Answer, p. 533

### Discussion questions

**A** If you hold an apple, does the apple exert a gravitational force on the earth? Is it much weaker than the earth's gravitational force on the apple? Why doesn't the earth seem to accelerate upward when you drop the apple?

**B** When astronauts travel from the earth to the moon, how does the gravitational force on them change as they progress?

**C** How would the gravity in the first-floor lobby of a massive skyscraper compare with the gravity in an open field outside of the city?

**D** In a few billion years, the sun will start undergoing changes that will eventually result in its puffing up into a red giant star. (Near the beginning of this process, the earth's oceans will boil off, and by the end, the sun will probably swallow the earth completely.) As the sun's surface starts to get closer and closer to the earth, how will the earth's orbit be affected?

## 10.5 Weighing the earth

Let's look more closely at the application of Newton's law of gravity to objects on the earth's surface. Since the earth's gravitational force is the same as if its mass was all concentrated at its center, the force on a falling object of mass  $m$  is given by

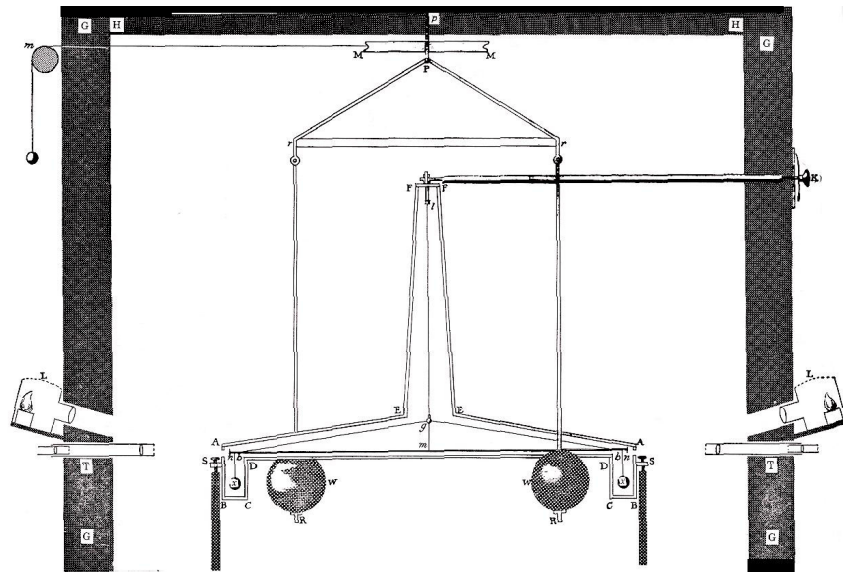
$$F = G M_{\text{earth}} m / r_{\text{earth}}^2 \quad .$$

The object's acceleration equals  $F/m$ , so the object's mass cancels out and we get the same acceleration for all falling objects, as we knew we should:

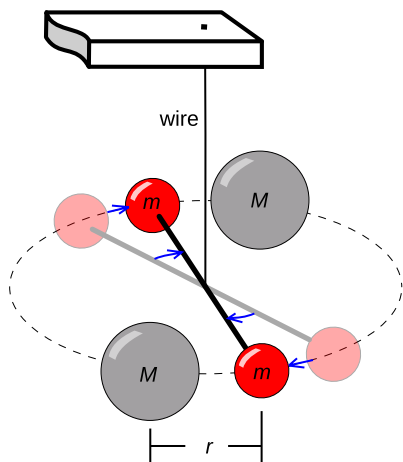
$$g = G M_{\text{earth}} / r_{\text{earth}}^2 \quad .$$

Newton knew neither the mass of the earth nor a numerical value for the constant  $G$ . But if someone could measure  $G$ , then it would be possible for the first time in history to determine the mass of the earth! The only way to measure  $G$  is to measure the gravitational force between two objects of known mass, but that's an exceedingly difficult task, because the force between any two objects of ordinary





n / Cavendish's apparatus. The two large balls are fixed in place, but the rod from which the two small balls hang is free to twist under the influence of the gravitational forces.



o / A simplified version of Cavendish's apparatus.

size is extremely small. The English physicist Henry Cavendish was the first to succeed, using the apparatus shown in figures n and o. The two larger balls were lead spheres 8 inches in diameter, and each one attracted the small ball near it. The two small balls hung from the ends of a horizontal rod, which itself hung by a thin thread. The frame from which the larger balls hung could be rotated by hand about a vertical axis, so that for instance the large ball on the right would pull its neighboring small ball toward us and while the small ball on the left would be pulled away from us. The thread from which the small balls hung would thus be twisted through a small angle, and by calibrating the twist of the thread with known forces, the actual gravitational force could be determined. Cavendish set up the whole apparatus in a room of his house, nailing all the doors shut to keep air currents from disturbing the delicate apparatus. The results had to be observed through telescopes stuck through holes drilled in the walls. Cavendish's experiment provided the first numerical values for  $G$  and for the mass of the earth. The presently accepted value of  $G$  is  $6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ .

Knowing  $G$  not only allowed the determination of the earth's mass but also those of the sun and the other planets. For instance, by observing the acceleration of one of Jupiter's moons, we can infer the mass of Jupiter. The following table gives the distances of the planets from the sun and the masses of the sun and planets. (Other data are given in the back of the book.)

	average distance from the sun, in units of the earth's average distance from the sun	mass, in units of the earth's mass
sun	—	330,000
mercury	0.38	0.056
venus	0.72	0.82
earth	1	1
mars	1.5	0.11
jupiter	5.2	320
saturn	9.5	95
uranus	19	14
neptune	30	17
pluto	39	0.002

### Discussion questions

**A** It would have been difficult for Cavendish to start designing an experiment without at least some idea of the order of magnitude of  $G$ . How could he estimate it in advance to within a factor of 10?

**B** Fill in the details of how one would determine Jupiter's mass by observing the acceleration of one of its moons. Why is it only necessary to know the acceleration of the moon, not the actual force acting on it? Why don't we need to know the mass of the moon? What about a planet that has no moons, such as Venus — how could its mass be found?

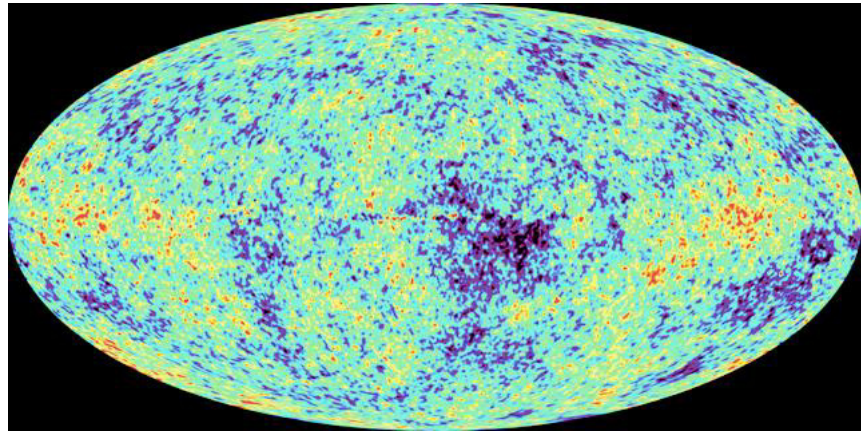
## 10.6 ★ Dark energy

Until recently, physicists thought they understood gravity fairly well. Einstein had modified Newton's theory, but certain characteristics of gravitational forces were firmly established. For one thing, they were always attractive. If gravity always attracts, then it is logical to ask why the universe doesn't collapse. Newton had answered this question by saying that if the universe was infinite in all directions, then it would have no geometric center toward which it would collapse; the forces on any particular star or planet exerted by distant parts of the universe would tend to cancel out by symmetry. More careful calculations, however, show that Newton's universe would have a tendency to collapse on smaller scales: any part of the universe that happened to be slightly more dense than average would contract further, and this contraction would result in stronger gravitational forces, which would cause even more rapid contraction, and so on.

When Einstein overhauled gravity, the same problem reared its ugly head. Like Newton, Einstein was predisposed to believe in a universe that was static, so he added a special repulsive term to his equations, intended to prevent a collapse. This term was not associated with any interaction of mass with mass, but represented merely an overall tendency for space itself to expand unless restrained by

the matter that inhabited it. It turns out that Einstein’s solution, like Newton’s, is unstable. Furthermore, it was soon discovered observationally that the universe was expanding, and this was interpreted by creating the Big Bang model, in which the universe’s current expansion is the aftermath of a fantastically hot explosion.<sup>1</sup> An expanding universe, unlike a static one, was capable of being explained with Einstein’s equations, without any repulsion term. The universe’s expansion would simply slow down over time due to the attractive gravitational forces. After these developments, Einstein said woefully that adding the repulsive term, known as the cosmological constant, had been the greatest blunder of his life.

p / The WMAP probe’s map of the cosmic microwave background is like a “baby picture” of the universe.



This was the state of things until 1999, when evidence began to turn up that the universe’s expansion has been speeding up rather than slowing down! The first evidence came from using a telescope as a sort of time machine: light from a distant galaxy may have taken billions of years to reach us, so we are seeing it as it was far in the past. Looking back in time, astronomers saw the universe expanding at speeds that were lower, rather than higher. At first they were mortified, since this was exactly the opposite of what had been expected. The statistical quality of the data was also not good enough to constitute ironclad proof, and there were worries about systematic errors. The case for an accelerating expansion has however been supported by high-precision mapping of the dim, sky-wide afterglow of the Big Bang, known as the cosmic microwave background. This is discussed in more detail in section 27.4.

So now Einstein’s “greatest blunder” has been resurrected. Since we don’t actually know whether or not this self-repulsion of space has a constant strength, the term “cosmological *constant*” has lost currency. Nowadays physicists usually refer to the phenomenon as “dark energy.” Picking an impressive-sounding name for it should not obscure the fact that we know absolutely nothing about the nature of the effect or why it exists.

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<sup>1</sup>Section section 19.5 presents some of the evidence for the Big Bang.

## Summary

### Selected vocabulary

ellipse . . . . .	a flattened circle; one of the conic sections
conic section . . .	a curve formed by the intersection of a plane and an infinite cone
hyperbola . . . .	another conic section; it does not close back on itself
period . . . . .	the time required for a planet to complete one orbit; more generally, the time for one repetition of some repeating motion
focus . . . . .	one of two special points inside an ellipse: the ellipse consists of all points such that the sum of the distances to the two foci equals a certain number; a hyperbola also has a focus

### Notation

$G$ . . . . .	the constant of proportionality in Newton's law of gravity; the gravitational force of attraction between two 1-kg spheres at a center-to-center distance of 1 m
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## Summary

Kepler deduced three empirical laws from data on the motion of the planets:

**Kepler's elliptical orbit law:** The planets orbit the sun in elliptical orbits with the sun at one focus.

**Kepler's equal-area law:** The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

**Kepler's law of periods:** The time required for a planet to orbit the sun is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

Newton was able to find a more fundamental explanation for these laws. Newton's law of gravity states that the magnitude of the attractive force between any two objects in the universe is given by

$$F = Gm_1m_2/r^2 \quad .$$

Weightlessness of objects in orbit around the earth is only apparent. An astronaut inside a spaceship is simply falling along with the spaceship. Since the spaceship is falling out from under the astronaut, it appears as though there was no gravity accelerating the astronaut down toward the deck.

Gravitational forces, like all other forces, add like vectors. A gravitational force such as we ordinarily feel is the vector sum of all

the forces exerted by all the parts of the earth. As a consequence of this, Newton proved the *shell theorem* for gravitational forces:

If an object lies outside a thin, uniform shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if all the shell's mass was concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** Roy has a mass of 60 kg. Laurie has a mass of 65 kg. They are 1.5 m apart.

(a) What is the magnitude of the gravitational force of the earth on Roy?

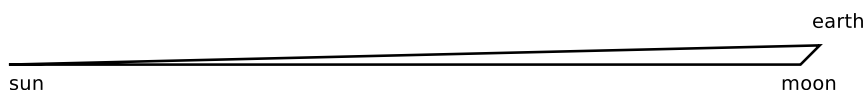
(b) What is the magnitude of Roy's gravitational force on the earth?

(c) What is the magnitude of the gravitational force between Roy and Laurie?

(d) What is the magnitude of the gravitational force between Laurie and the sun? ✓

**2** During a solar eclipse, the moon, earth and sun all lie on the same line, with the moon between the earth and sun. Define your coordinates so that the earth and moon lie at greater  $x$  values than the sun. For each force, give the correct sign as well as the magnitude. (a) What force is exerted on the moon by the sun? (b) On the moon by the earth? (c) On the earth by the sun? (d) What total force is exerted on the sun? (e) On the moon? (f) On the earth? ✓

**3** Suppose that on a certain day there is a crescent moon, and you can tell by the shape of the crescent that the earth, sun and moon form a triangle with a  $135^\circ$  interior angle at the moon's corner. What is the magnitude of the total gravitational force of the earth and the sun on the moon? (If you haven't done problem 2 already, you might want to try it first, since it's easier, and some of its results can be recycled in this problem.) ✓



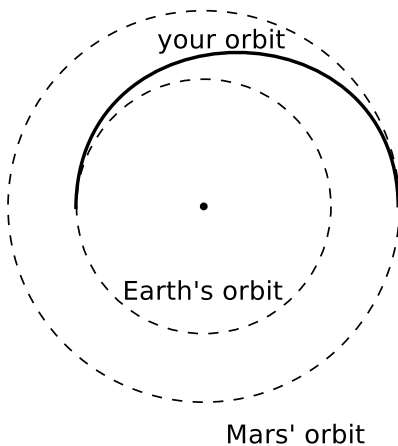
Problem 3.

**4** How high above the Earth's surface must a rocket be in order to have 1/100 the weight it would have at the surface? Express your answer in units of the radius of the Earth. ✓

**5** The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets' orbital radii? ✓

**6** In a Star Trek episode, the Enterprise is in a circular orbit around a planet when something happens to the engines. Spock then tells Kirk that the ship will spiral into the planet's surface unless they can fix the engines. Is this scientifically correct? Why?

- 7 (a) Suppose a rotating spherical body such as a planet has a radius  $r$  and a uniform density  $\rho$ , and the time required for one rotation is  $T$ . At the surface of the planet, the apparent acceleration of a falling object is reduced by the acceleration of the ground out from under it. Derive an equation for the apparent acceleration of gravity,  $g$ , at the equator in terms of  $r$ ,  $\rho$ ,  $T$ , and  $G$ . ✓
- (b) Applying your equation from a, by what fraction is your apparent weight reduced at the equator compared to the poles, due to the Earth's rotation? ✓
- (c) Using your equation from a, derive an equation giving the value of  $T$  for which the apparent acceleration of gravity becomes zero, i.e., objects can spontaneously drift off the surface of the planet. Show that  $T$  only depends on  $\rho$ , and not on  $r$ . ✓
- (d) Applying your equation from c, how long would a day have to be in order to reduce the apparent weight of objects at the equator of the Earth to zero? [Answer: 1.4 hours]
- (e) Astronomers have discovered objects they called pulsars, which emit bursts of radiation at regular intervals of less than a second. If a pulsar is to be interpreted as a rotating sphere beaming out a natural "searchlight" that sweeps past the earth with each rotation, use your equation from c to show that its density would have to be much greater than that of ordinary matter.
- (f) Astrophysicists predicted decades ago that certain stars that used up their sources of energy could collapse, forming a ball of neutrons with the fantastic density of  $\sim 10^{17} \text{ kg/m}^3$ . If this is what pulsars really are, use your equation from c to explain why no pulsar has ever been observed that flashes with a period of less than 1 ms or so.



Problem 8.

- 8 You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth's orbit at one end and tangent to Mars' orbit at the other. Your spacecraft's engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (Assume the orbits of Earth and Mars are circular.) ✓

- 9 (a) If the earth was of uniform density, would your weight be increased or decreased at the bottom of a mine shaft? Explain.
- (b) In real life, objects weigh slightly more at the bottom of a mine shaft. What does that allow us to infer about the Earth? ★

- 10 Ceres, the largest asteroid in our solar system, is a spherical body with a mass 6000 times less than the earth's, and a radius which is 13 times smaller. If an astronaut who weighs 400 N on earth is visiting the surface of Ceres, what is her weight?

▷ Solution, p. 525

**11** Prove, based on Newton's laws of motion and Newton's law of gravity, that all falling objects have the same acceleration if they are dropped at the same location on the earth and if other forces such as friction are unimportant. Do not just say, " $g = 9.8 \text{ m/s}^2$  – it's constant." You are supposed to be *proving* that  $g$  should be the same number for all objects. ▷ Solution, p. 525

**12** The figure shows an image from the Galileo space probe taken during its August 1993 flyby of the asteroid Ida. Astronomers were surprised when Galileo detected a smaller object orbiting Ida. This smaller object, the only known satellite of an asteroid in our solar system, was christened Dactyl, after the mythical creatures who lived on Mount Ida, and who protected the infant Zeus. For scale, Ida is about the size and shape of Orange County, and Dactyl the size of a college campus. Galileo was unfortunately unable to measure the time,  $T$ , required for Dactyl to orbit Ida. If it had, astronomers would have been able to make the first accurate determination of the mass and density of an asteroid. Find an equation for the density,  $\rho$ , of Ida in terms of Ida's known volume,  $V$ , the known radius,  $r$ , of Dactyl's orbit, and the lamentably unknown variable  $T$ . (This is the same technique that was used successfully for determining the masses and densities of the planets that have moons.) ▷ Solution, p. 525



Problem 12.

**13** If a bullet is shot straight up at a high enough velocity, it will never return to the earth. This is known as the escape velocity. We will discuss escape velocity using the concept of energy later in the course, but it can also be gotten at using straightforward calculus. In this problem, you will analyze the motion of an object of mass  $m$  whose initial velocity is *exactly* equal to escape velocity. We assume that it is starting from the surface of a spherically symmetric planet of mass  $M$  and radius  $b$ . The trick is to guess at the general form of the solution, and then determine the solution in more detail. Assume (as is true) that the solution is of the form  $r = kt^p$ , where  $r$  is the object's distance from the center of the planet at time  $t$ , and  $k$  and  $p$  are constants.

(a) Find the acceleration, and use Newton's second law and Newton's law of gravity to determine  $k$  and  $p$ . You should find that the result is independent of  $m$ . ✓

(b) What happens to the velocity as  $t$  approaches infinity?

(c) Determine escape velocity from the Earth's surface. ✓ ∫

**14** Astronomers have recently observed stars orbiting at very high speeds around an unknown object near the center of our galaxy. For stars orbiting at distances of about  $10^{14} \text{ m}$  from the object, the orbital velocities are about  $10^6 \text{ m/s}$ . Assuming the orbits are circular, estimate the mass of the object, in units of the mass of the sun,  $2 \times 10^{30} \text{ kg}$ . If the object was a tightly packed cluster of normal stars, it should be a very bright source of light. Since no visible light is detected coming from it, it is instead believed to be a supermassive black hole. ✓



**15** Astronomers have detected a solar system consisting of three planets orbiting the star Upsilon Andromedae. The planets have been named b, c, and d. Planet b's average distance from the star is 0.059 A.U., and planet c's average distance is 0.83 A.U., where an astronomical unit or A.U. is defined as the distance from the Earth to the sun. For technical reasons, it is possible to determine the ratios of the planets' masses, but their masses cannot presently be determined in absolute units. Planet c's mass is 3.0 times that of planet b. Compare the star's average gravitational force on planet c with its average force on planet b. [Based on a problem by Arnold Arons.]  
▷ Solution, p. 526

**16** Some communications satellites are in orbits called geosynchronous: the satellite takes one day to orbit the earth from west to east, so that as the earth spins, the satellite remains above the same point on the equator. What is such a satellite's altitude above the surface of the earth?  
▷ Solution, p. 526

**17** As is discussed in more detail in example 3 on p. 370, tidal interactions with the earth are causing the moon's orbit to grow gradually larger. Laser beams bounced off of a mirror left on the moon by astronauts have allowed a measurement of the moon's rate of recession, which is about 1 cm per year. This means that the gravitational force acting between earth and moon is decreasing. By what fraction does the force decrease with each 27-day orbit? [Based on a problem by Arnold Arons.]  
▷ Hint, p. 516 ▷ Solution, p. 526

**18** Suppose that we inhabited a universe in which, instead of Newton's law of gravity, we had  $F = k\sqrt{m_1 m_2}/r^2$ , where  $k$  is some constant with different units than  $G$ . (The force is still attractive.) However, we assume that  $a = F/m$  and the rest of Newtonian physics remains true, and we use  $a = F/m$  to define our mass scale, so that, e.g., a mass of 2 kg is one which exhibits half the acceleration when the same force is applied to it as to a 1 kg mass.

- (a) Is this new law of gravity consistent with Newton's third law?
- (b) Suppose you lived in such a universe, and you dropped two unequal masses side by side. What would happen?
- (c) Numerically, suppose a 1.0-kg object falls with an acceleration of  $10 \text{ m/s}^2$ . What would be the acceleration of a rain drop with a mass of 0.1 g? Would you want to go out in the rain?
- (d) If a falling object broke into two unequal pieces while it fell, what would happen?
- (e) Invent a law of gravity that results in behavior that is the opposite of what you found in part b. [Based on a problem by Arnold Arons.]

- 19** (a) A certain vile alien gangster lives on the surface of an asteroid, where his weight is 0.20 N. He decides he needs to lose weight without reducing his consumption of princesses, so he's going to move to a different asteroid where his weight will be 0.10 N. The real estate agent's database has asteroids listed by mass, however, not by surface gravity. Assuming that all asteroids are spherical and have the same density, how should the mass of his new asteroid compare with that of his old one?
- (b) Jupiter's mass is 318 times the Earth's, and its gravity is about twice Earth's. Is this consistent with the results of part a? If not, how do you explain the discrepancy? ▷ Solution, p. 526

- 20** Where would an object have to be located so that it would experience zero total gravitational force from the earth and moon? ✓

- 21** The planet Uranus has a mass of  $8.68 \times 10^{25}$  kg and a radius of  $2.56 \times 10^4$  km. The figure shows the relative sizes of Uranus and Earth.

- (a) Compute the ratio  $g_U/g_E$ , where  $g_U$  is the strength of the gravitational field at the surface of Uranus and  $g_E$  is the corresponding quantity at the surface of the Earth. ✓
- (b) What is surprising about this result? How do you explain it?



Problem 21.

- 22** The International Space Station orbits at an average altitude of about 370 km above sea level. Compute the value of  $g$  at that altitude. ✓

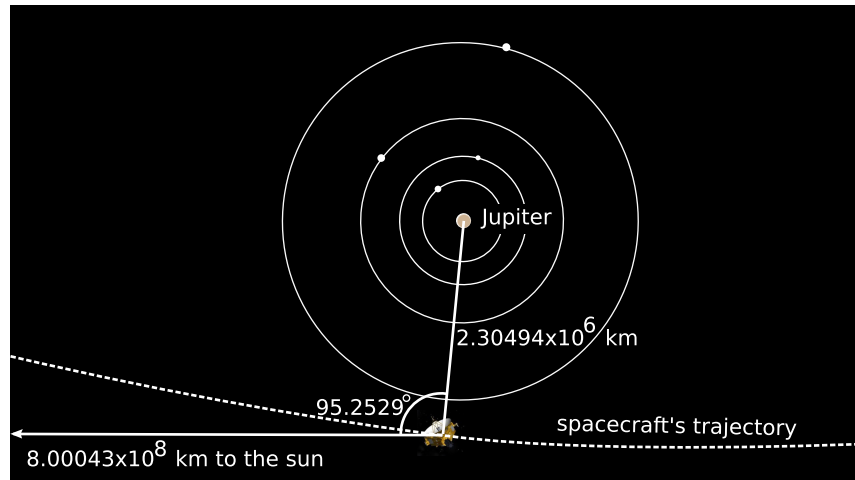
Problem 23: New Horizons at its closest approach to Jupiter. (Jupiter's four largest moons are shown for illustrative purposes.)

The masses are:

sun:  $1.9891 \times 10^{30}$  kg

Jupiter:  $1.8986 \times 10^{27}$  kg

New Horizons: 465.0 kg

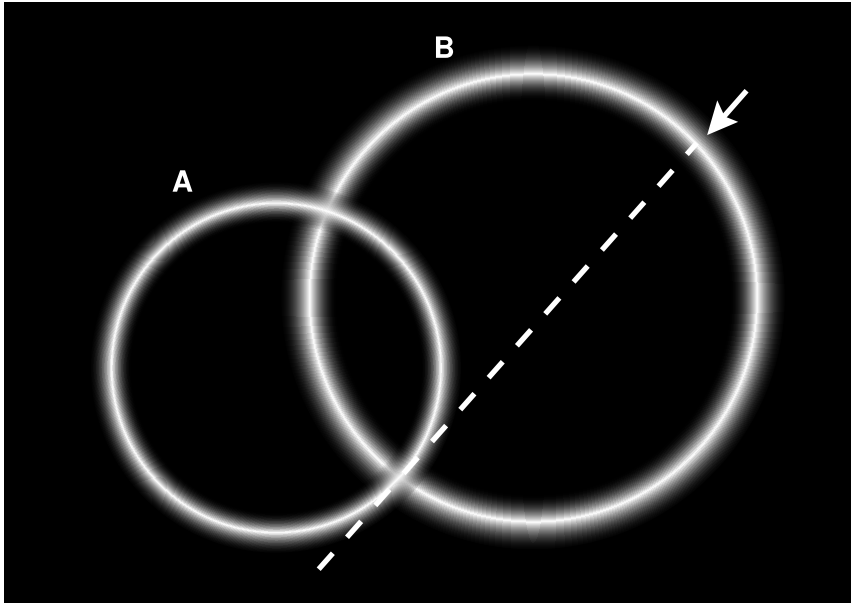


**23** On Feb. 28, 2007, the New Horizons space probe, on its way to a 2015 flyby of Pluto, passed by the planet Jupiter for a gravity-assisted maneuver that increased its speed and changed its course. The dashed line in the figure shows the spacecraft's trajectory, which is curved because of three forces: the force of the exhaust gases from the probe's own engines, the sun's gravitational force, and Jupiter's gravitational force. Find the magnitude of the total gravitational force acting on the probe. You will find that the sun's force is much smaller than Jupiter's, so that the magnitude of the total force is determined almost entirely by Jupiter's force. However, this is a high-precision problem, and you will find that the total force is slightly different from Jupiter's force. ✓

**24** On an airless body such as the moon, there is no atmospheric friction, so it should be possible for a satellite to orbit at a very low altitude, just high enough to keep from hitting the mountains. (a) Suppose that such a body is a smooth sphere of uniform density  $\rho$  and radius  $r$ . Find the velocity required for a ground-skimming orbit. ✓

(b) A typical asteroid has a density of about  $2 \text{ g/cm}^3$ , i.e., twice that of water. (This is a lot lower than the density of the earth's crust, probably indicating that the low gravity is not enough to compact the material very tightly, leaving lots of empty space inside.) Suppose that it is possible for an astronaut in a spacesuit to jump at  $2 \text{ m/s}$ . Find the radius of the largest asteroid on which it would be possible to jump into a ground-skimming orbit. ✓

**25** The figure shows a region of outer space in which two stars have exploded, leaving behind two overlapping spherical shells of gas, which we assume to remain at rest. The figure is a cross-section in a plane containing the shells' centers. A space probe is released with a very small initial speed at the point indicated by the arrow, initially moving in the direction indicated by the dashed line. Without any further information, predict as much as possible about the path followed by the probe and its changes in speed along that path. ★



Problem 25.

**26** Approximate the earth's density as being constant. Find the gravitational field at a point P inside the earth and half-way between the center and the surface. Express your result as a ratio  $g_P/g_S$  relative to the field we experience at the surface.

**27** The earth is divided into solid inner core, a liquid outer core, and a plastic mantle. Physical properties such as density change discontinuously at the boundaries between one layer and the next. Although the density is not completely constant within each region, we will approximate it as being so for the purposes of this problem. (We neglect the crust as well.) Let  $R$  be the radius of the earth as a whole and  $M$  its mass. The following table gives a model of some properties of the three layers, as determined by methods such as the observation of earthquake waves that have propagated from one side of the planet to the other.

<i>region</i>	<i>outer radius/R</i>	<i>mass/M</i>
mantle	1	0.69
outer core	0.55	0.29
inner core	0.19	0.017

The boundary between the mantle and the outer core is referred to as the Gutenberg discontinuity. Let  $g_s$  be the strength of the earth's gravitational field at its surface and  $g_G$  its value at the Gutenberg discontinuity. Find  $g_G/g_s$ . ✓

## Exercise 10: The shell theorem

This exercise is an approximate numerical test of the shell theorem. There are seven masses A-G, each being one kilogram. Masses A-F, each one meter from the center, form a shape like two Egyptian pyramids joined at their bases; this is a rough approximation to a six-kilogram spherical shell of mass. Mass G is five meters from the center of the main group. The class will divide into six groups and split up the work required in order to calculate the vector sum of the six gravitational forces exerted on mass G. Depending on the size of the class, more than one group may be assigned to deal with the contribution of the same mass to the total force, and the redundant groups can check each other's results.



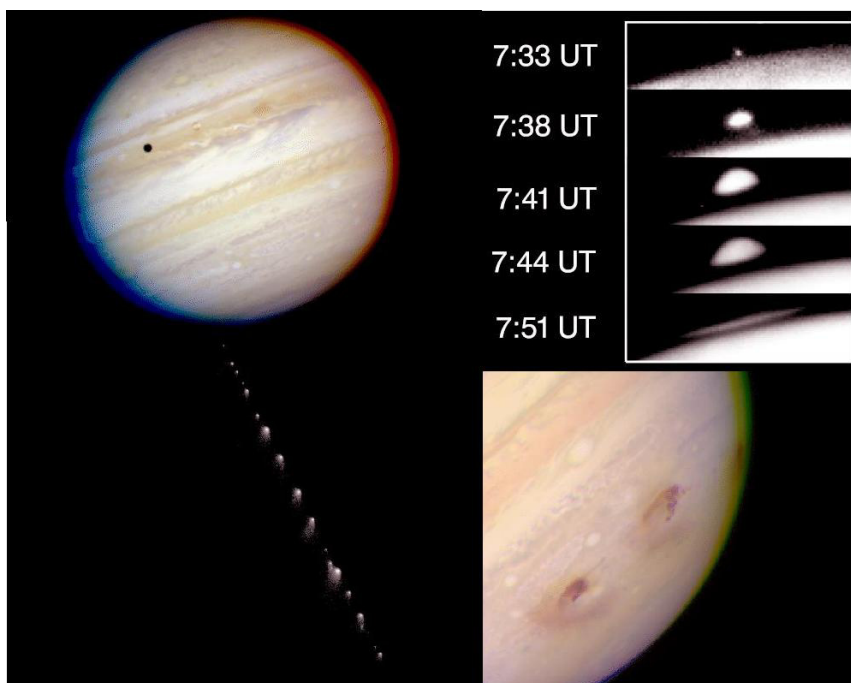
1. Discuss as a class what can be done to simplify the task of calculating the vector sum, and how to organize things so that each group can work in parallel with the others.
2. Each group should write its results on the board in units of piconewtons, retaining five significant figures of precision. Everyone will need to use the same value for the gravitational constant,  $G = 6.6743 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$ .
3. The class will determine the vector sum and compare with the result that would be obtained with the shell theorem.

# **Conservation laws**









In July of 1994, Comet Shoemaker-Levy struck the planet Jupiter, depositing  $7 \times 10^{22}$  joules of energy, and incidentally giving rise to a series of Hollywood movies in which our own planet is threatened by an impact by a comet or asteroid. There is evidence that such an impact caused the extinction of the dinosaurs. Left: Jupiter's gravitational force on the near side of the comet was greater than on the far side, and this difference in force tore up the comet into a string of fragments. Two separate telescope images have been combined to create the illusion of a point of view just behind the comet. (The colored fringes at the edges of Jupiter are artifacts of the imaging system.) Top: A series of images of the plume of superheated gas kicked up by the impact of one of the fragments. The plume is about the size of North America. Bottom: An image after all the impacts were over, showing the damage done.

## Chapter 11

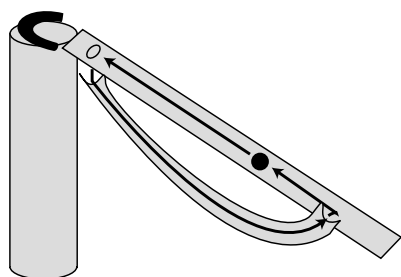
# Conservation of energy

### 11.1 The search for a perpetual motion machine

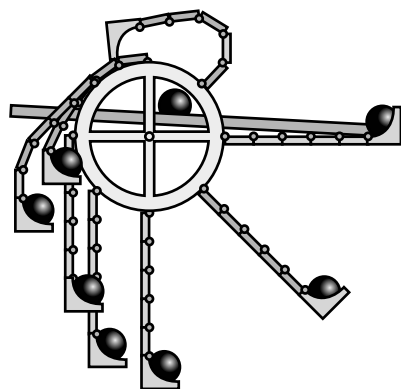
Don't underestimate greed and laziness as forces for progress. Modern chemistry was born from the collision of lust for gold with distaste for the hard work of finding it and digging it up. Failed efforts by generations of alchemists to turn lead into gold led finally to the conclusion that it could not be done: certain substances, the chem-

ical elements, are fundamental, and chemical reactions can neither increase nor decrease the amount of an element such as gold.

Now flash forward to the early industrial age. Greed and laziness have created the factory, the train, and the ocean liner, but in each of these is a boiler room where someone gets sweaty shoveling the coal to fuel the steam engine. Generations of inventors have tried to create a machine, called a perpetual motion machine, that would run forever without fuel. Such a machine is not forbidden by Newton's laws of motion, which are built around the concepts of force and inertia. Force is free, and can be multiplied indefinitely with pulleys, gears, or levers. The principle of inertia seems even to encourage the belief that a cleverly constructed machine might not ever run down.



a / The magnet draws the ball to the top of the ramp, where it falls through the hole and rolls back to the bottom.



b / As the wheel spins clockwise, the flexible arms sweep around and bend and unbend. By dropping off its ball on the ramp, the arm is supposed to make itself lighter and easier to lift over the top. Picking its own ball back up again on the right, it helps to pull the right side down.

Figures a and b show two of the innumerable perpetual motion machines that have been proposed. The reason these two examples don't work is not much different from the reason all the others have failed. Consider machine a. Even if we assume that a properly shaped ramp would keep the ball rolling smoothly through each cycle, friction would always be at work. The designer imagined that the machine would repeat the same motion over and over again, so that every time it reached a given point its speed would be exactly the same as the last time. But because of friction, the speed would actually be reduced a little with each cycle, until finally the ball would no longer be able to make it over the top.

Friction has a way of creeping into all moving systems. The rotating earth might seem like a perfect perpetual motion machine, since it is isolated in the vacuum of outer space with nothing to exert frictional forces on it. But in fact our planet's rotation has slowed drastically since it first formed, and the earth continues to slow its rotation, making today just a little longer than yesterday. The very subtle source of friction is the tides. The moon's gravity raises bulges in the earth's oceans, and as the earth rotates the bulges progress around the planet. Where the bulges encounter land, there is friction, which slows the earth's rotation very gradually.

## 11.2 Energy

The analysis based on friction is somewhat superficial, however. One could understand friction perfectly well and yet imagine the following situation. Astronauts bring back a piece of magnetic ore from the moon which does not behave like ordinary magnets. A normal bar magnet, c/1, attracts a piece of iron essentially directly toward it, and has no left- or right-handedness. The moon rock, however, exerts forces that form a whirlpool pattern around it, 2. NASA goes to a machine shop and has the moon rock put in a lathe and machined down to a smooth cylinder, 3. If we now release a ball bearing on the surface of the cylinder, the magnetic force whips it

around and around at ever higher speeds. Of course there is some friction, but there is a net gain in speed with each revolution.

Physicists would lay long odds against the discovery of such a moon rock, not just because it breaks the rules that magnets normally obey but because, like the alchemists, they have discovered a very deep and fundamental principle of nature which forbids certain things from happening. The first alchemist who deserved to be called a chemist was the one who realized one day, “In all these attempts to create gold where there was none before, all I’ve been doing is shuffling the same atoms back and forth among different test tubes. The only way to increase the amount of gold in my laboratory is to bring some in through the door.” It was like having some of your money in a checking account and some in a savings account. Transferring money from one account into the other doesn’t change the total amount.

We say that the number of grams of gold is a *conserved* quantity. In this context, the word “conserve” does not have its usual meaning of trying not to waste something. In physics, a conserved quantity is something that you wouldn’t be able to get rid of even if you wanted to. Conservation laws in physics always refer to a *closed system*, meaning a region of space with boundaries through which the quantity in question is not passing. In our example, the alchemist’s laboratory is a closed system because no gold is coming in or out through the doors.

#### Conservation of mass

#### example 1

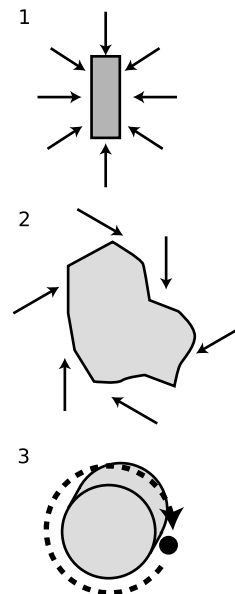
In figure d, the stream of water is fatter near the mouth of the faucet, and skinnier lower down. This is because the water speeds up as it falls. If the cross-sectional area of the stream was equal all along its length, then the rate of flow through a lower cross-section would be greater than the rate of flow through a cross-section higher up. Since the flow is steady, the amount of water between the two cross-sections stays constant. The cross-sectional area of the stream must therefore shrink in inverse proportion to the increasing speed of the falling water. This is an example of conservation of mass.

In general, the amount of any particular substance is not conserved. Chemical reactions can change one substance into another, and nuclear reactions can even change one element into another. The total mass of all substances is however conserved:

#### the law of conservation of mass

The total mass of a closed system always remains constant. Energy cannot be created or destroyed, but only transferred from one system to another.

A similar lightbulb eventually lit up in the heads of the people



c / A mysterious moon rock makes a perpetual motion machine.



d / Example 1.

who had been frustrated trying to build a perpetual motion machine. In perpetual motion machine a, consider the motion of one of the balls. It performs a cycle of rising and falling. On the way down it gains speed, and coming up it slows back down. Having a greater speed is like having more money in your checking account, and being high up is like having more in your savings account. The device is simply shuffling funds back and forth between the two. Having more balls doesn't change anything fundamentally. Not only that, but friction is always draining off money into a third "bank account:" heat. The reason we rub our hands together when we're cold is that kinetic friction heats things up. The continual buildup in the "heat account" leaves less and less for the "motion account" and "height account," causing the machine eventually to run down.

These insights can be distilled into the following basic principle of physics:

### **the law of conservation of energy**

It is possible to give a numerical rating, called energy, to the state of a physical system. The total energy is found by adding up contributions from characteristics of the system such as motion of objects in it, heating of the objects, and the relative positions of objects that interact via forces. The total energy of a closed system always remains constant. Energy cannot be created or destroyed, but only transferred from one system to another.

The moon rock story violates conservation of energy because the rock-cylinder and the ball together constitute a closed system. Once the ball has made one revolution around the cylinder, its position relative to the cylinder is exactly the same as before, so the numerical energy rating associated with its position is the same as before. Since the total amount of energy must remain constant, it is impossible for the ball to have a greater speed after one revolution. If it had picked up speed, it would have more energy associated with motion, the same amount of energy associated with position, and a little more energy associated with heating through friction. There cannot be a net increase in energy.

#### *Converting one form of energy to another* *example 2*

*Dropping a rock:* The rock loses energy because of its changing position with respect to the earth. Nearly all that energy is transformed into energy of motion, except for a small amount lost to heat created by air friction.

*Sliding in to home base:* The runner's energy of motion is nearly all converted into heat via friction with the ground.

*Accelerating a car:* The gasoline has energy stored in it, which is released as heat by burning it inside the engine. Perhaps 10%

of this heat energy is converted into the car's energy of motion. The rest remains in the form of heat, which is carried away by the exhaust.

*Cruising in a car:* As you cruise at constant speed in your car, all the energy of the burning gas is being converted into heat. The tires and engine get hot, and heat is also dissipated into the air through the radiator and the exhaust.

*Stepping on the brakes:* All the energy of the car's motion is converted into heat in the brake shoes.

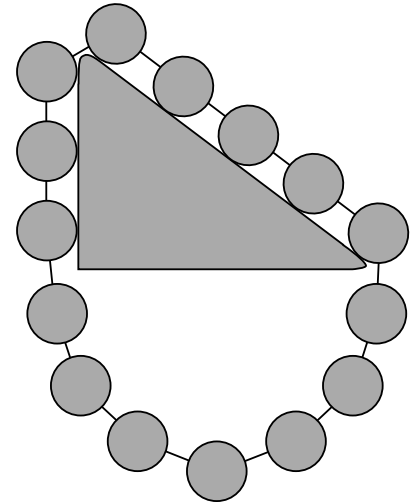
#### Stevin's machine

#### example 3

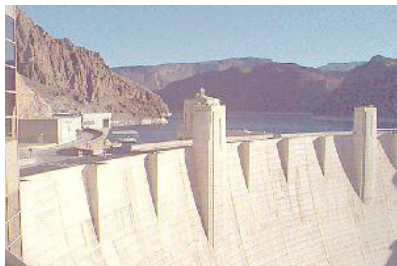
The Dutch mathematician and engineer Simon Stevin proposed the imaginary machine shown in figure e, which he had inscribed on his tombstone. This is an interesting example, because it shows a link between the force concept used earlier in this course, and the energy concept being developed now.

The point of the imaginary machine is to show the mechanical advantage of an inclined plane. In this example, the triangle has the proportions 3-4-5, but the argument works for any right triangle. We imagine that the chain of balls slides without friction, so that no energy is ever converted into heat. If we were to slide the chain clockwise by one step, then each ball would take the place of the one in front of it, and the overall configuration would be exactly the same. Since energy is something that only depends on the state of the system, the energy would have to be the same. Similarly for a counterclockwise rotation, no energy of position would be released by gravity. This means that if we place the chain on the triangle, and release it at rest, it can't start moving, because there would be no way for it to convert energy of position into energy of motion. Thus the chain must be perfectly balanced. Now by symmetry, the arc of the chain hanging underneath the triangle has equal tension at both ends, so removing this arc wouldn't affect the balance of the rest of the chain. This means that a weight of three units hanging vertically balances a weight of five units hanging diagonally along the hypotenuse.

The mechanical advantage of the inclined plane is therefore  $5/3$ , which is exactly the same as the result,  $1/\sin\theta$ , that we got on p. 211 by analyzing force vectors. What this shows is that Newton's laws and conservation laws are not logically separate, but rather are very closely related descriptions of nature. In the cases where Newton's laws are true, they give the same answers as the conservation laws. This is an example of a more general idea, called the correspondence principle, about how science progresses over time. When a newer, more general theory is proposed to replace an older theory, the new theory must agree with the old one in the realm of applicability of the old theory, since the old theory only became accepted as a valid theory by being ver-



e / Example 3.



Discussion question A. The water behind the Hoover Dam has energy because of its position relative to the planet earth, which is attracting it with a gravitational force. Letting water down to the bottom of the dam converts that energy into energy of motion. When the water reaches the bottom of the dam, it hits turbine blades that drive generators, and its energy of motion is converted into electrical energy.

ified experimentally in a variety of experiments. In other words, the new theory must be backward-compatible with the old one. Even though conservation laws can prove things that Newton's laws can't (that perpetual motion is impossible, for example), they aren't going to *disprove* Newton's laws when applied to mechanical systems where we already knew Newton's laws were valid.

### Discussion question

**A** Hydroelectric power (water flowing over a dam to spin turbines) appears to be completely free. Does this violate conservation of energy? If not, then what is the ultimate source of the electrical energy produced by a hydroelectric plant?

**B** How does the proof in example 3 fail if the assumption of a frictionless surface doesn't hold?

## 11.3 A numerical scale of energy

Energy comes in a variety of forms, and physicists didn't discover all of them right away. They had to start somewhere, so they picked one form of energy to use as a standard for creating a numerical energy scale. (In fact the history is complicated, and several different energy units were defined before it was realized that there was a single general energy concept that deserved a single consistent unit of measurement.) One practical approach is to define an energy unit based on heating water. The SI unit of energy is the joule, J, (rhymes with "cool"), named after the British physicist James Joule. One Joule is the amount of energy required in order to heat 0.24 g of water by  $1^{\circ}\text{C}$ . The number 0.24 is not worth memorizing.

Note that heat, which is a form of energy, is completely different from temperature, which is not. Twice as much heat energy is required to prepare two cups of coffee as to make one, but two cups of coffee mixed together don't have double the temperature. In other words, the temperature of an object tells us how hot it is, but the heat energy contained in an object also takes into account the object's mass and what it is made of.<sup>1</sup>

Later we will encounter other quantities that are conserved in physics, such as momentum and angular momentum, and the method for defining them will be similar to the one we have used for energy: pick some standard form of it, and then measure other forms by comparison with this standard. The flexible and adaptable nature of this procedure is part of what has made conservation laws such a durable basis for the evolution of physics.

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<sup>1</sup>In standard, formal terminology, there is another, finer distinction. The word "heat" is used only to indicate an amount of energy that is transferred, whereas "thermal energy" indicates an amount of energy contained in an object. I'm informal on this point, and refer to both as heat, but you should be aware of the distinction.

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*Heating a swimming pool* *example 4*

▷ If electricity costs 3.9 cents per MJ (1 MJ = 1 megajoule =  $10^6$  J), how much does it cost to heat a 26000-gallon swimming pool from  $10^\circ\text{C}$  to  $18^\circ\text{C}$ ?

▷ Converting gallons to  $\text{cm}^3$  gives

$$26000 \text{ gallons} \times \frac{3780 \text{ cm}^3}{1 \text{ gallon}} = 9.8 \times 10^7 \text{ cm}^3 \quad .$$

Water has a density of 1 gram per cubic centimeter, so the mass of the water is  $9.8 \times 10^7$  g. One joule is sufficient to heat 0.24 g by  $1^\circ\text{C}$ , so the energy needed to heat the swimming pool is

$$\begin{aligned} 1 \text{ J} \times \frac{9.8 \times 10^7 \text{ g}}{0.24 \text{ g}} \times \frac{8^\circ\text{C}}{1^\circ\text{C}} &= 3.3 \times 10^9 \text{ J} \\ &= 3.3 \times 10^3 \text{ MJ} \quad . \end{aligned}$$

The cost of the electricity is  $(3.3 \times 10^3 \text{ MJ})(\$0.039/\text{MJ}) = \$130$ .

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*Irish coffee* *example 5*

▷ You make a cup of Irish coffee out of 300 g of coffee at  $100^\circ\text{C}$  and 30 g of pure ethyl alcohol at  $20^\circ\text{C}$ . One Joule is enough energy to produce a change of  $1^\circ\text{C}$  in 0.42 g of ethyl alcohol (i.e., alcohol is easier to heat than water). What temperature is the final mixture?

▷ Adding up all the energy after mixing has to give the same result as the total before mixing. We let the subscript  $i$  stand for the initial situation, before mixing, and  $f$  for the final situation, and use subscripts  $c$  for the coffee and  $a$  for the alcohol. In this notation, we have

total initial energy = total final energy

$$E_{ci} + E_{ai} = E_{cf} + E_{af} \quad .$$

We assume coffee has the same heat-carrying properties as water. Our information about the heat-carrying properties of the two substances is stated in terms of the *change* in energy required for a certain *change* in temperature, so we rearrange the equation to express everything in terms of energy differences:

$$E_{af} - E_{ai} = E_{ci} - E_{cf} \quad .$$

Using the given ratios of temperature change to energy change, we have

$$\begin{aligned} E_{ci} - E_{cf} &= (T_{ci} - T_{cf})(m_c)/(0.24 \text{ g}) \\ E_{af} - E_{ai} &= (T_{af} - T_{ai})(m_a)/(0.42 \text{ g}) \end{aligned}$$

Setting these two quantities to be equal, we have

$$(T_{af} - T_{ai})(m_a)/(0.42 \text{ g}) = (T_{ci} - T_{cf})(m_c)/(0.24 \text{ g}) \quad .$$



In the final mixture the two substances must be at the same temperature, so we can use a single symbol  $T_f = T_{cf} = T_{af}$  for the two quantities previously represented by two different symbols,

$$(T_f - T_{ai})(m_a)/(0.42 \text{ g}) = (T_{ci} - T_f)(m_c)/(0.24 \text{ g}) \quad .$$

Solving for  $T_f$  gives

$$\begin{aligned} T_f &= \frac{T_{ci} \frac{m_c}{0.24} + T_{ai} \frac{m_a}{0.42}}{\frac{m_c}{0.24} + \frac{m_a}{0.42}} \\ &= 96^\circ\text{C} \quad . \end{aligned}$$

Once a numerical scale of energy has been established for some form of energy such as heat, it can easily be extended to other types of energy. For instance, the energy stored in one gallon of gasoline can be determined by putting some gasoline and some water in an insulated chamber, igniting the gas, and measuring the rise in the water's temperature. (The fact that the apparatus is known as a "bomb calorimeter" will give you some idea of how dangerous these experiments are if you don't take the right safety precautions.) Here are some examples of other types of energy that can be measured using the same units of joules:

type of energy	example
chemical energy released by burning	About 50 MJ are released by burning a kg of gasoline.
energy required to break an object	When a person suffers a spiral fracture of the thighbone (a common type in skiing accidents), about 2 J of energy go into breaking the bone.
energy required to melt a solid substance	7 MJ are required to melt 1 kg of tin.
chemical energy released by digesting food	A bowl of Cheerios with milk provides us with about 800 kJ of usable energy.
raising a mass against the force of gravity	Lifting 1.0 kg through a height of 1.0 m requires 9.8 J.
nuclear energy released in fission	1 kg of uranium oxide fuel consumed by a reactor releases $2 \times 10^{12}$ J of stored nuclear energy.

It is interesting to note the disproportion between the megajoule energies we consume as food and the joule-sized energies we expend in physical activities. If we could perceive the flow of energy around us the way we perceive the flow of water, eating a bowl of cereal would be like swallowing a bathtub's worth of energy, the continual loss of body heat to one's environment would be like an energy-hose left on all day, and lifting a bag of cement would be like flicking it with a few tiny energy-drops. The human body is tremendously

inefficient. The calories we “burn” in heavy exercise are almost all dissipated directly as body heat.

*You take the high road and I'll take the low road.*      *example 6*

▷ Figure f shows two ramps which two balls will roll down. Compare their final speeds, when they reach point B. Assume friction is negligible.

▷ Each ball loses some energy because of its decreasing height above the earth, and conservation of energy says that it must gain an equal amount of energy of motion (minus a little heat created by friction). The balls lose the same amount of height, so their final speeds must be equal.

It's impressive to note the complete impossibility of solving this problem using only Newton's laws. Even if the shape of the track had been given mathematically, it would have been a formidable task to compute the balls' final speed based on vector addition of the normal force and gravitational force at each point along the way.

### How new forms of energy are discovered

Textbooks often give the impression that a sophisticated physics concept was created by one person who had an inspiration one day, but in reality it is more in the nature of science to rough out an idea and then gradually refine it over many years. The idea of energy was tinkered with from the early 1800's on, and new types of energy kept getting added to the list.

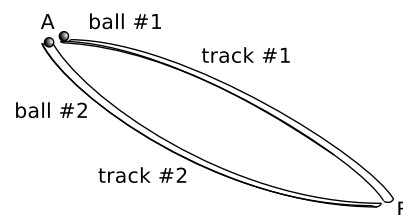
To establish the existence of a new form of energy, a physicist has to

(1) show that it could be converted to and from other forms of energy; and

(2) show that it related to some definite measurable property of the object, for example its temperature, motion, position relative to another object, or being in a solid or liquid state.

For example, energy is released when a piece of iron is soaked in water, so apparently there is some form of energy already stored in the iron. The release of this energy can also be related to a definite measurable property of the chunk of metal: it turns reddish-orange. There has been a chemical change in its physical state, which we call rusting.

Although the list of types of energy kept getting longer and longer, it was clear that many of the types were just variations on a theme. There is an obvious similarity between the energy needed to melt ice and to melt butter, or between the rusting of iron and many other chemical reactions. The topic of the next chapter is how this process of simplification reduced all the types of energy to a very small number (four, according to the way I've chosen to



f / Example 6.

count them).

It might seem that if the principle of conservation of energy ever appeared to be violated, we could fix it up simply by inventing some new type of energy to compensate for the discrepancy. This would be like balancing your checkbook by adding in an imaginary deposit or withdrawal to make your figures agree with the bank's statements. Step (2) above guards against this kind of chicanery. In the 1920s there were experiments that suggested energy was not conserved in radioactive processes. Precise measurements of the energy released in the radioactive decay of a given type of atom showed inconsistent results. One atom might decay and release, say,  $1.1 \times 10^{-10}$  J of energy, which had presumably been stored in some mysterious form in the nucleus. But in a later measurement, an atom of exactly the same type might release  $1.2 \times 10^{-10}$  J. Atoms of the same type are supposed to be identical, so both atoms were thought to have started out with the same energy. If the amount released was random, then apparently the total amount of energy was not the same after the decay as before, i.e., energy was not conserved.

Only later was it found that a previously unknown particle, which is very hard to detect, was being spewed out in the decay. The particle, now called a neutrino, was carrying off some energy, and if this previously unsuspected form of energy was added in, energy was found to be conserved after all. The discovery of the energy discrepancies is seen with hindsight as being step (1) in the establishment of a new form of energy, and the discovery of the neutrino was step (2). But during the decade or so between step (1) and step (2) (the accumulation of evidence was gradual), physicists had the admirable honesty to admit that the cherished principle of conservation of energy might have to be discarded.

#### *self-check A*

How would you carry out the two steps given above in order to establish that some form of energy was stored in a stretched or compressed spring?

▷ Answer, p. 533

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#### **Mass Into Energy**

Einstein showed that mass itself could be converted to and from energy, according to his celebrated equation  $E = mc^2$ , in which  $c$  is the speed of light. We thus speak of mass as simply another form of energy, and it is valid to measure it in units of joules. The mass of a 15-gram pencil corresponds to about  $1.3 \times 10^{15}$  J. The issue is largely academic in the case of the pencil, because very violent processes such as nuclear reactions are required in order to convert any significant fraction of an object's mass into energy. Cosmic rays, however, are continually striking you and your surroundings and converting part of their energy of motion into the mass of newly created particles. A single high-energy cosmic ray can create a "shower" of millions of previously nonexistent particles when it strikes the atmosphere. Einstein's theories are discussed later in this book.

Even today, when the energy concept is relatively mature and stable, a new form of energy has been proposed based on observations of distant galaxies whose light began its voyage to us billions of years ago. Astronomers have found that the universe's continuing expansion, resulting from the Big Bang, has not been decelerating as rapidly in the last few billion years as would have been expected from gravitational forces. They suggest that a new form of energy may be at work.

### Discussion question

**A** I'm not making this up. XS Energy Drink has ads that read like this: *All the "Energy" ... Without the Sugar! Only 8 Calories!* Comment on this.

## 11.4 Kinetic energy

The technical term for the energy associated with motion is kinetic energy, from the Greek word for motion. (The root is the same as the root of the word "cinema" for a motion picture, and in French the term for kinetic energy is "énergie cinétique.") To find how much kinetic energy is possessed by a given moving object, we must convert all its kinetic energy into heat energy, which we have chosen as the standard reference type of energy. We could do this, for example, by firing projectiles into a tank of water and measuring the increase in temperature of the water as a function of the projectile's mass and velocity. Consider the following data from a series of three such experiments:

<b>m</b> (kg)	<b>v</b> (m/s)	<b>energy</b> (J)
1.00	1.00	0.50
1.00	2.00	2.00
2.00	1.00	1.00

Comparing the first experiment with the second, we see that doubling the object's velocity doesn't just double its energy, it quadruples it. If we compare the first and third lines, however, we find that doubling the mass only doubles the energy. This suggests that kinetic energy is proportional to mass and to the square of velocity,  $KE \propto mv^2$ , and further experiments of this type would indeed establish such a general rule. The proportionality factor equals 0.5 because of the design of the metric system, so the kinetic energy of a moving object is given by

$$KE = \frac{1}{2}mv^2 \quad .$$

The metric system is based on the meter, kilogram, and second, with other units being derived from those. Comparing the units on the left and right sides of the equation shows that the joule can be reexpressed in terms of the basic units as  $\text{kg} \cdot \text{m}^2/\text{s}^2$ .

Students are often mystified by the occurrence of the factor of 1/2, but it is less obscure than it looks. The metric system was

designed so that some of the equations relating to energy would come out looking simple, at the expense of some others, which had to have inconvenient conversion factors in front. If we were using the old British Engineering System of units in this course, then we'd have the British Thermal Unit (BTU) as our unit of energy. In that system, the equation you'd learn for kinetic energy would have an inconvenient proportionality constant,  $KE = (1.29 \times 10^{-3}) mv^2$ , with  $KE$  measured in units of BTUs,  $v$  measured in feet per second, and so on. At the expense of this inconvenient equation for kinetic energy, the designers of the British Engineering System got a simple rule for calculating the energy required to heat water: one BTU per degree Fahrenheit per pound. The inventor of kinetic energy, Thomas Young, actually defined it as  $KE = mv^2$ , which meant that all his other equations had to be different from ours by a factor of two. All these systems of units work just fine as long as they are not combined with one another in an inconsistent way.

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*Energy released by a comet impact* *example 7*

▷ Comet Shoemaker-Levy, which struck the planet Jupiter in 1994, had a mass of roughly  $4 \times 10^{13}$  kg, and was moving at a speed of 60 km/s. Compare the kinetic energy released in the impact to the total energy in the world's nuclear arsenals, which is  $2 \times 10^{19}$  J. Assume for the sake of simplicity that Jupiter was at rest.

▷ Since we assume Jupiter was at rest, we can imagine that the comet stopped completely on impact, and 100% of its kinetic energy was converted to heat and sound. We first convert the speed to mks units,  $v = 6 \times 10^4$  m/s, and then plug in to the equation to find that the comet's kinetic energy was roughly  $7 \times 10^{22}$  J, or about 3000 times the energy in the world's nuclear arsenals.

Is there any way to derive the equation  $KE = (1/2)mv^2$  mathematically from first principles? No, it is purely empirical. The factor of 1/2 in front is definitely not derivable, since it is different in different systems of units. The proportionality to  $v^2$  is not even quite correct; experiments have shown deviations from the  $v^2$  rule at high speeds, an effect that is related to Einstein's theory of relativity. Only the proportionality to  $m$  is inevitable. The whole energy concept is based on the idea that we add up energy contributions from all the objects within a system. Based on this philosophy, it is logically necessary that a 2-kg object moving at 1 m/s have the same kinetic energy as two 1-kg objects moving side-by-side at the same speed.

### Energy and relative motion

Although I mentioned Einstein's theory of relativity above, it's more relevant right now to consider how conservation of energy relates to the simpler Galilean idea, which we've already studied, that motion is relative. Galileo's Aristotelian enemies (and it is no exaggeration to call them enemies!) would probably have objected to

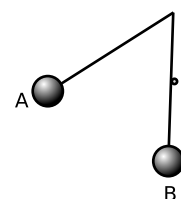
conservation of energy. After all, the Galilean idea that an object in motion will continue in motion indefinitely in the absence of a force is not so different from the idea that an object's kinetic energy stays the same unless there is a mechanism like frictional heating for converting that energy into some other form.

More subtly, however, it's not immediately obvious that what we've learned so far about energy is strictly mathematically consistent with the principle that motion is relative. Suppose we verify that a certain process, say the collision of two pool balls, conserves energy as measured in a certain frame of reference: the sum of the balls' kinetic energies before the collision is equal to their sum after the collision. (In reality we'd need to add in other forms of energy, like heat and sound, that are liberated by the collision, but let's keep it simple.) But what if we were to measure everything in a frame of reference that was in a different state of motion? A particular pool ball might have less kinetic energy in this new frame; for example, if the new frame of reference was moving right along with it, its kinetic energy in that frame would be zero. On the other hand, some other balls might have a greater kinetic energy in the new frame. It's not immediately obvious that the total energy before the collision will still equal the total energy after the collision. After all, the equation for kinetic energy is fairly complicated, since it involves the square of the velocity, so it would be surprising if everything still worked out in the new frame of reference. It *does* still work out. Homework problem 13 in this chapter gives a simple numerical example, and the general proof is taken up in problem 15 on p. 363 (with the solution given in the back of the book).

### Discussion questions

**A** Suppose that, like Young or Einstein, you were trying out different equations for kinetic energy to see if they agreed with the experimental data. Based on the meaning of positive and negative signs of velocity, why would you suspect that a proportionality to  $mv$  would be less likely than  $mv^2$ ?

**B** The figure shows a pendulum that is released at A and caught by a peg as it passes through the vertical, B. To what height will the bob rise on the right?



Discussion question B

## 11.5 Power

A car may have plenty of energy in its gas tank, but still may not be able to increase its kinetic energy rapidly. A Porsche doesn't necessarily have more energy in its gas tank than a Hyundai, it is just able to transfer it more quickly. The rate of transferring energy from one form to another is called *power*. The definition can be written as an equation,

$$P = \frac{\Delta E}{\Delta t} \quad ,$$

where the use of the delta notation in the symbol  $\Delta E$  has the usual interpretation: the final amount of energy in a certain form minus the initial amount that was present in that form. Power has units of J/s, which are abbreviated as watts, W (rhymes with “lots”).

If the rate of energy transfer is not constant, the power at any instant can be defined as the slope of the tangent line on a graph of  $E$  versus  $t$ . Likewise  $\Delta E$  can be extracted from the area under the  $P$ -versus- $t$  curve.

---

*Converting kilowatt-hours to joules*

*example 8*

▷ The electric company bills you for energy in units of kilowatt-hours (kilowatts multiplied by hours) rather than in SI units of joules. How many joules is a kilowatt-hour?

▷ 1 kilowatt-hour = (1 kW)(1 hour) = (1000 J/s)(3600 s) = 3.6 MJ.

---

*Human wattage*

*example 9*

▷ A typical person consumes 2000 kcal of food in a day, and converts nearly all of that directly to heat. Compare the person's heat output to the rate of energy consumption of a 100-watt lightbulb.

▷ Looking up the conversion factor from calories to joules, we find

$$\Delta E = 2000 \text{ kcal} \times \frac{1000 \text{ cal}}{1 \text{ kcal}} \times \frac{4.18 \text{ J}}{1 \text{ cal}} = 8 \times 10^6 \text{ J}$$

for our daily energy consumption. Converting the time interval likewise into mks,

$$\Delta t = 1 \text{ day} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{60 \text{ min}}{1 \text{ hour}} \times \frac{60 \text{ s}}{1 \text{ min}} = 9 \times 10^4 \text{ s} \quad .$$

Dividing, we find that our power dissipated as heat is  $90 \text{ J/s} = 90 \text{ W}$ , about the same as a lightbulb.

It is easy to confuse the concepts of force, energy, and power, especially since they are synonyms in ordinary speech. The table on the following page may help to clear this up:

	<b>force</b>	<b>energy</b>	<b>power</b>
<b>conceptual definition</b>	A force is an interaction between two objects that causes a push or a pull. A force can be defined as anything that is capable of changing an object's state of motion.	Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of energy.	Power is the rate at which energy is transformed from one form to another or transferred from one object to another.
<b>operational definition</b>	A spring scale can be used to measure force.	If we define a unit of energy as the amount required to heat a certain amount of water by a 1°C, then we can measure any other quantity of energy by transferring it into heat in water and measuring the temperature increase.	Measure the change in the amount of some form of energy possessed by an object, and divide by the amount of time required for the change to occur.
<b>scalar or vector?</b>	vector — has a direction in space which is the direction in which it pulls or pushes	scalar — has no direction in space	scalar — has no direction in space
<b>unit</b>	newtons (N)	joules (J)	watts (W) = joules/s
<b>Can it run out? Does it cost money?</b>	No. I don't have to pay a monthly bill for the meganewtons of force required to hold up my house.	Yes. We pay money for gasoline, electrical energy, batteries, etc., because they contain energy.	More power means you are paying money at a higher rate. A 100-W lightbulb costs a certain number of cents per hour.
<b>Can it be a property of an object?</b>	No. A force is a relationship between two interacting objects. A home-run baseball doesn't "have" force.	Yes. What a home-run baseball has is kinetic energy, not force.	Not really. A 100-W lightbulb doesn't "have" 100 W. 100 J/s is the rate at which it converts electrical energy into light.



## Summary

### Selected vocabulary

energy . . . . .	A numerical scale used to measure the heat, motion, or other properties that would require fuel or physical effort to put into an object; a scalar quantity with units of joules (J).
power . . . . .	The rate of transferring energy; a scalar quantity with units of watts (W).
kinetic energy . .	The energy an object possesses because of its motion.
heat . . . . .	A form of energy that relates to temperature. Heat is different from temperature because an object with twice as much mass requires twice as much heat to increase its temperature by the same amount. Heat is measured in joules, temperature in degrees. (In standard terminology, there is another, finer distinction between heat and thermal energy, which is discussed below. In this book, I informally refer to both as heat.)
temperature . . .	What a thermometer measures. Objects left in contact with each other tend to reach the same temperature. Cf. heat. As discussed in more detail in chapter 2, temperature is essentially a measure of the average kinetic energy per molecule.

### Notation

$E$ . . . . .	energy
$J$ . . . . .	joules, the SI unit of energy
$KE$ . . . . .	kinetic energy
$P$ . . . . .	power
$W$ . . . . .	watts, the SI unit of power; equivalent to J/s

### Other terminology and notation

$Q$ or $\Delta Q$ . . . . .	the amount of heat transferred into or out of an object
$K$ or $T$ . . . . .	alternative symbols for kinetic energy, used in the scientific literature and in most advanced textbooks
thermal energy .	Careful writers make a distinction between heat and thermal energy, but the distinction is often ignored in casual speech, even among physicists. Properly, thermal energy is used to mean the total amount of energy possessed by an object, while heat indicates the amount of thermal energy transferred in or out. The term heat is used in this book to include both meanings.

## Summary

Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of *energy*. The SI unit of energy is the Joule. The reason why energy is a useful and important quantity is that it is always conserved. That is, it cannot be created or destroyed but only transferred between objects or changed from one form to another. Conservation of energy is the most important and broadly applicable of all the laws of physics, more fundamental and general even than Newton's laws of motion.

Heating an object requires a certain amount of energy per degree of temperature and per unit mass, which depends on the substance of which the object consists. Heat and temperature are completely different things. Heat is a form of energy, and its SI unit is the joule (J). Temperature is not a measure of energy. Heating twice as much of something requires twice as much heat, but double the amount of a substance does not have double the temperature.

The energy that an object possesses because of its motion is called kinetic energy. Kinetic energy is related to the mass of the object and the magnitude of its velocity vector by the equation

$$KE = \frac{1}{2}mv^2 \quad .$$

Power is the rate at which energy is transformed from one form to another or transferred from one object to another,

$$P = \frac{\Delta E}{\Delta t} \quad . \quad \text{[only for constant power]}$$

The SI unit of power is the watt (W).

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

1 This problem is now problem 14 in chapter 12, on page 305.

2 Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

3 Estimate the kinetic energy of an Olympic sprinter.

4 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds? ✓

5 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes. ✓

6 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world's coastal cities. The total sunlight that falls on Antarctica amounts to about  $1 \times 10^{16}$  watts. In the absence of natural or human-caused climate change, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires

$3 \times 10^3$  J.

**7** A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy?

▷ Solution, p. 526

**8** Experiments show that the power consumed by a boat's engine is approximately proportional to third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is cruising at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat's cruising speed increased? [Based on a problem by Arnold Arons.]

▷ Solution, p. 527

**9** Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B's kinetic energy? [Based on a problem by Arnold Arons.]

▷ Solution, p. 527

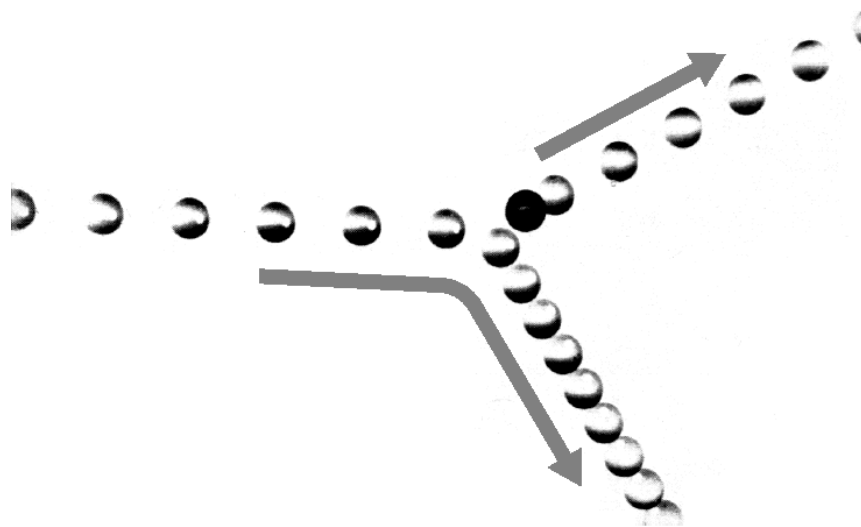
**10** The moon doesn't really just orbit the Earth. By Newton's third law, the moon's gravitational force on the earth is the same as the earth's force on the moon, and the earth must respond to the moon's force by accelerating. If we consider the earth and moon in isolation and ignore outside forces, then Newton's first law says their common center of mass doesn't accelerate, i.e., the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon's mass is 81 times smaller than the earth's. Compare the kinetic energies of the earth and moon. (We know that the center of mass is a kind of balance point, so it must be closer to the earth than to the moon. In fact, the distance from the earth to the center of mass is 1/81 of the distance from the moon to the center of mass, which makes sense intuitively, and can be proved rigorously using the equation on page 348.)

**11** My 1.25 kW microwave oven takes 126 seconds to bring 250 g of water from room temperature to a boil. What percentage of the power is being wasted? Where might the rest of the energy be going?

▷ Solution, p. 527

**12** The multiframe photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making *measurements* on the figure, determine *numerically* whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]

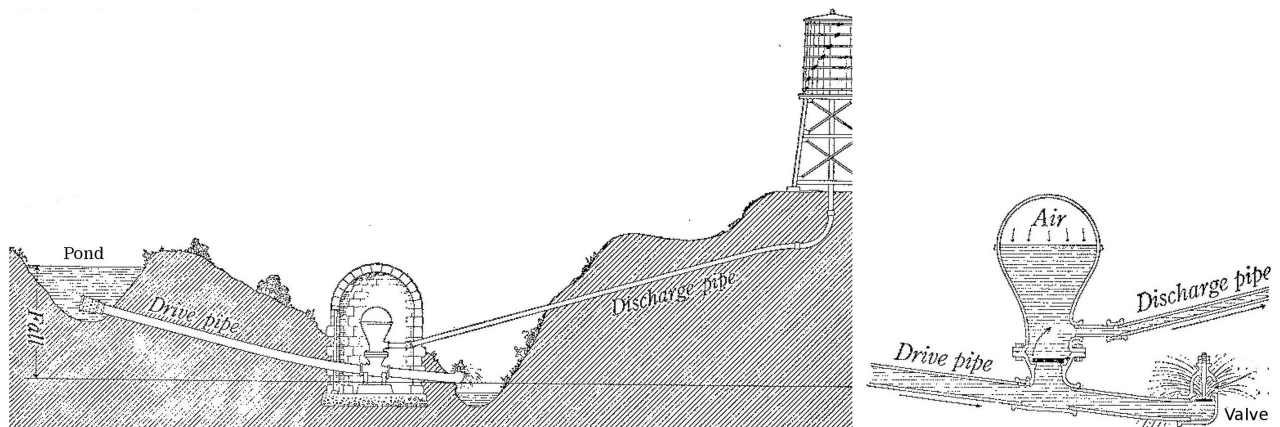
Problem 12.



**13** This problem is a numerical example of the imaginary experiment discussed on p. 282 regarding the relationship between energy and relative motion. Let's say that the pool balls both have masses of 1.00 kg. Suppose that in the frame of reference of the pool table, the cue ball moves at a speed of 1.00 m/s toward the eight ball, which is initially at rest. The collision is head-on, and as you can verify for yourself the next time you're playing pool, the result of such a collision is that the incoming ball stops dead and the ball that was struck takes off with the same speed originally possessed by the incoming ball. (This is actually a bit of an idealization. To keep things simple, we're ignoring the spin of the balls, and we assume that no energy is liberated by the collision as heat or sound.) (a) Calculate the total initial kinetic energy and the total final kinetic energy, and verify that they are equal. (b) Now carry out the whole calculation again in the frame of reference that is moving in the same direction that the cue ball was initially moving, but at a speed of 0.50 m/s. In this frame of reference, both balls have nonzero initial and final velocities, which are different from what they were in the table's frame. [See also problem 15 on p. 363.]

**14** One theory about the destruction of the space shuttle Columbia in 2003 is that one of its wings had been damaged on liftoff by a chunk of foam insulation that fell off of one of its external fuel tanks. The New York Times reported on June 5, 2003, that NASA engineers had recreated the impact to see if it would damage a mock-up of the shuttle's wing. "Before last week's test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball." In fact, the 1.7-pound piece of foam, moving at 531 miles per hour, did serious damage. A member of the board investigating the disaster said this demonstrated that "people's intuitive sense of physics is sometimes way off." (a) Compute the kinetic energy of the foam, and (b) compare with the energy of a 170-pound boulder moving at 5.3 miles per hour (the speed it would have if you dropped it from about knee-level). ✓

(c) The boulder is a hundred times more massive, but its speed is a hundred times smaller, so what's counterintuitive about your results?



**15** The figure above is from a classic 1920 physics textbook by Millikan and Gale. It represents a method for raising the water from the pond up to the water tower, at a higher level, without using a pump. Water is allowed into the drive pipe, and once it is flowing fast enough, it forces the valve at the bottom closed. Explain how this works in terms of conservation of mass and energy. (Cf. example 1 on page 273.)

**16** The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

heat 1 g of ice by $1^{\circ}\text{C}$	2.05 J
melt 1 g of ice	333 J
heat 1 g of liquid by $1^{\circ}\text{C}$	4.19 J
boil 1 g of water	2500 J
heat 1 g of steam by $1^{\circ}\text{C}$	2.01 J

- (a) How much energy is required in order to convert 1.00 g of ice at  $-20^{\circ}\text{C}$  into steam at  $137^{\circ}\text{C}$ ?  $\checkmark$
- (b) What is the minimum amount of hot water that could melt 1.00 g of ice?  $\checkmark$



Do these forms of energy have anything in common?

## Chapter 12

# Simplifying the energy zoo

Variety is the spice of life, not of science. The figure shows a few examples from the bewildering array of forms of energy that surrounds us. The physicist's psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we've arrived in the study of energy is analogous to the period in the 1960's when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the "particle zoo," and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the



new particles being whipped up were simply clusters of a previously unsuspected set of more fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, “Three quarks for Master Mark.”) The energy zoo can also be simplified, and it is the purpose of this chapter to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.

a / A vivid demonstration that heat is a form of motion. A small amount of boiling water is poured into the empty can, which rapidly fills up with hot steam. The can is then sealed tightly, and soon crumples. This can be explained as follows. The high temperature of the steam is interpreted as a high average speed of random motions of its molecules. Before the lid was put on the can, the rapidly moving steam molecules pushed their way out of the can, forcing the slower air molecules out of the way. As the steam inside the can thinned out, a stable situation was soon achieved, in which the force from the less dense steam molecules moving at high speed balanced against the force from the more dense but slower air molecules outside. The cap was put on, and after a while the steam inside the can reached the same temperature as the air outside. The force from the cool, thin steam no longer matched the force from the cool, dense air outside, and the imbalance of forces crushed the can.



## 12.1 Heat is kinetic energy

What is heat really? Is it an invisible fluid that your bare feet soak up from a hot sidewalk? Can one ever remove all the heat from an object? Is there a maximum to the temperature scale?

The theory of heat as a fluid seemed to explain why colder objects absorbed heat from hotter ones, but once it became clear that heat was a form of energy, it began to seem unlikely that a material substance could transform itself into and out of all those other forms of energy like motion or light. For instance, a compost pile gets hot, and we describe this as a case where, through the action of bacteria, chemical energy stored in the plant cuttings is transformed into heat energy. The heating occurs even if there is no nearby warmer object that could have been leaking “heat fluid” into the pile.

An alternative interpretation of heat was suggested by the theory that matter is made of atoms. Since gases are thousands of times less dense than solids or liquids, the atoms (or clusters of atoms called molecules) in a gas must be far apart. In that case, what is keeping all the air molecules from settling into a thin film on the floor of the room in which you are reading this book? The simplest explanation is that they are moving very rapidly, continually ricocheting off of

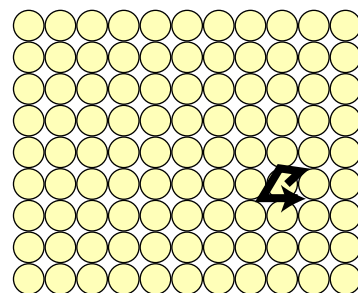
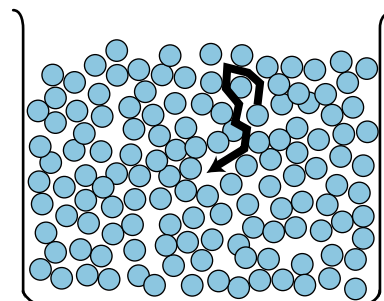
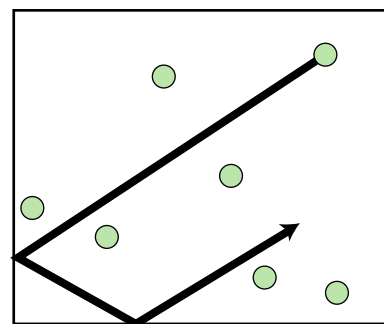
the floor, walls, and ceiling. Though bizarre, the cloud-of-bullets image of a gas did give a natural explanation for the surprising ability of something as tenuous as a gas to exert huge forces. Your car's tires can hold it up because you have pumped extra molecules into them. The inside of the tire gets hit by molecules more often than the outside, forcing it to stretch and stiffen.

The outward forces of the air in your car's tires increase even further when you drive on the freeway for a while, heating up the rubber and the air inside. This type of observation leads naturally to the conclusion that hotter matter differs from colder in that its atoms' random motion is more rapid. In a liquid, the motion could be visualized as people in a milling crowd shoving past each other more quickly. In a solid, where the atoms are packed together, the motion is a random vibration of each atom as it knocks against its neighbors.

We thus achieve a great simplification in the theory of heat. Heat is simply a form of kinetic energy, the total kinetic energy of random motion of all the atoms in an object. With this new understanding, it becomes possible to answer at one stroke the questions posed at the beginning of the section. Yes, it is at least theoretically possible to remove all the heat from an object. The coldest possible temperature, known as absolute zero, is that at which all the atoms have zero velocity, so that their kinetic energies,  $(1/2)mv^2$ , are all zero. No, there is no maximum amount of heat that a certain quantity of matter can have, and no maximum to the temperature scale, since arbitrarily large values of  $v$  can create arbitrarily large amounts of kinetic energy per atom.

The kinetic theory of heat also provides a simple explanation of the true nature of temperature. Temperature is a measure of the amount of energy per molecule, whereas heat is the total amount of energy possessed by all the molecules in an object.

There is an entire branch of physics, called thermodynamics, that deals with heat and temperature and forms the basis for technologies such as refrigeration. Thermodynamics is discussed in more detail in optional chapter 16, and I have provided here only a brief overview of the thermodynamic concepts that relate directly to energy, glossing over at least one point that would be dealt with more carefully in a thermodynamics course: it is really only true for a gas that all the heat is in the form of kinetic energy. In solids and liquids, the atoms are close enough to each other to exert intense electrical forces on each other, and there is therefore another type of energy involved, the energy associated with the atoms' distances from each other. Strictly speaking, heat energy is defined not as energy associated with random motion of molecules but as any form of energy that can be conducted between objects in contact, without any force.



b / Random motion of atoms in a gas, a liquid, and a solid.

## 12.2 Potential energy: energy of distance or closeness

We have already seen many examples of energy related to the distance between interacting objects. When two objects participate in an attractive noncontact force, energy is required to bring them farther apart. In both of the perpetual motion machines that started off the previous chapter, one of the types of energy involved was the energy associated with the distance between the balls and the earth, which attract each other gravitationally. In the perpetual motion machine with the magnet on the pedestal, there was also energy associated with the distance between the magnet and the iron ball, which were attracting each other.

The opposite happens with repulsive forces: two socks with the same type of static electric charge will repel each other, and cannot be pushed closer together without supplying energy.

In general, the term *potential energy*, with algebra symbol  $PE$ , is used for the energy associated with the distance between two objects that attract or repel each other via a force that depends on the distance between them. Forces that are not determined by distance do not have potential energy associated with them. For instance, the normal force acts only between objects that have zero distance between them, and depends on other factors besides the fact that the distance is zero. There is no potential energy associated with the normal force.

The following are some commonplace examples of potential energy:



c / The skater has converted all his kinetic energy into potential energy on the way up the side of the pool.

**gravitational potential energy:** The skateboarder in the photo has risen from the bottom of the pool, converting kinetic energy into gravitational potential energy. After being at rest for an instant, he will go back down, converting PE back into KE.

**magnetic potential energy:** When a magnetic compass needle is allowed to rotate, the poles of the compass change their distances from the earth's north and south magnetic poles, converting magnetic potential energy into kinetic energy. (Eventually the kinetic energy is all changed into heat by friction, and the needle settles down in the position that minimizes its potential energy.)

**electrical potential energy:** Socks coming out of the dryer cling together because of attractive electrical forces. Energy is required in order to separate them.

**potential energy of bending or stretching:** The force between the two ends of a spring depends on the distance between

them, i.e., on the length of the spring. If a car is pressed down on its shock absorbers and then released, the potential energy stored in the spring is transformed into kinetic and gravitational potential energy as the car bounces back up.

I have deliberately avoided introducing the term potential energy up until this point, because it tends to produce unfortunate connotations in the minds of students who have not yet been inoculated with a careful description of the construction of a numerical energy scale. Specifically, there is a tendency to generalize the term inappropriately to apply to any situation where there is the “potential” for something to happen: “I took a break from digging, but I had potential energy because I knew I’d be ready to work hard again in a few minutes.”

### An equation for gravitational potential energy

All the vital points about potential energy can be made by focusing on the example of gravitational potential energy. For simplicity, we treat only vertical motion, and motion close to the surface of the earth, where the gravitational force is nearly constant. (The generalization to the three dimensions and varying forces is more easily accomplished using the concept of work, which is the subject the next chapter.)

To find an equation for gravitational PE, we examine the case of free fall, in which energy is transformed between kinetic energy and gravitational PE. Whatever energy is lost in one form is gained in an equal amount in the other form, so using the notation  $\Delta KE$  to stand for  $KE_f - KE_i$  and a similar notation for PE, we have

$$[1] \quad \Delta KE = -\Delta PE_{grav} \quad .$$

It will be convenient to refer to the object as falling, so that PE is being changed into KE, but the math applies equally well to an object slowing down on its way up. We know an equation for kinetic energy,

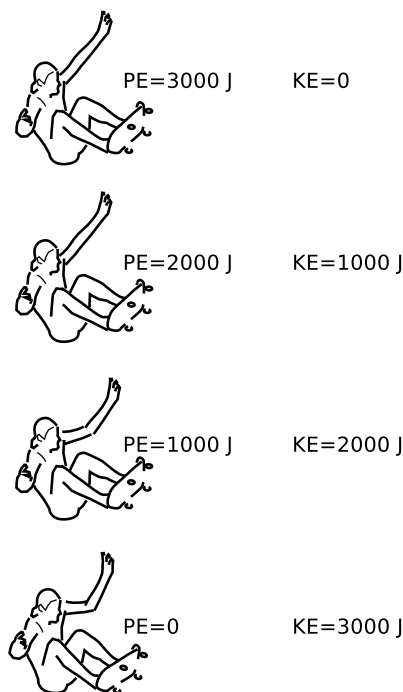
$$[2] \quad KE = \frac{1}{2}mv^2 \quad ,$$

so if we can relate  $v$  to height,  $y$ , we will be able to relate  $\Delta PE$  to  $y$ , which would tell us what we want to know about potential energy. The  $y$  component of the velocity can be connected to the height via the constant acceleration equation

$$[3] \quad v_f^2 = v_i^2 + 2a\Delta y \quad ,$$

and Newton’s second law provides the acceleration,

$$[4] \quad a = F/m \quad ,$$



d / As the skater free-falls, his PE is converted into KE. (The numbers would be equally valid as a description of his motion on the way up.)

in terms of the gravitational force.

The algebra is simple because both equation [2] and equation [3] have velocity to the second power. Equation [2] can be solved for  $v^2$  to give  $v^2 = 2KE/m$ , and substituting this into equation [3], we find

$$2\frac{KE_f}{m} = 2\frac{KE_i}{m} + 2a\Delta y \quad .$$

Making use of equations [1] and [4] gives the simple result

$$\Delta PE_{grav} = -F\Delta y \quad . \quad \begin{array}{l} \text{[change in gravitational PE} \\ \text{resulting from a change in height } \Delta y; \\ F \text{ is the gravitational force on the object,} \\ \text{i.e., its weight; valid only near the surface} \\ \text{of the earth, where } F \text{ is constant]} \end{array}$$

---

*Dropping a rock*

*example 1*

- ▷ If you drop a 1-kg rock from a height of 1 m, how many joules of KE does it have on impact with the ground? (Assume that any energy transformed into heat by air friction is negligible.)
- ▷ If we choose the  $y$  axis to point up, then  $F_y$  is negative, and equals  $-(1 \text{ kg})(g) = -9.8 \text{ N}$ . A decrease in  $y$  is represented by a negative value of  $\Delta y$ ,  $\Delta y = -1 \text{ m}$ , so the change in potential energy is  $-(-9.8 \text{ N})(-1 \text{ m}) \approx -10 \text{ J}$ . (The proof that newtons multiplied by meters give units of joules is left as a homework problem.) Conservation of energy says that the loss of this amount of PE must be accompanied by a corresponding increase in KE of 10 J.

It may be dismaying to note how many minus signs had to be handled correctly even in this relatively simple example: a total of four. Rather than depending on yourself to avoid any mistakes with signs, it is better to check whether the final result make sense physically. If it doesn't, just reverse the sign.

Although the equation for gravitational potential energy was derived by imagining a situation where it was transformed into kinetic energy, the equation can be used in any context, because all the types of energy are freely convertible into each other.

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*Gravitational PE converted directly into heat*

*example 2*

- ▷ A 50-kg firefighter slides down a 5-m pole at constant velocity. How much heat is produced?
- ▷ Since she slides down at constant velocity, there is no change in KE. Heat and gravitational PE are the only forms of energy that change. Ignoring plus and minus signs, the gravitational force on

her body equals  $mg$ , and the amount of energy transformed is

$$(mg)(5 \text{ m}) = 2500 \text{ J} \quad .$$

On physical grounds, we know that there must have been an increase (positive change) in the heat energy in her hands and in the flagpole.

Here are some questions and answers about the interpretation of the equation  $\Delta PE_{\text{grav}} = -F\Delta y$  for gravitational potential energy.

**Question:** In a nutshell, why is there a minus sign in the equation?

**Answer:** It is because we increase the PE by moving the object in the *opposite* direction compared to the gravitational force.

**Question:** Why do we only get an equation for the *change* in potential energy? Don't I really want an equation for the potential energy itself?

**Answer:** No, you really don't. This relates to a basic fact about potential energy, which is that it is not a well defined quantity in the absolute sense. Only changes in potential energy are unambiguously defined. If you and I both observe a rock falling, and agree that it deposits 10 J of energy in the dirt when it hits, then we will be forced to agree that the 10 J of KE must have come from a loss of 10 joules of PE. But I might claim that it started with 37 J of PE and ended with 27, while you might swear just as truthfully that it had 109 J initially and 99 at the end. It is possible to pick some specific height as a reference level and say that the PE is zero there, but it's easier and safer just to work with changes in PE and avoid absolute PE altogether.

**Question:** You referred to potential energy as the energy that *two* objects have because of their distance from each other. If a rock falls, the object is the rock. Where's the other object?

**Answer:** Newton's third law guarantees that there will always be two objects. The other object is the planet earth.

**Question:** If the other object is the earth, are we talking about the distance from the rock to the center of the earth or the distance from the rock to the surface of the earth?

**Answer:** It doesn't matter. All that matters is the change in distance,  $\Delta y$ , not  $y$ . Measuring from the earth's center or its surface are just two equally valid choices of a reference point for defining absolute PE.

**Question:** Which object contains the PE, the rock or the earth?

**Answer:** We may refer casually to the PE of the rock, but technically the PE is a relationship between the earth and the rock, and we should refer to the earth and the rock together as possessing the PE.

**Question:** How would this be any different for a force other than gravity?

**Answer:** It wouldn't. The result was derived under the assumption of constant force, but the result would be valid for any other situation where two objects interacted through a constant force. Gravity is unusual, however, in that the gravitational force on an object is so nearly constant under ordinary conditions. The magnetic force between a magnet and a refrigerator, on the other hand, changes drastically with distance. The math is a little more complex for a varying force, but the concepts are the same.

**Question:** Suppose a pencil is balanced on its tip and then falls over. The pencil is simultaneously changing its height and rotating, so the height change is different for different parts of the object. The bottom of the pencil doesn't lose any height at all. What do you do in this situation?

**Answer:** The general philosophy of energy is that an object's energy is found by adding up the energy of every little part of it. You could thus add up the changes in potential energy of all the little parts of the pencil to find the total change in potential energy. Luckily there's an easier way! The derivation of the equation for gravitational potential energy used Newton's second law, which deals with the acceleration of the object's center of mass (i.e., its balance point). If you just define  $\Delta y$  as the height change of the center of mass, everything works out. A huge Ferris wheel can be rotated without putting in or taking out any PE, because its center of mass is staying at the same height.

#### self-check A

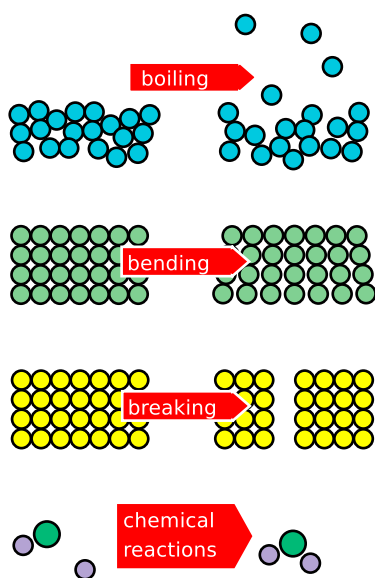
A ball thrown straight up will have the same speed on impact with the ground as a ball thrown straight down at the same speed. How can this be explained using potential energy? ▷ Answer, p. 533

#### Discussion question

**A** You throw a steel ball up in the air. How can you prove based on conservation of energy that it has the same speed when it falls back into your hand? What if you throw a feather up — is energy not conserved in this case?

## 12.3 All energy is potential or kinetic

In the same way that we found that a change in temperature is really only a change in kinetic energy at the atomic level, we now find that every other form of energy turns out to be a form of potential energy. Boiling, for instance, means knocking some of the atoms (or molecules) out of the liquid and into the space above, where they constitute a gas. There is a net attractive force between essentially any two atoms that are next to each other, which is why matter always prefers to be packed tightly in the solid or liquid state unless we supply enough potential energy to pull it apart into a gas. This explains why water stops getting hotter when it reaches the



e / All these energy transformations turn out at the atomic level to be changes in potential energy resulting from changes in the distances between atoms.



boiling point: the power being pumped into the water by your stove begins going into potential energy rather than kinetic energy.

As shown in figure e, every stored form of energy that we encounter in everyday life turns out to be a form of potential energy at the atomic level. The forces between atoms are electrical and magnetic in nature, so these are actually electrical and magnetic potential energies.

Although light is a topic of the second half of this course, it is useful to have a preview of how it fits in here. Light is a wave composed of oscillating electric and magnetic fields, so we can include it under the category of electrical and magnetic potential energy.

Even if we wish to include nuclear reactions in the picture, there still turn out to be only four fundamental types of energy:

**kinetic energy** (including heat)

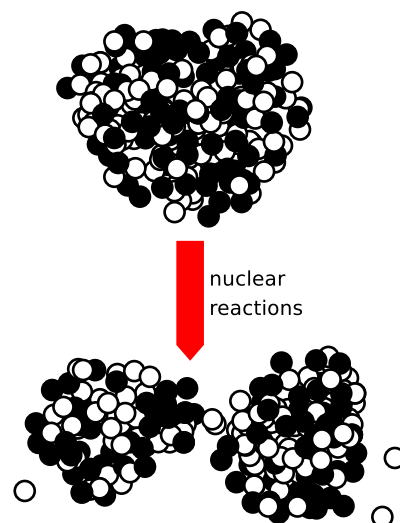
**gravitational potential energy**

**electrical and magnetic potential energy**

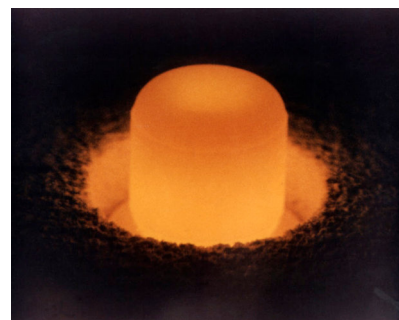
**nuclear potential energy**

### Discussion question

**A** Referring back to the pictures at the beginning of the chapter, how do all these forms of energy fit into the shortened list of categories given above?



f / This figure looks similar to the previous ones, but the scale is a million times smaller. The little balls are the neutrons and protons that make up the tiny nucleus at the center of the uranium atom. When the nucleus splits (fissions), the potential energy change is partly electrical and partly a change in the potential energy derived from the force that holds atomic nuclei together (known as the strong nuclear force).



g / A pellet of plutonium-238 glows with its own heat. Its nuclear potential energy is being converted into heat, a form of kinetic energy. Pellets of this type are used as power supplies on some space probes.



# Summary

## Selected vocabulary

potential energy      the energy having to do with the distance between two objects that interact via a noncontact force

## Notation

PE . . . . . potential energy

## Other terminology and notation

$U$  or  $V$  . . . . . symbols used for potential energy in the scientific literature and in most advanced textbooks

## Summary

Historically, the energy concept was only invented to include a few phenomena, but it was later generalized more and more to apply to new situations, for example nuclear reactions. This generalizing process resulted in an undesirably long list of types of energy, each of which apparently behaved according to its own rules.

The first step in simplifying the picture came with the realization that heat was a form of random motion on the atomic level, i.e., heat was nothing more than the kinetic energy of atoms.

A second and even greater simplification was achieved with the realization that all the other apparently mysterious forms of energy actually had to do with changing the distances between atoms (or similar processes in nuclei). This type of energy, which relates to the distance between objects that interact via a force, is therefore of great importance. We call it potential energy.

Most of the important ideas about potential energy can be understood by studying the example of gravitational potential energy. The change in an object's gravitational potential energy is given by

$$\Delta PE_{grav} = -F_{grav}\Delta y \quad , \quad \begin{array}{l} \text{[if } F_{grav} \text{ is constant, i.e., the} \\ \text{the motion is all near the Earth's surface]} \end{array}$$

The most important thing to understand about potential energy is that there is no unambiguous way to define it in an absolute sense. The only thing that everyone can agree on is how much the potential energy has changed from one moment in time to some later moment in time.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** Can gravitational potential energy ever be negative? Note that the question refers to  $PE$ , not  $\Delta PE$ , so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

**2** A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational potential energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

**3** (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic potential energy increase, or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic potential energy increase, or decrease? Explain.

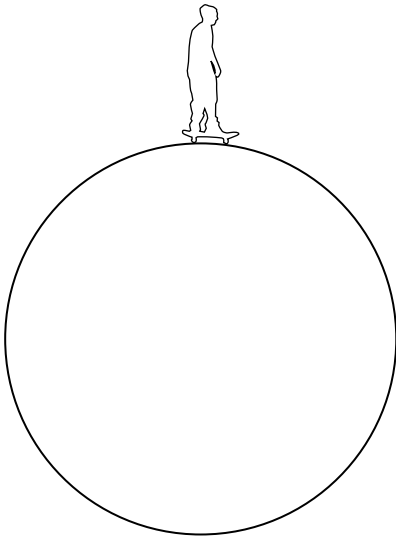
**4** Let  $E_b$  be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to  $100^\circ\text{C}$ . [Numerical check, not for credit: Plugging in  $E_b = 2.3 \text{ MJ/kg}$  should give a result of 230 km.] ✓

(b) Show that the units of your answer in part a come out right based on the units given for  $E_b$ .

**5** A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?

▷ Solution, p. 527

**6** A person on a bicycle is to coast down a ramp of height  $h$  and then pass through a circular loop of radius  $r$ . What is the smallest value of  $h$  for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.) ✓



Problem 7.

**7** A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe. ▷ Solution, p. 527 ★

**8** (a) A circular hoop of mass  $m$  and radius  $r$  spins like a wheel while its center remains at rest. Its period (time required for one revolution) is  $T$ . Show that its kinetic energy equals  $2\pi^2 mr^2/T^2$ .

(b) If such a hoop rolls with its center moving at velocity  $v$ , its kinetic energy equals  $(1/2)mv^2$ , plus the amount of kinetic energy found in the first part of this problem. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have. ★

**9** Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE. ▷ Solution, p. 528

**10** Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.] ✓

**11** Make an order-of-magnitude estimate of the power represented by the loss of gravitational energy of the water going over Niagara Falls. If the hydroelectric plant at the bottom of the falls could convert 100% of this to electrical power, roughly how many households could be powered? ▷ Solution, p. 528

**12** When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares

cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

**13** At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to “try” many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.)  $\checkmark$

**14** Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don’t you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

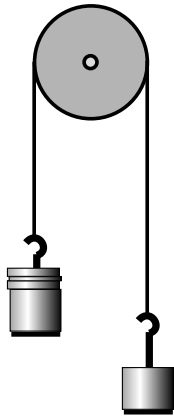
**15** Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

**16** (a) A circular hoop of mass  $m$  and radius  $r$  spins like a wheel while its center remains at rest. Let  $\omega$  (Greek letter omega) be the number of radians it covers per unit time, i.e.,  $\omega = 2\pi/T$ , where the period,  $T$ , is the time for one revolution. Show that its kinetic energy equals  $(1/2)m\omega^2r^2$ .

(b) Show that the answer to part a has the right units. (Note that radians aren’t really units, since the definition of a radian is a unitless ratio of two lengths.)

(c) If such a hoop rolls with its center moving at velocity  $v$ , its kinetic energy equals  $(1/2)mv^2$ , plus the amount of kinetic energy found in part a. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

★



**17** The figure shows two unequal masses,  $M$  and  $m$ , connected by a string running over a pulley. This system was analyzed previously in problem 10 on p. 171, using Newton's laws.

(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance  $h$ . ✓

(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem. ✓

Problem 17.



## Chapter 13

# Work: the transfer of mechanical energy

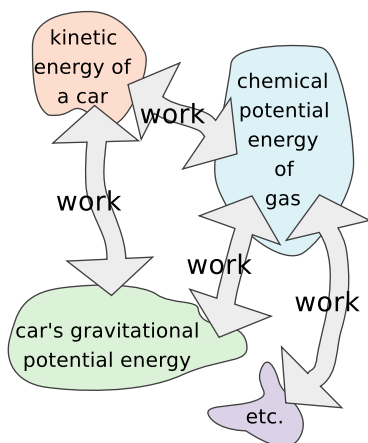
### 13.1 Work: the transfer of mechanical energy

#### The concept of work

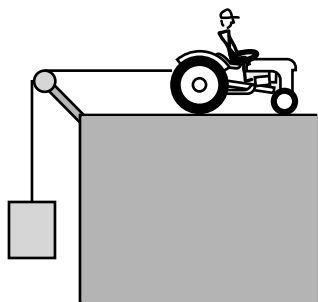
The mass contained in a closed system is a conserved quantity, but if the system is not closed, we also have ways of measuring the amount of mass that goes in or out. The water company does this with a meter that records your water use.

Likewise, we often have a system that is not closed, and would like to know how much energy comes in or out. Energy, however, is not a physical substance like water, so energy transfer cannot be measured with the same kind of meter. How can we tell, for instance, how much useful energy a tractor can “put out” on one tank of gas?

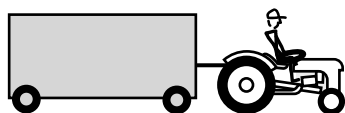
The law of conservation of energy guarantees that all the chem-



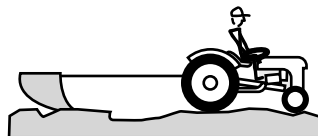
a / Work is a transfer of energy.



b / The tractor raises the weight over the pulley, increasing its gravitational potential energy.



c / The tractor accelerates the trailer, increasing its kinetic energy.



d / The tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow.

ical energy in the gasoline will reappear in some form, but not necessarily in a form that is useful for doing farm work. Tractors, like cars, are extremely inefficient, and typically 90% of the energy they consume is converted directly into heat, which is carried away by the exhaust and the air flowing over the radiator. We wish to distinguish the energy that comes out directly as heat from the energy that serves to accelerate a trailer or to plow a field, so we define a technical meaning of the ordinary word “work” to express the distinction:

### definition of work

Work is the amount of energy transferred into or out of a system, not counting energy transferred by heat conduction.

### self-check A

Based on this definition, is work a vector, or a scalar? What are its units?

▷ Answer, p. 533

The conduction of heat is to be distinguished from heating by friction. When a hot potato heats up your hands by conduction, the energy transfer occurs without any force, but when friction heats your car’s brake shoes, there is a force involved. The transfer of energy with and without a force are measured by completely different methods, so we wish to include heat transfer by frictional heating under the definition of work, but not heat transfer by conduction. The definition of work could thus be restated as the amount of energy transferred by forces.

### Calculating work as force multiplied by distance

The examples in figures b-d show that there are many different ways in which energy can be transferred. Even so, all these examples have two things in common:

1. A force is involved.
2. The tractor travels some distance as it does the work.

In b, the increase in the height of the weight,  $\Delta y$ , is the same as the distance the tractor travels, which we’ll call  $d$ . For simplicity, we discuss the case where the tractor raises the weight at constant speed, so that there is no change in the kinetic energy of the weight, and we assume that there is negligible friction in the pulley, so that the force the tractor applies to the rope is the same as the rope’s upward force on the weight. By Newton’s first law, these forces are also of the same magnitude as the earth’s gravitational force on the weight. The increase in the weight’s potential energy is given by  $F\Delta y$ , so the work done by the tractor on the weight equals  $Fd$ , the product of the force and the distance moved:

$$W = Fd \quad .$$

In example c, the tractor's force on the trailer accelerates it, increasing its kinetic energy. If frictional forces on the trailer are negligible, then the increase in the trailer's kinetic energy can be found using the same algebra that was used on page 297 to find the potential energy due to gravity. Just as in example b, we have

$$W = Fd \quad .$$

Does this equation always give the right answer? Well, sort of. In example d, there are two quantities of work you might want to calculate: the work done by the tractor on the plow and the work done by the plow on the dirt. These two quantities can't both equal  $Fd$ . Most of the energy transmitted through the cable goes into frictional heating of the plow and the dirt. The work done by the plow on the dirt is less than the work done by the tractor on the plow, by an amount equal to the heat absorbed by the plow. It turns out that the equation  $W = Fd$  gives the work done by the tractor, not the work done by the plow. How are you supposed to know when the equation will work and when it won't? The somewhat complex answer is postponed until section 13.6. Until then, we will restrict ourselves to examples in which  $W = Fd$  gives the right answer; essentially the reason the ambiguities come up is that when one surface is slipping past another,  $d$  may be hard to define, because the two surfaces move different distances.



e / The baseball pitcher put kinetic energy into the ball, so he did work on it. To do the greatest possible amount of work, he applied the greatest possible force over the greatest possible distance.

We have also been using examples in which the force is in the same direction as the motion, and the force is constant. (If the force was not constant, we would have to represent it with a function, not a symbol that stands for a number.) To summarize, we have:

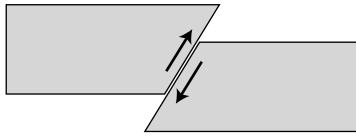
#### **rule for calculating work (simplest version)**

The work done by a force can be calculated as

$$W = Fd \quad ,$$

if the force is constant and in the same direction as the motion. Some ambiguities are encountered in cases such as kinetic friction.





f / Example 1.

### Mechanical work done in an earthquake

example 1

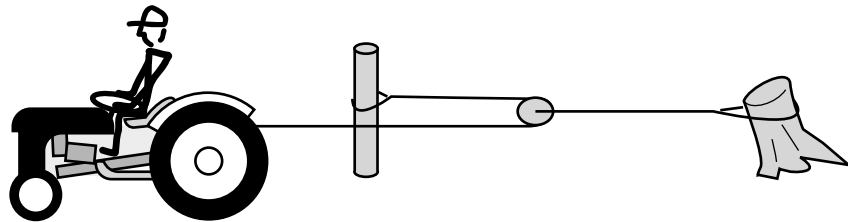
▷ In 1998, geologists discovered evidence for a big prehistoric earthquake in Pasadena, between 10,000 and 15,000 years ago. They found that the two sides of the fault moved 6.7 m relative to one another, and estimated that the force between them was  $1.3 \times 10^{17}$  N. How much energy was released?

▷ Multiplying the force by the distance gives  $9 \times 10^{17}$  J. For comparison, the Northridge earthquake of 1994, which killed 57 people and did 40 billion dollars of damage, released 22 times less energy.

### machines can increase force, but not work.

Figure g shows a pulley arrangement for doubling the force supplied by the tractor (book 1, section 5.6). The tension in the left-hand rope is equal throughout, assuming negligible friction, so there are two forces pulling the pulley to the left, each equal to the original force exerted by the tractor on the rope. This doubled force is transmitted through the right-hand rope to the stump.

g / The pulley doubles the force the tractor can exert on the stump.



It might seem as though this arrangement would also double the work done by the tractor, but look again. As the tractor moves forward 2 meters, 1 meter of rope comes around the pulley, and the pulley moves 1 m to the left. Although the pulley exerts double the force on the stump, the pulley and stump only move half as far, so the work done on the stump is no greater than it would have been without the pulley.

The same is true for any mechanical arrangement that increases or decreases force, such as the gears on a ten-speed bike. You can't get out more work than you put in, because that would violate conservation of energy. If you shift gears so that your force on the pedals is amplified, the result is that you just have to spin the pedals more times.

### No work is done without motion.

It strikes most students as nonsensical when they are told that if they stand still and hold a heavy bag of cement, they are doing no work on the bag. Even if it makes sense mathematically that  $W = Fd$  gives zero when  $d$  is zero, it seems to violate common sense. You would certainly become tired! The solution is simple.

Physicists have taken over the common word “work” and given it a new technical meaning, which is the transfer of energy. The energy of the bag of cement is not changing, and that is what the physicist means by saying no work is done on the bag.

There is a transformation of energy, but it is taking place entirely within your own muscles, which are converting chemical energy into heat. Physiologically, a human muscle is not like a tree limb, which can support a weight indefinitely without the expenditure of energy. Each muscle cell’s contraction is generated by zillions of little molecular machines, which take turns supporting the tension. When a particular molecule goes on or off duty, it moves, and since it moves while exerting a force, it is doing work. There is work, but it is work done by one molecule in a muscle cell on another.

### Positive and negative work

When object A transfers energy to object B, we say that A does positive work on B. B is said to do negative work on A. In other words, a machine like a tractor is defined as doing positive work. This use of the plus and minus signs relates in a logical and consistent way to their use in indicating the directions of force and motion in one dimension. In figure h, suppose we choose a coordinate system with the  $x$  axis pointing to the right. Then the force the spring exerts on the ball is always a positive number. The ball’s motion, however, changes directions. The symbol  $d$  is really just a shorter way of writing the familiar quantity  $\Delta x$ , whose positive and negative signs indicate direction.

While the ball is moving to the left, we use  $d < 0$  to represent its direction of motion, and the work done by the spring,  $Fd$ , comes out negative. This indicates that the spring is taking kinetic energy out of the ball, and accepting it in the form of its own potential energy.

As the ball is reaccelerated to the right, it has  $d > 0$ ,  $Fd$  is positive, and the spring does positive work on the ball. Potential energy is transferred out of the spring and deposited in the ball as kinetic energy.

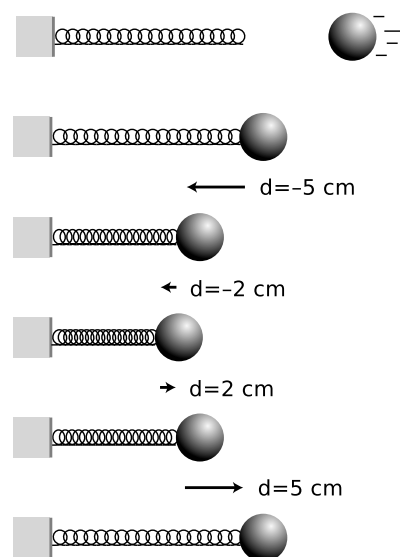
In summary:

#### rule for calculating work (including cases of negative work)

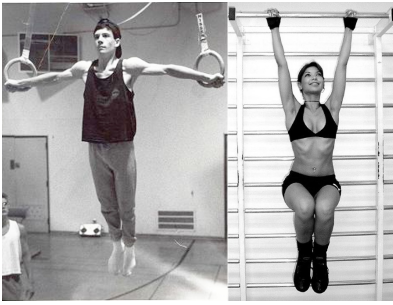
The work done by a force can be calculated as

$$W = Fd \quad ,$$

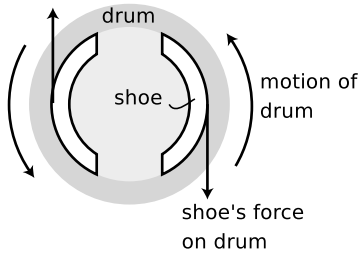
if the force is constant and along the same line as the motion. The quantity  $d$  is to be interpreted as a synonym for  $\Delta x$ , i.e., positive and negative signs are used to indicate the direction of motion. Some ambiguities are encountered in cases such as kinetic friction.



h / Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).



i / *Left:* No mechanical work occurs in the man's body while he holds himself motionless. There is a transformation of chemical energy into heat, but this happens at the microscopic level inside the tensed muscles. *Right:* When the woman lifts herself, her arms do positive work on her body, transforming chemical energy into gravitational potential energy and heat. On the way back down, the arms' work is negative; gravitational potential energy is transformed into heat. (In exercise physiology, the man is said to be doing isometric exercise, while the woman's is concentric and then concentric.)



j / Because the force is in the opposite direction compared to the motion, the brake shoe does negative work on the drum, i.e., accepts energy from it in the form of heat.

### self-check B

In figure h, what about the work done by the ball on the spring?

▷ Answer, p. 533

There are many examples where the transfer of energy out of an object cancels out the transfer of energy in. When the tractor pulls the plow with a rope, the rope does negative work on the tractor and positive work on the plow. The total work done by the rope is zero, which makes sense, since it is not changing its energy.

It may seem that when your arms do negative work by lowering a bag of cement, the cement is not really transferring energy into your body. If your body was storing potential energy like a compressed spring, you would be able to raise and lower a weight all day, recycling the same energy. The bag of cement does transfer energy into your body, but your body accepts it as heat, not as potential energy. The tension in the muscles that control the speed of the motion also results in the conversion of chemical energy to heat, for the same physiological reasons discussed previously in the case where you just hold the bag still.

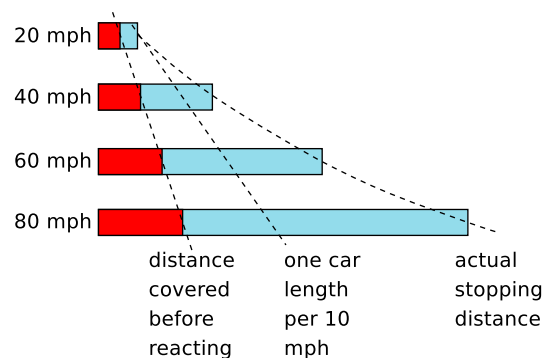
One of the advantages of electric cars over gasoline-powered cars is that it is just as easy to put energy back in a battery as it is to take energy out. When you step on the brakes in a gas car, the brake shoes do negative work on the rest of the car. The kinetic energy of the car is transmitted through the brakes and accepted by the brake shoes in the form of heat. The energy cannot be recovered. Electric cars, however, are designed to use regenerative braking. The brakes don't use friction at all. They are electrical, and when you step on the brake, the negative work done by the brakes means they accept the energy and put it in the battery for later use. This is one of the reasons why an electric car is far better for the environment than a gas car, even if the ultimate source of the electrical energy happens to be the burning of oil in the electric company's plant. The electric car recycles the same energy over and over, and only dissipates heat due to air friction and rolling resistance, not braking. (The electric company's power plant can also be fitted with expensive pollution-reduction equipment that would be prohibitively expensive or bulky for a passenger car.)

## Discussion questions

**A** Besides the presence of a force, what other things differentiate the processes of frictional heating and heat conduction?

**B** Criticize the following incorrect statement: "A force doesn't do any work unless it's causing the object to move."

**C** To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?



Discussion question C.

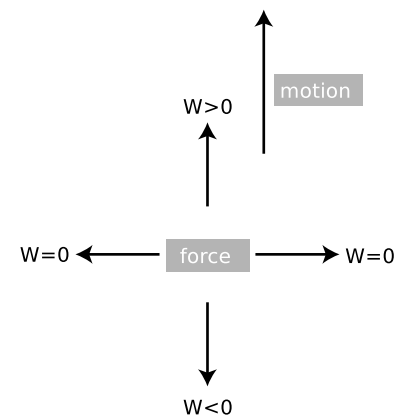
## 13.2 Work in three dimensions

### A force perpendicular to the motion does no work.

Suppose work is being done to change an object's kinetic energy. A force in the same direction as its motion will speed it up, and a force in the opposite direction will slow it down. As we have already seen, this is described as doing positive work or doing negative work on the object. All the examples discussed up until now have been of motion in one dimension, but in three dimensions the force can be at any angle  $\theta$  with respect to the direction of motion.

What if the force is perpendicular to the direction of motion? We have already seen that a force perpendicular to the motion results in circular motion at constant speed. The kinetic energy does not change, and we conclude that no work is done when the force is perpendicular to the motion.

So far we have been reasoning about the case of a single force acting on an object, and changing only its kinetic energy. The result is more generally true, however. For instance, imagine a hockey puck sliding across the ice. The ice makes an upward normal force, but does not transfer energy to or from the puck.



A force can do positive, negative, or zero work, depending on its direction relative to the direction of the motion.

## Forces at other angles

Suppose the force is at some other angle with respect to the motion, say  $\theta = 45^\circ$ . Such a force could be broken down into two components, one along the direction of the motion and the other perpendicular to it. The force vector equals the vector sum of its two components, and the principle of vector addition of forces thus tells us that the work done by the total force cannot be any different than the sum of the works that would be done by the two forces by themselves. Since the component perpendicular to the motion does no work, the work done by the force must be

$$W = F_{\parallel} |\mathbf{d}| \quad , \quad [\text{work done by a constant force}]$$

where the vector  $\mathbf{d}$  is simply a less cumbersome version of the notation  $\Delta \mathbf{r}$ . This result can be rewritten via trigonometry as

$$W = |\mathbf{F}| |\mathbf{d}| \cos \theta \quad . \quad [\text{work done by a constant force}]$$

Even though this equation has vectors in it, it depends only on their magnitudes, and the magnitude of a vector is a scalar. Work is therefore still a scalar quantity, which only makes sense if it is defined as the transfer of energy. Ten gallons of gasoline have the ability to do a certain amount of mechanical work, and when you pull in to a full-service gas station you don't have to say "Fill 'er up with 10 gallons of south-going gas."

Students often wonder why this equation involves a cosine rather than a sine, or ask if it would ever be a sine. In vector addition, the treatment of sines and cosines seemed more equal and democratic, so why is the cosine so special now? The answer is that if we are going to describe, say, a velocity vector, we must give both the component *parallel* to the  $x$  axis and the component *perpendicular* to the  $x$  axis (i.e., the  $y$  component). In calculating work, however, the force component perpendicular to the motion is irrelevant — it changes the direction of motion without increasing or decreasing the energy of the object on which it acts. In this context, it is *only* the parallel force component that matters, so only the cosine occurs.

### self-check C

(a) Work is the transfer of energy. According to this definition, is the horse in the picture doing work on the pack? (b) If you calculate work by the method described in this section, is the horse in figure n doing work on the pack?

▷ Answer, p. 534

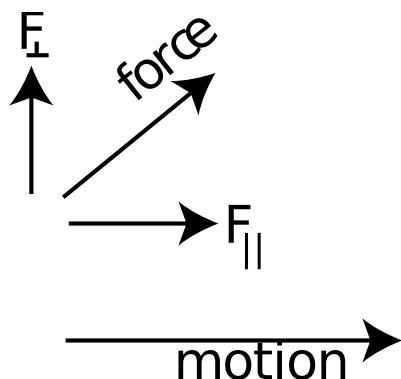
### Pushing a broom

### example 2

▷ If you exert a force of 21 N on a push broom, at an angle 35 degrees below horizontal, and walk for 5.0 m, how much work do you do? What is the physical significance of this quantity of work?

▷ Using the second equation above, the work done equals

$$(21 \text{ N})(5.0 \text{ m})(\cos 35^\circ) = 86 \text{ J} \quad .$$



m / Work is only done by the component of the force parallel to the motion.



n / Self-check. (Breaking Trail, by Walter E. Bohl.)

The form of energy being transferred is heat in the floor and the broom's bristles. This comes from the chemical energy stored in your body. (The majority of the calories you burn are dissipated directly as heat inside your body rather than doing any work on the broom. The 86 J is only the amount of energy transferred through the broom's handle.)

*A violin*

*example 3*

As a violinist draws the bow across a string, the bow hairs exert both a normal force and a kinetic frictional force on the string. The normal force is perpendicular to the direction of motion, and does no work. However, the frictional force is in the same direction as the motion of the bow, so it does work: energy is transferred to the string, causing it to vibrate.

One way of playing a violin more loudly is to use longer strokes. Since  $W = Fd$ , the greater distance results in more work.

A second way of getting a louder sound is to press the bow more firmly against the strings. This increases the normal force, and although the normal force itself does no work, an increase in the normal force has the side effect of increasing the frictional force, thereby increasing  $W = Fd$ .

The violinist moves the bow back and forth, and sound is produced on both the “up-bow” (the stroke toward the player's left) and the “down-bow” (to the right). One may, for example, play a series of notes in alternation between up-bows and down-bows. However, if the notes are of unequal length, the up and down motions tend to be unequal, and if the player is not careful, she can run out of bow in the middle of a note! To keep this from happening, one can move the bow more quickly on the shorter notes, but the resulting increase in  $d$  will make the shorter notes louder than they should be. A skilled player compensates by reducing the force.

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation  $W = |\mathbf{F}||\mathbf{d}|\cos\theta$  is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the *scalar product* of the vectors  $\mathbf{F}$  and  $\mathbf{d}$ . There is a standard shorthand notation for this operation,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta \quad , \quad \begin{array}{l} \text{[definition of the notation } \mathbf{A} \cdot \mathbf{B}; \\ \theta \text{ is the angle between vectors } \mathbf{A} \text{ and } \mathbf{B}] \end{array}$$

and because of this notation, a more common term for this operation

is the *dot product*. In dot product notation, the equation for work is simply

$$W = \mathbf{F} \cdot \mathbf{d} \quad .$$

The dot product has the following geometric interpretation:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}|(\text{component of } \mathbf{B} \text{ parallel to } \mathbf{A}) \\ &= |\mathbf{B}|(\text{component of } \mathbf{A} \text{ parallel to } \mathbf{B}) \end{aligned}$$

The dot product has some of the properties possessed by ordinary multiplication of numbers,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ (c\mathbf{A}) \cdot \mathbf{B} &= c(\mathbf{A} \cdot \mathbf{B}) \quad , \end{aligned}$$

but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily calculate their dot product as follows:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad .$$

(This can be proved by first analyzing the special case where each vector has only an  $x$  component, and the similar cases for  $y$  and  $z$ . We can then use the rule  $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$  to make a generalization by writing each vector as the sum of its  $x$ ,  $y$ , and  $z$  components. See homework problem 17.)

---

*Magnitude expressed with a dot product* *example 4*  
 If we take the dot product of any vector  $\mathbf{b}$  with itself, we find

$$\begin{aligned} \mathbf{b} \cdot \mathbf{b} &= (b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}) \cdot (b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}) \\ &= b_x^2 + b_y^2 + b_z^2 \quad , \end{aligned}$$

so its magnitude can be expressed as

$$|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} \quad .$$

We will often write  $b^2$  to mean  $\mathbf{b} \cdot \mathbf{b}$ , when the context makes it clear what is intended. For example, we could express kinetic energy as  $(1/2)m|\mathbf{v}|^2$ ,  $(1/2)m\mathbf{v} \cdot \mathbf{v}$ , or  $(1/2)mv^2$ . In the third version, nothing but context tells us that  $v$  really stands for the magnitude of some vector  $\mathbf{v}$ .

---

*Towing a barge* *example 5*  
 ▷ A mule pulls a barge with a force  $\mathbf{F} = (1100 \text{ N})\hat{\mathbf{x}} + (400 \text{ N})\hat{\mathbf{y}}$ , and the total distance it travels is  $(1000 \text{ m})\hat{\mathbf{x}}$ . How much work does it do?

▷ The dot product is  $1.1 \times 10^6 \text{ N} \cdot \text{m} = 1.1 \times 10^6 \text{ J}$ .

## 13.3 Varying force

Up until now we have done no actual calculations of work in cases where the force was not constant. The question of how to treat such cases is mathematically analogous to the issue of how to generalize the equation (distance) = (velocity)(time) to cases where the velocity was not constant. There, we found that the correct generalization was to find the area under the graph of velocity versus time. The equivalent thing can be done with work:

### general rule for calculating work

The work done by a force  $F$  equals the area under the curve on a graph of  $F_{\parallel}$  versus  $x$ . (Some ambiguities are encountered in cases such as kinetic friction.)

The examples in this section are ones in which the force is varying, but is always along the same line as the motion, so  $F$  is the same as  $F_{\parallel}$ .

#### self-check D

In which of the following examples would it be OK to calculate work using  $Fd$ , and in which ones would you have to use the area under the  $F - x$  graph?

- (a) A fishing boat cruises with a net dragging behind it.
- (b) A magnet leaps onto a refrigerator from a distance.
- (c) Earth's gravity does work on an outward-bound space probe.

Answer, p. 534

An important and straightforward example is the calculation of the work done by a spring that obeys Hooke's law,

$$F \approx -k(x - x_0) \quad .$$

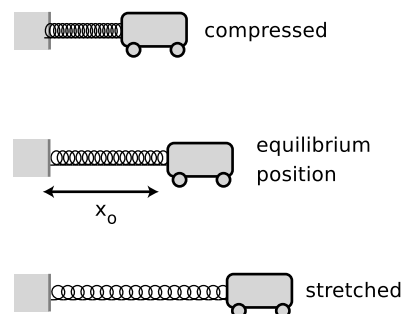
The minus sign is because this is the force being exerted by the spring, not the force that would have to act on the spring to keep it at this position. That is, if the position of the cart in figure o is to the right of equilibrium, the spring pulls back to the left, and vice-versa.

We calculate the work done when the spring is initially at equilibrium and then decelerates the car as the car moves to the right. The work done by the spring on the cart equals the minus area of the shaded triangle, because the triangle hangs below the  $x$  axis. The area of a triangle is half its base multiplied by its height, so

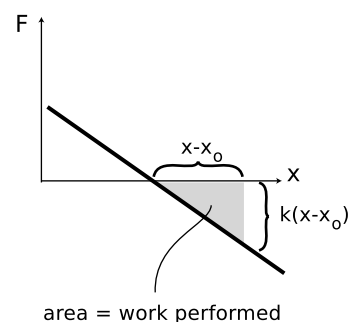
$$W = -\frac{1}{2}k(x - x_0)^2 \quad .$$

This is the amount of kinetic energy lost by the cart as the spring decelerates it.

It was straightforward to calculate the work done by the spring in this case because the graph of  $F$  versus  $x$  was a straight line, giving



o / The spring does work on the cart. (Unlike the ball in section 13.1, the cart is attached to the spring.)

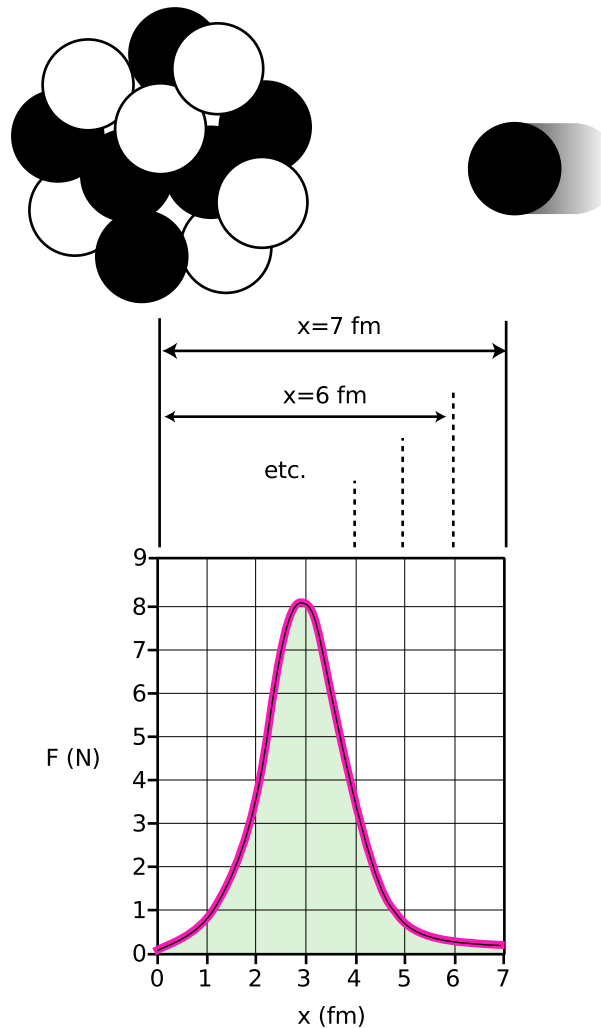


p / The area of the shaded triangle gives the work done by the spring as the cart moves from the equilibrium position to position  $x$ .



a triangular area. But if the curve had not been so geometrically simple, it might not have been possible to find a simple equation for the work done, or an equation might have been derivable only using calculus. Optional section 13.4 gives an important example of such an application of calculus.

q / Example 6.



#### *Energy production in the sun*

*example 6*

The sun produces energy through nuclear reactions in which nuclei collide and stick together. The figure depicts one such reaction, in which a single proton (hydrogen nucleus) collides with a carbon nucleus, consisting of six protons and six neutrons. Neutrons and protons attract other neutrons and protons via the strong nuclear force, so as the proton approaches the carbon nucleus it is accelerated. In the language of energy, we say that it loses nuclear potential energy and gains kinetic energy. Together, the seven protons and six neutrons make a nitrogen nucleus. Within the newly put-together nucleus, the neutrons and

protons are continually colliding, and the new proton's extra kinetic energy is rapidly shared out among all the neutrons and protons. Soon afterward, the nucleus calms down by releasing some energy in the form of a gamma ray, which helps to heat the sun.

The graph shows the force between the carbon nucleus and the proton as the proton is on its way in, with the distance in units of femtometers ( $1 \text{ fm} = 10^{-15} \text{ m}$ ). Amusingly, the force turns out to be a few newtons: on the same order of magnitude as the forces we encounter ordinarily on the human scale. Keep in mind, however, that a force this big exerted on a single subatomic particle such as a proton will produce a truly fantastic acceleration (on the order of  $10^{27} \text{ m/s}^2$ !).

Why does the force have a peak around  $x = 3 \text{ fm}$ , and become smaller once the proton has actually merged with the nucleus? At  $x = 3 \text{ fm}$ , the proton is at the edge of the crowd of protons and neutrons. It feels many attractive forces from the left, and none from the right. The forces add up to a large value. However if it later finds itself at the center of the nucleus,  $x = 0$ , there are forces pulling it from all directions, and these force vectors cancel out.

We can now calculate the energy released in this reaction by using the area under the graph to determine the amount of mechanical work done by the carbon nucleus on the proton. (For simplicity, we assume that the proton came in "aimed" at the center of the nucleus, and we ignore the fact that it has to shove some neutrons and protons out of the way in order to get there.) The area under the curve is about 17 squares, and the work represented by each square is

$$(1 \text{ N})(10^{-15} \text{ m}) = 10^{-15} \text{ J} \quad ,$$

so the total energy released is about

$$(10^{-15} \text{ J/square})(17 \text{ squares}) = 1.7 \times 10^{-14} \text{ J} \quad .$$

This may not seem like much, but remember that this is only a reaction between the nuclei of two out of the zillions of atoms in the sun. For comparison, a typical *chemical* reaction between two atoms might transform on the order of  $10^{-19} \text{ J}$  of electrical potential energy into heat — 100,000 times less energy!

As a final note, you may wonder why reactions such as these only occur in the sun. The reason is that there is a repulsive electrical force between nuclei. When two nuclei are close together, the electrical forces are typically about a million times weaker than the nuclear forces, but the nuclear forces fall off much more quickly with distance than the electrical forces, so the electrical force is the dominant one at longer ranges. The sun is a very hot gas, so

the random motion of its atoms is extremely rapid, and a collision between two atoms is sometimes violent enough to overcome this initial electrical repulsion.

## 13.4 $\int$ Applications of calculus

The student who has studied integral calculus will recognize that the graphical rule given in the previous section can be reexpressed as an integral,

$$W = \int_{x_1}^{x_2} F dx \quad .$$

We can then immediately find by the fundamental theorem of calculus that force is the derivative of work with respect to position,

$$F = \frac{dW}{dx} \quad .$$

For example, a crane raising a one-ton block on the moon would be transferring potential energy into the block at only one sixth the rate that would be required on Earth, and this corresponds to one sixth the force.

Although the work done by the spring could be calculated without calculus using the area of a triangle, there are many cases where the methods of calculus are needed in order to find an answer in closed form. The most important example is the work done by gravity when the change in height is not small enough to assume a constant force. Newton's law of gravity is

$$F = \frac{GMm}{r^2} \quad ,$$

which can be integrated to give

$$\begin{aligned} W &= \int_{r_1}^{r_2} \frac{GMm}{r^2} dr \\ &= -GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \quad . \end{aligned}$$

## 13.5 Work and potential energy

The techniques for calculating work can also be applied to the calculation of potential energy. If a certain force depends only on the distance between the two participating objects, then the energy released by changing the distance between them is defined as the potential energy, and the amount of potential energy lost equals minus the work done by the force,

$$\Delta PE = -W \quad .$$

The minus sign occurs because positive work indicates that the potential energy is being expended and converted to some other form.

It is sometimes convenient to pick some arbitrary position as a reference position, and derive an equation for once and for all that gives the potential energy relative to this position

$$PE_x = -W_{\text{ref} \rightarrow x} \quad . \quad [\text{potential energy at a point } x]$$

To find the energy transferred into or out of potential energy, one then subtracts two different values of this equation.

These equations might almost make it look as though work and energy were the same thing, but they are not. First, potential energy measures the energy that a system *has* stored in it, while work measures how much energy is *transferred* in or out. Second, the techniques for calculating work can be used to find the amount of energy transferred in many situations where there is no potential energy involved, as when we calculate the amount of kinetic energy transformed into heat by a car's brake shoes.

### A toy gun

### example 7

▷ A toy gun uses a spring with a spring constant of 10 N/m to shoot a ping-pong ball of mass 5 g. The spring is compressed to 10 cm shorter than its equilibrium length when the gun is loaded. At what speed is the ball released?

▷ The equilibrium point is the natural choice for a reference point. Using the equation found previously for the work, we have

$$PE_x = \frac{1}{2} k (x - x_0)^2 \quad .$$

The spring loses contact with the ball at the equilibrium point, so the final potential energy is

$$PE_f = 0 \quad .$$

The initial potential energy is

$$\begin{aligned} PE_i &= \frac{1}{2} (10 \text{ N/m}) (0.10 \text{ m})^2 \quad . \\ &= 0.05 \text{ J}. \end{aligned}$$

The loss in potential energy of 0.05 J means an increase in kinetic energy of the same amount. The velocity of the ball is found by solving the equation  $KE = (1/2)mv^2$  for  $v$ ,

$$\begin{aligned} v &= \sqrt{\frac{2KE}{m}} \\ &= \sqrt{\frac{(2)(0.05 \text{ J})}{0.005 \text{ kg}}} \\ &= 4 \text{ m/s} \quad . \end{aligned}$$

#### Gravitational potential energy

example 8

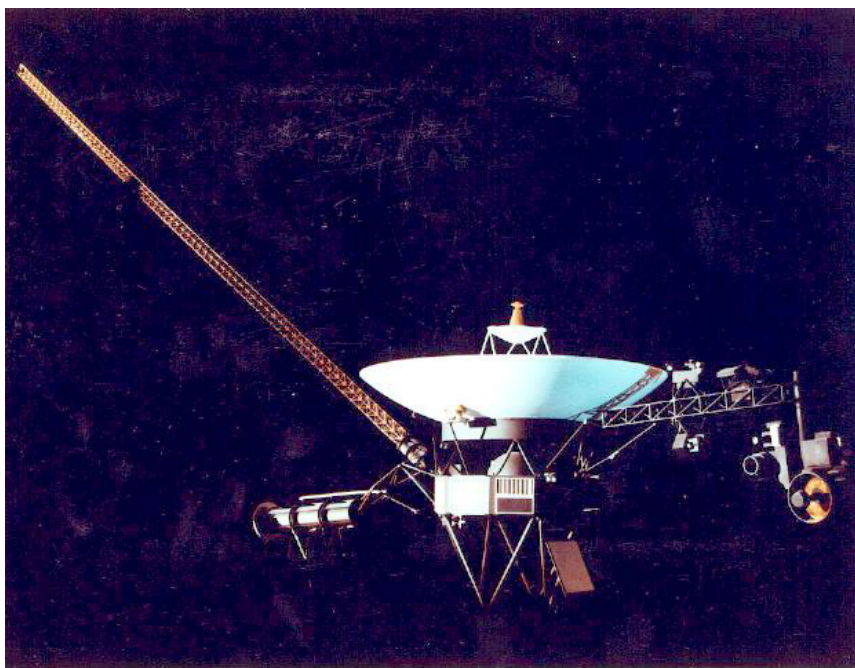
▷ We have already found the equation  $\Delta PE = -F\Delta y$  for the gravitational potential energy when the change in height is not enough to cause a significant change in the gravitational force  $F$ . What if the change in height is enough so that this assumption is no longer valid? Use the equation  $W = GMm(1/r_2 - 1/r_1)$  derived in section 13.4 to find the potential energy, using  $r = \infty$  as a reference point.

▷ The potential energy equals minus the work that would have to be done to bring the object from  $r_1 = \infty$  to  $r = r_2$ , which is

$$PE = -\frac{GMm}{r} \quad .$$

This is simpler than the equation for the work, which is an example of why it is advantageous to record an equation for potential energy relative to some reference point, rather than an equation for work.

Although the equations derived in the previous two examples may seem arcane and not particularly useful except for toy designers and rocket scientists, their usefulness is actually greater than it appears. The equation for the potential energy of a spring can be adapted to any other case in which an object is compressed, stretched, twisted, or bent. While you are not likely to use the equation for gravitational potential energy for anything practical, it is directly analogous to an equation that is extremely useful in chemistry, which is the equation for the potential energy of an electron at a distance  $r$  from the nucleus of its atom. As discussed in more detail later in the course, the electrical force between the electron and the nucleus is proportional to  $1/r^2$ , just like the gravitational force between two masses. Since the equation for the force is of the same form, so is the equation for the potential energy.



$r$  / The twin Voyager space probes were perhaps the greatest scientific successes of the space program. Over a period of decades, they flew by all the planets of the outer solar system, probably accomplishing more of scientific interest than the entire space shuttle program at a tiny fraction of the cost. Both Voyager probes completed their final planetary flybys with speeds greater than the escape velocity at that distance from the sun, and so headed on out of the solar system on hyperbolic orbits, never to return. Radio contact has been lost, and they are now likely to travel interstellar space for billions of years without colliding with anything or being detected by any intelligent species.

### Discussion questions

**A** What does the graph of  $PE = (1/2)k(x - x_0)^2$  look like as a function of  $x$ ? Discuss the physical significance of its features.

**B** What does the graph of  $PE = -GMm/r$  look like as a function of  $r$ ? Discuss the physical significance of its features. How would the equation and graph change if some other reference point was chosen rather than  $r = \infty$ ?

**C** Starting at a distance  $r$  from a planet of mass  $M$ , how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

**D** Does a spring have an “escape velocity?”

**E** Calculus-based question: If the form of energy being transferred is potential energy, then the equations  $F = dW/dx$  and  $W = \int Fdx$  become  $F = -dPE/dx$  and  $PE = -\int Fdx$ . How would you then apply the following calculus concepts: zero derivative at minima and maxima, and the second derivative test for concavity up or down.

## 13.6 ★ When does work equal force times distance?

In the example of the tractor pulling the plow discussed on page 309, the work did not equal  $Fd$ . The purpose of this section is to explain more fully how the quantity  $Fd$  can and cannot be used. To simplify things, I write  $Fd$  throughout this section, but more generally everything said here would be true for the area under the

graph of  $F_{\parallel}$  versus  $d$ .

The following two theorems allow most of the ambiguity to be cleared up.

### **the work-kinetic-energy theorem**

The change in kinetic energy associated with the motion of an object's center of mass is related to the total force acting on it and to the distance traveled by its center of mass according to the equation  $\Delta KE_{cm} = F_{total}d_{cm}$ .

This can be proved based on Newton's second law and the equation  $KE = (1/2)mv^2$ . Note that despite the traditional name, it does not necessarily tell the amount of work done, since the forces acting on the object could be changing other types of energy besides the KE associated with its center of mass motion.

The second theorem does relate directly to work:

When a contact force acts between two objects and the two surfaces do not slip past each other, the work done equals  $Fd$ , where  $d$  is the distance traveled by the point of contact.

This one has no generally accepted name, so we refer to it simply as the second theorem.

A great number of physical situations can be analyzed with these two theorems, and often it is advantageous to apply both of them to the same situation.

---

#### *An ice skater pushing off from a wall*

*example 9*

The work-kinetic energy theorem tells us how to calculate the skater's kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.

The second theorem tells us that the wall does no work on the skater. This makes sense, since the wall does not have any source of energy.

---

#### *Absorbing an impact without recoiling?*

*example 10*

▷ Is it possible to absorb an impact without recoiling? For instance, would a brick wall "give" at all if hit by a ping-pong ball?

▷ There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat and vibration. The second theorem tells us that we can only have nonzero work if the distance traveled by the point of contact is nonzero.

---

*Dragging a refrigerator at constant velocity* *example 11*

Newton's first law tells us that the total force on the refrigerator must be zero: your force is canceling the floor's kinetic frictional force. The work-kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator's kinetic energy doesn't change.

The second theorem tells us that the work you do equals your hand's force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.

---

*Accelerating a cart* *example 12*

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand's force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

Applying the second theorem to your force tells us how to calculate the work you do.

Applying the second theorem to the floor's force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, since the floor does not have any source of energy.

The work-kinetic energy theorem refers to the total force, and because the floor's backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart's center of mass. The rest goes into rotation of the wheels.

## 13.7 ★ The dot product

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation  $W = |\mathbf{F}||\mathbf{d}|\cos\theta$  is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the *scalar product* of the vectors  $\mathbf{F}$  and  $\mathbf{d}$ . There is a standard shorthand notation for



this operation,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta \quad , \quad \begin{array}{l} \text{[definition of the notation } \mathbf{A} \cdot \mathbf{B}; \\ \theta \text{ is the angle between vectors } \mathbf{A} \text{ and } \mathbf{B}] \end{array}$$

and because of this notation, a more common term for this operation is the *dot product*. In dot product notation, the equation for work is simply

$$W = \mathbf{F} \cdot \mathbf{d} \quad .$$

The dot product has the following geometric interpretation:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}|(\text{component of } \mathbf{B} \text{ parallel to } \mathbf{A}) \\ &= |\mathbf{B}|(\text{component of } \mathbf{A} \text{ parallel to } \mathbf{B}) \end{aligned}$$

The dot product has some of the properties possessed by ordinary multiplication of numbers,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ (c\mathbf{A}) \cdot \mathbf{B} &= c(\mathbf{A} \cdot \mathbf{B}) \quad , \end{aligned}$$

but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily calculate their dot product as follows:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad .$$

(This can be proved by first analyzing the special case where each vector has only an  $x$  component, and the similar cases for  $y$  and  $z$ . We can then use the rule  $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$  to make a generalization by writing each vector as the sum of its  $x$ ,  $y$ , and  $z$  components. See homework problem 17.)

---

*Magnitude expressed with a dot product* *example 13*

If we take the dot product of any vector  $\mathbf{b}$  with itself, we find

$$\begin{aligned} \mathbf{b} \cdot \mathbf{b} &= (b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}) \cdot (b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}) \\ &= b_x^2 + b_y^2 + b_z^2 \quad , \end{aligned}$$

so its magnitude can be expressed as

$$|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} \quad .$$

We will often write  $b^2$  to mean  $\mathbf{b} \cdot \mathbf{b}$ , when the context makes it clear what is intended. For example, we could express kinetic energy as  $(1/2)m|\mathbf{v}|^2$ ,  $(1/2)m\mathbf{v} \cdot \mathbf{v}$ , or  $(1/2)mv^2$ . In the third version, nothing but context tells us that  $v$  really stands for the magnitude of some vector  $\mathbf{v}$ .

---

*Towing a barge* *example 14*

▷ A mule pulls a barge with a force  $\mathbf{F} = (1100 \text{ N})\hat{\mathbf{x}} + (400 \text{ N})\hat{\mathbf{y}}$ , and the total distance it travels is  $(1000 \text{ m})\hat{\mathbf{x}}$ . How much work does it do?

▷ The dot product is  $1.1 \times 10^6 \text{ N} \cdot \text{m} = 1.1 \times 10^6 \text{ J}$ .

## Summary

### Selected vocabulary

work . . . . . the amount of energy transferred into or out of a system, excluding energy transferred by heat conduction

### Notation

$W$  . . . . . work

### Summary

Work is a measure of the transfer of mechanical energy, i.e., the transfer of energy by a force rather than by heat conduction. When the force is constant, work can usually be calculated as

$$W = F_{\parallel} |\mathbf{d}| \quad , \quad [\text{only if the force is constant}]$$

where  $\mathbf{d}$  is simply a less cumbersome notation for  $\Delta \mathbf{r}$ , the vector from the initial position to the final position. Thus,

- A force in the same direction as the motion does positive work, i.e., transfers energy into the object on which it acts.
- A force in the opposite direction compared to the motion does negative work, i.e., transfers energy out of the object on which it acts.
- When there is no motion, no mechanical work is done. The human body burns calories when it exerts a force without moving, but this is an internal energy transfer of energy within the body, and thus does not fall within the scientific definition of work.
- A force perpendicular to the motion does no work.

When the force is not constant, the above equation should be generalized as the area under the graph of  $F_{\parallel}$  versus  $d$ .

Machines such as pulleys, levers, and gears may increase or decrease a force, but they can never increase or decrease the amount of work done. That would violate conservation of energy unless the machine had some source of stored energy or some way to accept and store up energy.

There are some situations in which the equation  $W = F_{\parallel} |\mathbf{d}|$  is ambiguous or not true, and these issues are discussed rigorously in section 13.6. However, problems can usually be avoided by analyzing the types of energy being transferred before plunging into the math. In any case there is no substitute for a physical understanding of the processes involved.

The techniques developed for calculating work can also be applied to the calculation of potential energy. We fix some position

as a reference position, and calculate the potential energy for some other position,  $x$ , as

$$PE_x = -W_{\text{ref} \rightarrow x} \quad .$$

The following two equations for potential energy have broader significance than might be suspected based on the limited situations in which they were derived:

$$PE = \frac{1}{2}k(x - x_o)^2 \quad .$$

[potential energy of a spring having spring constant  $k$ , when stretched or compressed from the equilibrium position  $x_o$ ; analogous equations apply for the twisting, bending, compression, or stretching of any object.]

$$PE = -\frac{GMm}{r}$$

[gravitational potential energy of objects of masses  $M$  and  $m$ , separated by a distance  $r$ ; an analogous equation applies to the electrical potential energy of an electron in an atom.]

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it's important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats' stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

**2** In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck's bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

**3** In the earth's atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the  $O_2$  molecules is the same as the average KE of the  $N_2$  molecules. (a) If the mass of an  $O_2$  molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 21 first, for insight.

**4** Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you'll need to discuss what specific type of energy is involved in each case.

**5** In one of his more flamboyant moments, Galileo wrote "Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon." Find the speed of an ant that falls to earth from the distance of the moon at the

moment when it is about to enter the atmosphere. Assume it is released from a point that is not actually near the moon, so the moon's gravity is negligible. You will need the result of example 8 on p. 322. ✓

6 [Problem 6 has been deleted.]

7 (a) The crew of an 18th century warship is raising the anchor. The anchor has a mass of 5000 kg. The water is 30 m deep. The chain to which the anchor is attached has a mass per unit length of 150 kg/m. Before they start raising the anchor, what is the total weight of the anchor plus the portion of the chain hanging out of the ship? (Assume that the buoyancy of the anchor and is negligible.)

(b) After they have raised the anchor by 1 m, what is the weight they are raising?

(c) Define  $y = 0$  when the anchor is resting on the bottom, and  $y = +30$  m when it has been raised up to the ship. Draw a graph of the force the crew has to exert to raise the anchor and chain, as a function of  $y$ . (Assume that they are raising it slowly, so water resistance is negligible.) It will not be a constant! Now find the area under the graph, and determine the work done by the crew in raising the anchor, in joules.

(d) Convert your answer from (c) into units of kcal. ✓

8 In the power stroke of a car's gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture's force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by  $F = a/x$ , where  $x$  is the distance from the cylinder to the piston head, and  $a$  is a constant with units of N·m. (Actually  $a/x^{1.4}$  would be more accurate, but the problem works out more nicely with  $a/x$ !) The piston begins its stroke at  $x = x_1$ , and ends at  $x = x_2$ . The 1965 Rambler had six cylinders, each with  $a = 220$  N·m,  $x_1 = 1.2$  cm, and  $x_2 = 10.2$  cm.

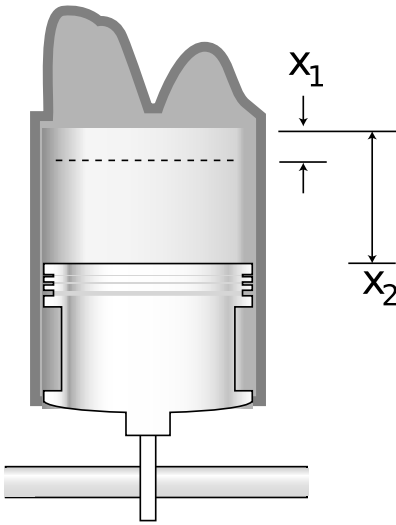
(a) Draw a neat, accurate graph of  $F$  vs  $x$ , on graph paper.

(b) From the area under the curve, derive the amount of work done in one stroke by one cylinder. ✓

(c) Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine's power, in units of horsepower (1 hp=746 W). ✓

(d) The compression ratio of an engine is defined as  $x_2/x_1$ . Explain in words why the car's power would be exactly the same if  $x_1$  and  $x_2$  were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would *not* quite be true with the more realistic force equation  $F = a/x^{1.4}$ .

9 The magnitude of the force between two magnets separated by a distance  $r$  can be approximated as  $kr^{-3}$  for large values of  $r$ . The constant  $k$  depends on the strengths of the magnets and the



Problem 8: A cylinder from the 1965 Rambler's engine. The piston is shown in its pushed out position. The two bulges at the top are for the valves that let fresh air-gas mixture in. Based on a figure from Motor Service's Automotive Encyclopedia, Toboldt and Purvis.

relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance  $r_i$ , and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance  $r_f$ ? (Ignore friction.)  $\checkmark \int$

**10** A car starts from rest at  $t = 0$ , and starts speeding up with constant acceleration. (a) Find the car's kinetic energy in terms of its mass,  $m$ , acceleration,  $a$ , and the time,  $t$ . (b) Your answer in the previous part also equals the amount of work,  $W$ , done from  $t = 0$  until time  $t$ . Take the derivative of the previous expression to find the power expended by the car at time  $t$ . (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.)  $\checkmark \int$

**11** A space probe of mass  $m$  is dropped into a previously unexplored spherical cloud of gas and dust, and accelerates toward the center of the cloud under the influence of the cloud's gravity. Measurements of its velocity allow its potential energy,  $PE$ , to be determined as a function of the distance  $r$  from the cloud's center. The mass in the cloud is distributed in a spherically symmetric way, so its density,  $\rho(r)$ , depends only on  $r$  and not on the angular coordinates. Show that by finding  $PE$ , one can infer  $\rho(r)$  as follows:

$$\rho(r) = \frac{1}{4\pi Gmr^2} \frac{d}{dr} \left( r^2 \frac{dPE}{dr} \right) \quad .$$

$\int \star$

**12** A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by  $F = be^{-cx}$ , where  $x$  is the altitude,  $b$  and  $c$  are constants, and  $e$  is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.)  $\checkmark \int$

**13** A certain binary star system consists of two stars with masses  $m_1$  and  $m_2$ , separated by a distance  $b$ . A comet, originally nearly at rest in deep space, drops into the system and at a certain point

in time arrives at the midpoint between the two stars. For that moment in time, find its velocity,  $v$ , symbolically in terms of  $b$ ,  $m_1$ ,  $m_2$ , and fundamental constants. ✓

**14** An airplane flies in the positive direction along the  $x$  axis, through crosswinds that exert a force  $\mathbf{F} = (a + bx)\hat{\mathbf{x}} + (c + dx)\hat{\mathbf{y}}$ . Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position  $x$ .  $\int$

**15** In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance  $r$ , was  $PE(r) = gr^{-1}e^{-r/a}$ , where  $g$  parametrizes the strength of the interaction,  $e$  is the base of natural logarithms, and  $a$  is about  $10^{-15}$  m. Find the force between two nucleons that would be consistent with this equation for the potential energy. ✓  $\int$

**16** Prove that the dot product defined in section 13.7 is rotationally invariant in the sense of section 7.5.

**17** Fill in the details of the proof of  $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$  on page 326.

**18** Does it make sense to say that work is conserved?

▷ Solution, p. 528

**19** (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work. (b) Now answer the question based on the  $W = Fd$  rule.

**20** A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the oven doesn't heat the solid very quickly, although eventually melting begins in one small spot. Once a melted spot forms, it grows rapidly, while the rest of the solid remains solid. In other words, it appears based on this experiment that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids. (See, e.g., figure b on page 295.)

Please don't repeat the following common mistakes in your explanation:

*In a solid, the atoms are packed more tightly and have less space between them.* Not true. Ice floats because it's *less* dense than water.

*In a liquid, the atoms are moving much faster.* No, the difference in average speed between ice at  $-1^\circ\text{C}$  and water at  $1^\circ\text{C}$  is only 0.4%.

**21** Starting at a distance  $r$  from a planet of mass  $M$ , how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it? ✓

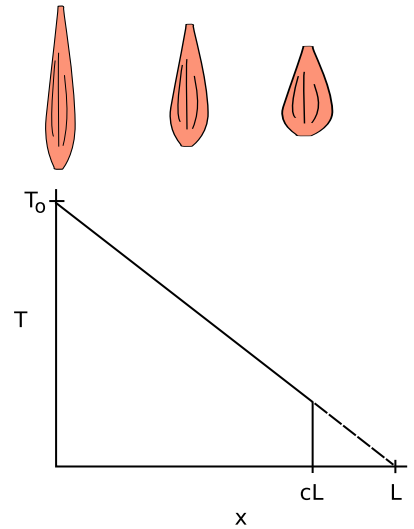
**22** The figure, redrawn from *Gray's Anatomy*, shows the tension of which a muscle is capable. The variable  $x$  is defined as the contraction of the muscle from its maximum length  $L$ , so that at  $x = 0$  the muscle has length  $L$ , and at  $x = L$  the muscle would theoretically have zero length. In reality, the muscle can only contract to  $x = cL$ , where  $c$  is less than 1. When the muscle is extended to its maximum length, at  $x = 0$ , it is capable of the greatest tension,  $T_0$ . As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at  $x = L$ . (a) Find the maximum work the muscle can do in one contraction, in terms of  $c$ ,  $L$ , and  $T_0$ . ✓

(b) Show that your answer to part a has the right units.

(c) Show that your answer to part a has the right behavior when  $c = 0$  and when  $c = 1$ .

(d) Gray also states that the absolute maximum tension  $T_0$  has been found to be approximately proportional to the muscle's cross-sectional area  $A$  (which is presumably measured at  $x = 0$ ), with proportionality constant  $k$ . Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume,  $V$ , eliminating  $L$  and  $A$ . ✓

(e) Evaluate your result numerically for a biceps muscle with a volume of  $200 \text{ cm}^3$ , with  $c = 0.8$  and  $k = 100 \text{ N/cm}^2$  as estimated by Gray. ✓



Problem 22.

**23** A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will “burn rubber” (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.

(a) Show that if a force  $F$  is applied to an object moving at speed  $v$ , the power required is given by  $P = vF$ .

(b) Find the speed  $v$  at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car's power  $P$ , its mass  $m$ , the coefficient of static friction  $\mu_s$ , and  $g$ . ✓

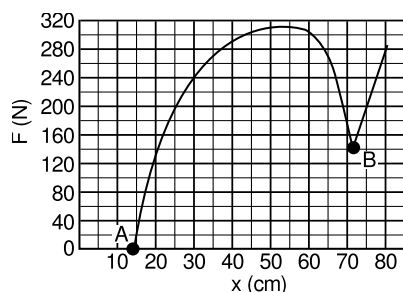
(c) Show that your answer to part b has units that make sense.



(d) Show that the dependence of your answer on each of the four variables makes sense physically.

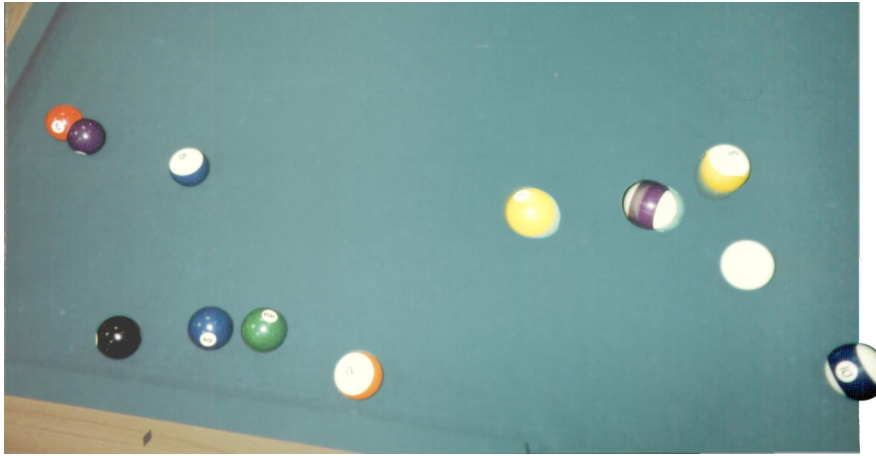
(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of  $3.23 \times 10^5$  W (433 horsepower) and a mass (including a 50-kg driver) of  $2.03 \times 10^3$  kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is at moving at  $v$ .) Rubber on asphalt has  $\mu_s \approx 0.9$ . Find  $v$  for this car. Answer: 18 m/s, or about 40 miles per hour.

(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to  $v^2$ . The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine's maximum power is being used to overcome air resistance when the car is moving at the speed  $v$  found in part e. Answer: 1%



Problem 24.

**24** Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where  $F$  is the force required to pull the string back, and  $x$  is the distance between the string and the center of the bow's body. It is not a linear Hooke's-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can't be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a 26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.) ✓



Pool balls exchange momentum.

## Chapter 14

# Conservation of momentum

In many subfields of physics these days, it is possible to read an entire issue of a journal without ever encountering an equation involving force or a reference to Newton's laws of motion. In the last hundred and fifty years, an entirely different framework has been developed for physics, based on conservation laws.

The new approach is not just preferred because it is in fashion. It applies inside an atom or near a black hole, where Newton's laws do not. Even in everyday situations the new approach can be superior. We have already seen how perpetual motion machines could be designed that were too complex to be easily debunked by Newton's laws. The beauty of conservation laws is that they tell us something must remain the same, regardless of the complexity of the process.

So far we have discussed only two conservation laws, the laws of conservation of mass and energy. Is there any reason to believe that further conservation laws are needed in order to replace Newton's laws as a complete description of nature? Yes. Conservation of mass and energy do not relate in any way to the three dimensions of space, because both are scalars. Conservation of energy, for instance, does not prevent the planet earth from abruptly making a 90-degree turn and heading straight into the sun, because kinetic energy does not depend on direction. In this chapter, we develop a new conserved quantity, called momentum, which is a vector.

## 14.1 Momentum

### A conserved quantity of motion

Your first encounter with conservation of momentum may have come as a small child unjustly confined to a shopping cart. You spot something interesting to play with, like the display case of imported wine down at the end of the aisle, and decide to push the cart over there. But being imprisoned by Dad in the cart was not the only injustice that day. There was a far greater conspiracy to thwart your young id, one that originated in the laws of nature. Pushing forward did nudge the cart forward, but it pushed you backward. If the wheels of the cart were well lubricated, it wouldn't matter how you jerked, yanked, or kicked off from the back of the cart. You could not cause any overall forward motion of the entire system consisting of the cart with you inside.

In the Newtonian framework, we describe this as arising from Newton's third law. The cart made a force on you that was equal and opposite to your force on it. In the framework of conservation laws, we cannot attribute your frustration to conservation of energy. It would have been perfectly possible for you to transform some of the internal chemical energy stored in your body to kinetic energy of the cart and your body.

The following characteristics of the situation suggest that there may be a new conservation law involved:

**A closed system is involved.** All conservation laws deal with closed systems. You and the cart are a closed system, since the well-oiled wheels prevent the floor from making any forward force on you.

**Something remains unchanged.** The overall velocity of the system started out being zero, and you cannot change it. This vague reference to "overall velocity" can be made more precise: it is the velocity of the system's center of mass that cannot be changed.

**Something can be transferred back and forth without changing the total amount.** If we define forward as positive and backward as negative, then one part of the system can gain positive motion if another part acquires negative motion. If we don't want to worry about positive and negative signs, we can imagine that the whole cart was initially gliding forward on its well-oiled wheels. By kicking off from the back of the cart, you could increase your own velocity, but this inevitably causes the cart to slow down.

It thus appears that there is some numerical measure of an object's quantity of motion that is conserved when you add up all the objects within a system.

## Momentum

Although velocity has been referred to, it is not the total velocity of a closed system that remains constant. If it was, then firing a gun would cause the gun to recoil at the same velocity as the bullet! The gun does recoil, but at a much lower velocity than the bullet. Newton's third law tells us

$$F_{\text{gun on bullet}} = -F_{\text{bullet on gun}} \quad ,$$

and assuming a constant force for simplicity, Newton's second law allows us to change this to

$$m_{\text{bullet}} \frac{\Delta v_{\text{bullet}}}{\Delta t} = -m_{\text{gun}} \frac{\Delta v_{\text{gun}}}{\Delta t} \quad .$$

Thus if the gun has 100 times more mass than the bullet, it will recoil at a velocity that is 100 times smaller and in the opposite direction, represented by the opposite sign. The quantity  $mv$  is therefore apparently a useful measure of motion, and we give it a name, *momentum*, and a symbol,  $p$ . (As far as I know, the letter "p" was just chosen at random, since "m" was already being used for mass.) The situations discussed so far have been one-dimensional, but in three-dimensional situations it is treated as a vector.

### definition of momentum for material objects

The momentum of a material object, i.e., a piece of matter, is defined as

$$\mathbf{p} = m\mathbf{v} \quad ,$$

the product of the object's mass and its velocity vector.

The units of momentum are  $\text{kg}\cdot\text{m/s}$ , and there is unfortunately no abbreviation for this clumsy combination of units.

The reasoning leading up to the definition of momentum was all based on the search for a conservation law, and the only reason why we bother to define such a quantity is that experiments show it is conserved:

### the law of conservation of momentum

In any closed system, the vector sum of all the momenta remains constant,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} + \dots = \mathbf{p}_{1f} + \mathbf{p}_{2f} + \dots \quad ,$$

where  $i$  labels the initial and  $f$  the final momenta. (A closed system is one on which no external forces act.)

This chapter first addresses the one-dimensional case, in which the direction of the momentum can be taken into account by using plus and minus signs. We then pass to three dimensions, necessitating the use of vector addition.

A subtle point about conservation laws is that they all refer to “closed systems,” but “closed” means different things in different cases. When discussing conservation of mass, “closed” means a system that doesn’t have matter moving in or out of it. With energy, we mean that there is no work or heat transfer occurring across the boundary of the system. For momentum conservation, “closed” means there are no external *forces* reaching into the system.

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*A cannon* *example 1*

▷ A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

▷ The law of conservation of momentum tells us that

$$p_{\text{cannon},i} + p_{\text{shell},i} = p_{\text{cannon},f} + p_{\text{shell},f} \quad .$$

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

$$p_{\text{cannon},i} = 0$$

$$p_{\text{shell},i} = 0$$

$$p_{\text{shell},f} = 2000 \text{ kg} \cdot \text{m/s} \quad .$$

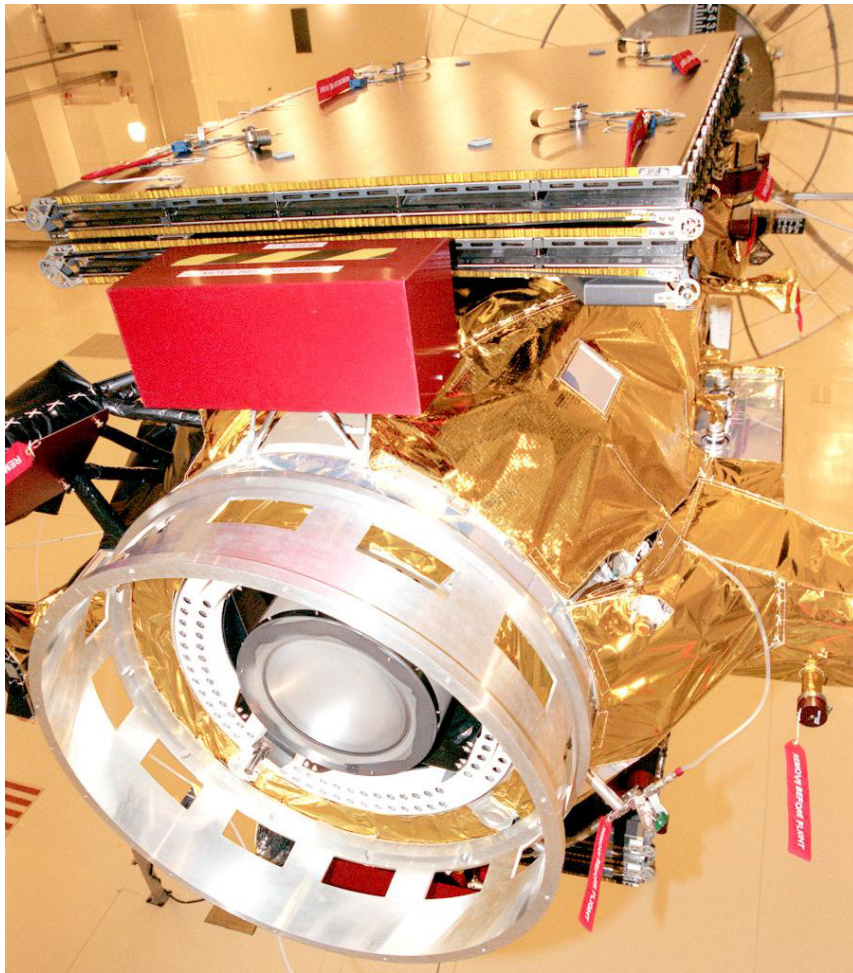
We must have  $p_{\text{cannon},f} = -2000 \text{ kg} \cdot \text{m/s}$ , so the recoil velocity of the cannon is  $-2 \text{ m/s}$ .

---

*Ion drive for propelling spacecraft* *example 2*

▷ The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel (“reaction mass”) available for pushing out the back as exhaust?

▷ Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive’s exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after



a / The ion drive engine of the NASA Deep Space 1 probe, shown under construction (left) and being tested in a vacuum chamber (right) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless carried out a successful scientific program that included a flyby of a comet.

all the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

### Generalization of the momentum concept

As with all the conservation laws, the law of conservation of momentum has evolved over time. In the 1800's it was found that a beam of light striking an object would give it some momentum, even though light has no mass, and would therefore have no momentum

according to the above definition. Rather than discarding the principle of conservation of momentum, the physicists of the time decided to see if the definition of momentum could be extended to include momentum carried by light. The process is analogous to the process outlined on page 279 for identifying new forms of energy. The first step was the discovery that light could impart momentum to matter, and the second step was to show that the momentum possessed by light could be related in a definite way to observable properties of the light. They found that conservation of momentum could be successfully generalized by attributing to a beam of light a momentum vector in the direction of the light's motion and having a magnitude proportional to the amount of energy the light possessed. The momentum of light is negligible under ordinary circumstances, e.g., a flashlight left on for an hour would only absorb about  $10^{-5}$  kg·m/s of momentum as it recoiled.



b / Steam and other gases boiling off of the nucleus of Halley's comet. This close-up photo was taken by the European Giotto space probe, which passed within 596 km of the nucleus on March 13, 1986.



c / Halley's comet, in a much less magnified view from a ground-based telescope.

#### *The tail of a comet*

#### *example 3*

Momentum is not always equal to  $mv$ . Like many comets, Halley's comet has a very elongated elliptical orbit. About once per century, its orbit brings it close to the sun. The comet's head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight boils off steam and dust, b. The sunlight does not just carry energy, however — it also carries momentum. The momentum of the sunlight impacting on the smaller dust particles pushes them away from the sun, forming a tail, c. By analogy with matter, for which momentum equals  $mv$ , you would expect that massless light would have zero momentum, but the equation  $p = mv$  is not the correct one for light, and light does have momentum. (The gases typically form a second, distinct tail whose motion is controlled by the sun's magnetic field.)

The reason for bringing this up is not so that you can plug numbers into a formulas in these exotic situations. The point is that the conservation laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. Newton's laws are no longer at the center of the stage of physics because they did not have the same adaptability. More generally, the moral of this story is the provisional nature of scientific truth.

It should also be noted that conservation of momentum is not a consequence of Newton's laws, as is often asserted in textbooks. Newton's laws do not apply to light, and therefore could not possibly be used to prove anything about a concept as general as the conservation of momentum in its modern form.

### **Momentum compared to kinetic energy**

Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibnitz controversy over who invented calculus was an argument over whether  $mv$  (i.e., momentum) or  $mv^2$  (i.e., kinetic energy without the  $1/2$  in front)



was the “true” measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700’s. The following table highlights their differences.

<b>kinetic energy ...</b>	<b>momentum ...</b>
is a scalar.	is a vector
is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.	is changed by any force, since a change in either the magnitude or the direction of the velocity vector will result in a change in the momentum vector.
is always positive, and cannot cancel out.	cancels with momentum in the opposite direction.
can be traded for other forms of energy that do not involve motion. KE is not a conserved quantity by itself.	is always conserved in a closed system.
is quadrupled if the velocity is doubled.	is doubled if the velocity is doubled.

*A spinning top* *example 4*

A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

*Momentum and kinetic energy in firing a rifle* *example 5*

The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle’s backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun’s “backward” kinetic energy does not refrigerate the shooter’s shoulder!

*The wobbly earth* *example 6*

As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth’s gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little “orbit” about a point below its surface on the line connecting it and the moon. The two bodies’ momentum vectors always point in opposite directions and cancel each other out.



---

*The earth and moon get a divorce*

*example 7*

Why can't the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon's newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

---

*Momentum and kinetic energy of a glacier*

*example 8*

A cubic-kilometer glacier would have a mass of about  $10^{12}$  kg. If it moves at a speed of  $10^{-5}$  m/s, then its momentum is  $10^7$  kg·m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.



d / This Hubble Space Telescope photo shows a small galaxy (yellow blob in the lower right) that has collided with a larger galaxy (spiral near the center), producing a wave of star formation (blue track) due to the shock waves passing through the galaxies' clouds of gas. This is considered a collision in the physics sense, even though it is statistically certain that no star in either galaxy ever struck a star in the other. (This is because the stars are very small compared to the distances between them.)

### Discussion questions

- A** If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum as well as conservation of energy?
- B** A refrigerator has coils in back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn't the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?

## 14.2 Collisions in one dimension

Physicists employ the term “collision” in a broader sense than ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a radioactively emitted particle damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun exerted gravitational forces on each other.

The reason for broadening the term “collision” in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In the first example, conservation of momentum is all that is required.

▷ Ms. Chang is rear-ended at a stop light by Mr. Nelson, and sues to make him pay her medical bills. He testifies that he was only going 35 miles per hour when he hit Ms. Chang. She thinks he was going much faster than that. The cars skidded together after the impact, and measurements of the length of the skid marks and the coefficient of friction show that their joint velocity immediately after the impact was 19 miles per hour. Mr. Nelson's Nissan weighs 3100 pounds, and Ms. Chang's Cadillac weighs 5200 pounds. Is Mr. Nelson telling the truth?

▷ Since the cars skidded together, we can write down the equation for conservation of momentum using only two velocities,  $v$  for Mr. Nelson's velocity before the crash, and  $v'$  for their joint velocity afterward:

$$m_N v = m_N v' + m_C v' \quad .$$

Solving for the unknown,  $v$ , we find

$$v = \left( 1 + \frac{m_C}{m_N} \right) v' \quad .$$

Although we are given the weights in pounds, a unit of force, the ratio of the masses is the same as the ratio of the weights, and we find  $v = 51$  miles per hour. He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many collisions, part of the kinetic energy that was present before the collision is used to create heat or sound, or to break the objects or permanently bend them. Cars, in fact, are carefully designed to crumple in a collision. Crumpling the car uses up energy, and that's good because the goal is to get rid of all that kinetic energy in a relatively safe and controlled way. At the opposite extreme, a superball is "super" because it emerges from a collision with almost all its original kinetic energy, having only stored it briefly as potential energy while it was being squashed by the impact.

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have  $KE_f = KE_i$ , as opposed to the less useful inequality  $KE_f < KE_i$  for a case like a tennis ball bouncing on grass.

▷ Two pool balls collide head-on, so that the collision is restricted to one dimension. Pool balls are constructed so as to lose as little kinetic energy as possible in a collision, so under the assumption that no kinetic energy is converted to any other form of energy, what can we predict about the results of such a collision?

▷ Pool balls have identical masses, so we use the same symbol  $m$  for both. Conservation of momentum and no loss of kinetic energy give us the two equations

$$mv_{1i} + mv_{2i} = mv_{1f} + mv_{2f}$$

$$\frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$$

The masses and the factors of  $1/2$  can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols  $v_{1i}, \dots$  with the symbols  $A, B, C$ , and  $D$ :

$$A + B = C + D$$

$$A^2 + B^2 = C^2 + D^2 \quad .$$

A little experimentation with numbers shows that given values of  $A$  and  $B$ , it is impossible to find  $C$  and  $D$  that satisfy these equations unless  $C$  and  $D$  equal  $A$  and  $B$ , or  $C$  and  $D$  are the same as  $A$  and  $B$  but swapped around. A formal proof of this fact is given in the sidebar. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.

Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at pool notices a case where her cue ball hits an initially stationary ball and stops dead. “Wow, what a good trick,” she thinks. “I bet I could never do that again in a million years.” But she tries again, and finds that she can’t help doing it even if she doesn’t want to. Luckily she has just learned about collisions in her physics course. Once she has written down the equations for conservation of energy and no loss of kinetic energy, she really doesn’t have to complete the algebra. She knows that she has two equations in two unknowns, so there must be a well-defined solution. Once she has seen the result of one such collision, she knows that the same thing must happen every time. The same thing would happen with colliding marbles or croquet balls. It doesn’t matter if the masses or velocities are different, because that just multiplies both equations by some constant factor.

#### Gory Details of the Proof in Example 10

The equation  $A + B = C + D$  says that the change in one ball’s velocity is equal and opposite to the change in the other’s. We invent a symbol  $x = C - A$  for the change in ball 1’s velocity. The second equation can then be rewritten as  $A^2 + B^2 = (A+x)^2 + (B-x)^2$ . Squaring out the quantities in parentheses and then simplifying, we get  $0 = Ax - Bx + x^2$ . The equation has the trivial solution  $x = 0$ , i.e., neither ball’s velocity is changed, but this is physically impossible because the balls can’t travel through each other like ghosts. Assuming  $x \neq 0$ , we can divide by  $x$  and solve for  $x = B - A$ . This means that ball 1 has gained an amount of velocity exactly right to match ball 2’s initial velocity, and vice-versa. The balls must have swapped velocities.

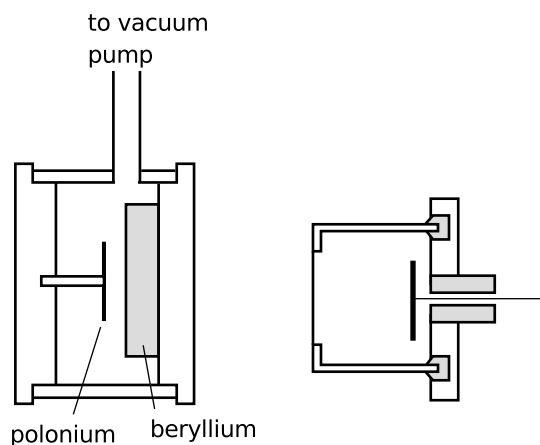
## The discovery of the neutron

This was the type of reasoning employed by James Chadwick in his 1932 discovery of the neutron. At the time, the atom was imagined to be made out of two types of fundamental particles, protons and electrons. The protons were far more massive, and clustered together in the atom's core, or nucleus. Attractive electrical forces caused the electrons to orbit the nucleus in circles, in much the same way that gravitational forces kept the planets from cruising out of the solar system. Experiments showed that the helium nucleus, for instance, exerted exactly twice as much electrical force on an electron as a nucleus of hydrogen, the smallest atom, and this was explained by saying that helium had two protons to hydrogen's one. The trouble was that according to this model, helium would have two electrons and two protons, giving it precisely twice the mass of a hydrogen atom with one of each. In fact, helium has about four times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two additional particles of a new type, which did not participate in electrical forces at all, i.e., were electrically neutral. If these particles had very nearly the same mass as protons, then the four-to-one mass ratio of helium and hydrogen could be explained. In 1930, a new type of radiation was discovered that seemed to fit this description. It was electrically neutral, and seemed to be coming from the nuclei of light elements that had been exposed to other types of radiation. At this time, however, reports of new types of particles were a dime a dozen, and most of them turned out to be either clusters made of previously known particles or else previously known particles with higher energies. Many physicists believed that the "new" particle that had attracted Chadwick's interest was really a previously known particle called a gamma ray, which was electrically neutral. Since gamma rays have no mass, Chadwick decided to try to determine the new particle's mass and see if it was nonzero and approximately equal to the mass of a proton.

Unfortunately a subatomic particle is not something you can just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy



Chadwick's subatomic pool table. A disk of the naturally occurring metal polonium provides a source of radiation capable of kicking neutrons out of the beryllium nuclei. The type of radiation emitted by the polonium is easily absorbed by a few mm of air, so the air has to be pumped out of the left-hand chamber. The neutrons, Chadwick's mystery particles, penetrate matter far more readily, and fly out through the wall and into the chamber on the right, which is filled with nitrogen or hydrogen gas. When a neutron collides with a nitrogen or hydrogen nucleus, it kicks it out of its atom at high speed, and this recoiling nucleus then rips apart thousands of other atoms of the gas. The result is an electrical pulse that can be detected in the wire on the right. Physicists had already calibrated this type of apparatus so that they could translate the strength of the electrical pulse into the velocity of the recoiling nucleus. The whole apparatus shown in the figure would fit in the palm of your hand, in dramatic contrast to today's giant particle accelerators.

to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:

equation #1: conservation of momentum

equation #2: no loss of kinetic energy

unknown #1: mass of the mystery particle

unknown #2: initial velocity of the mystery particle

unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

equation #3: conservation of momentum in the new collision

equation #4: no loss of kinetic energy in the new collision

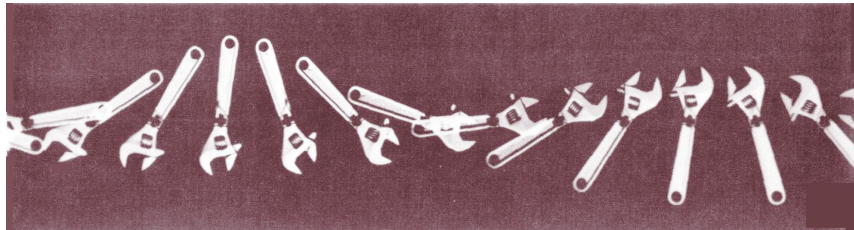
unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

### Discussion question

**A** Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example above?

## 14.3 ★ Relationship of momentum to the center of mass

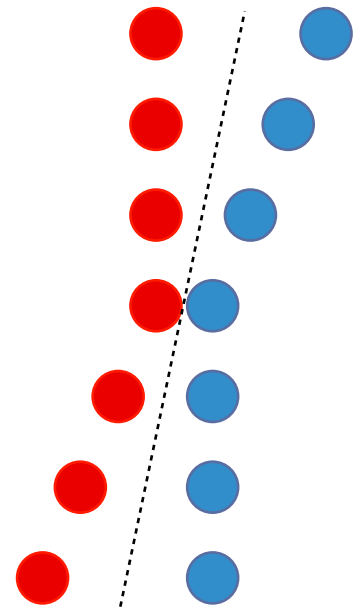


f / In this multiple-flash photograph, we see the wrench from above as it flies through the air, rotating as it goes. Its center of mass, marked with the black cross, travels along a straight line, unlike the other points on the wrench, which execute loops.

We have already discussed the idea of the center of mass on p. 67, but using the concept of momentum we can now find a mathematical method for defining the center of mass, explain why the motion of an object's center of mass usually exhibits simpler motion than any other point, and gain a very simple and powerful way of understanding collisions.

The first step is to realize that the center of mass concept can be applied to systems containing more than one object. Even something like a wrench, which we think of as one object, is really made of many atoms. The center of mass is particularly easy to visualize in the case shown on the left, where two identical hockey pucks collide. It is clear on grounds of symmetry that their center of mass must be at the midpoint between them. After all, we previously defined the center of mass as the balance point, and if the two hockey pucks were joined with a very lightweight rod whose own mass was negligible, they would obviously balance at the midpoint. It doesn't matter that the hockey pucks are two separate objects. It is still true that the motion of their center of mass is exceptionally simple, just like that of the wrench's center of mass.

The  $x$  coordinate of the hockey pucks' center of mass is thus given by  $x_{cm} = (x_1 + x_2)/2$ , i.e., the arithmetic average of their  $x$  coordinates. Why is its motion so simple? It has to do with conservation of momentum. Since the hockey pucks are not being acted on by any net external force, they constitute a closed system,



g / Two hockey pucks collide. Their mutual center of mass traces the straight path shown by the dashed line.

and their total momentum is conserved. Their total momentum is

$$\begin{aligned}
 mv_1 + mv_2 &= m(v_1 + v_2) \\
 &= m \left( \frac{\Delta x_1}{\Delta t} + \frac{\Delta x_2}{\Delta t} \right) \\
 &= \frac{m}{\Delta t} \Delta (x_1 + x_2) \\
 &= m \frac{2\Delta x_{cm}}{\Delta t} \\
 &= m_{total} v_{cm}
 \end{aligned}$$

In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point. Since the total momentum is conserved, the  $x$  component of the center of mass's velocity vector cannot change. The same is also true for the other components, so the center of mass must move along a straight line at constant speed.

The above relationship between the total momentum and the motion of the center of mass applies to any system, even if it is not closed.

#### **total momentum related to center of mass motion**

The total momentum of any system is related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total} \mathbf{v}_{cm} \quad .$$

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots}{m_1 + m_2 + \dots} \quad ,$$

with similar equations for the  $y$  and  $z$  coordinates.

#### **Momentum in different frames of reference**

Absolute motion is supposed to be undetectable, i.e., the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation  $\mathbf{p}_{total} = m_{total} \mathbf{v}_{cm}$ . If the velocity of frame B relative to frame A is  $\mathbf{v}_{BA}$ , then the only effect of changing frames of reference is to change  $\mathbf{v}_{cm}$  from its original value to  $\mathbf{v}_{cm} + \mathbf{v}_{BA}$ . This adds a constant onto the momentum vector, which has no effect on conservation of momentum.

## The center of mass frame of reference

A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.

### *A collision of pool balls viewed in the c.m. frame* example 11

If you move your head so that your eye is always above the point halfway in between the two pool balls, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.

### *The slingshot effect* example 12

It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if arrives in the opposite direction compared to the planet's motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Figure i shows such a "collision," with a space probe whipping around Jupiter. In the sun's frame of reference, Jupiter is moving.

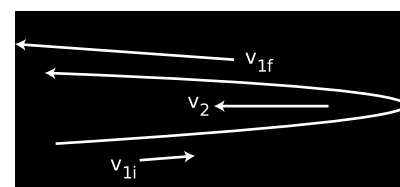
What about the center of mass frame? Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter's center, and Jupiter has zero velocity in the center of mass frame, as shown in figure j. The c.m. frame is moving to the left compared to the sun-fixed frame used in i, so the spacecraft's initial velocity is greater in this frame.

Things are simpler in the center of mass frame, because it is more symmetric. In the complicated sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have  $v_{1f} = -v_{1i}$ . Going back to the sun-fixed frame, the spacecraft's final velocity is increased by the frames' motion relative to each other. In the sun-fixed frame, the spacecraft's velocity has increased greatly.

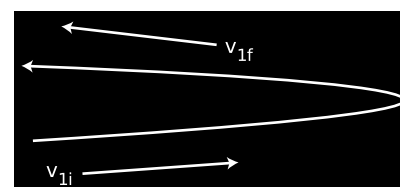
The result can also be understood in terms of work and energy. In Jupiter's frame, Jupiter is not doing any work on the spacecraft as it rounds the back of the planet, because the motion is perpendicular to the force. But in the sun's frame, the spacecraft's



h / Moving your head so that you are always looking down from right above the center of mass, you observe the collision of the two hockey pucks in the center of mass frame.



i / The slingshot effect viewed in the sun's frame of reference. Jupiter is moving to the left, and the collision is head-on.



j / The slingshot viewed in the frame of the center of mass of the Jupiter-spacecraft system.



velocity vector at the same moment has a large component to the left, so Jupiter is doing work on it.

### Discussion questions

**A** Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation  $p_{total} = m_{total} v_{cm}$  over a time interval of one second.

**B** A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter's hands do not make any force large enough to have a significant effect over the short duration of the impact.

## 14.4 Momentum transfer

### The rate of change of momentum

As with conservation of energy, we need a way to measure and calculate the transfer of momentum into or out of a system when the system is not closed. In the case of energy, the answer was rather complicated, and entirely different techniques had to be used for measuring the transfer of mechanical energy (work) and the transfer of heat by conduction. For momentum, the situation is far simpler.

In the simplest case, the system consists of a single object acted on by a constant external force. Since it is only the object's velocity that can change, not its mass, the momentum transferred is

$$\Delta \mathbf{p} = m \Delta \mathbf{v} \quad ,$$

which with the help of  $\mathbf{a} = \mathbf{F}/m$  and the constant-acceleration equation  $\mathbf{a} = \Delta \mathbf{v}/\Delta t$  becomes

$$\begin{aligned} \Delta \mathbf{p} &= m \mathbf{a} \Delta t \\ &= \mathbf{F} \Delta t \quad . \end{aligned}$$

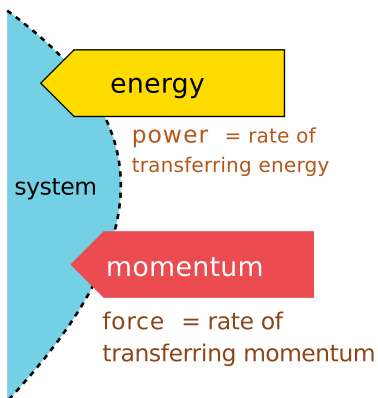
Thus the rate of transfer of momentum, i.e., the number of kg·m/s absorbed per second, is simply the external force,

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} \quad .$$

[relationship between the force on an object and the rate of change of its momentum; valid only if the force is constant]

This is just a restatement of Newton's second law, and in fact Newton originally stated it this way. As shown in figure k, the relationship between force and momentum is directly analogous to that between power and energy.

The situation is not materially altered for a system composed of many objects. There may be forces between the objects, but the



k / Power and force are the rates at which energy and momentum are transferred.

internal forces cannot change the system's momentum. (If they did, then removing the external forces would result in a closed system that could change its own momentum, like the mythical man who could pull himself up by his own bootstraps. That would violate conservation of momentum.) The equation above becomes

$$\mathbf{F}_{total} = \frac{\Delta \mathbf{p}_{total}}{\Delta t} \quad .$$

[relationship between the total external force on a system and the rate of change of its total momentum; valid only if the force is constant]

#### *Walking into a lamppost*

#### *example 13*

▷ Starting from rest, you begin walking, bringing your momentum up to  $100 \text{ kg} \cdot \text{m/s}$ . You walk straight into a lamppost. Why is the momentum change of  $-100 \text{ kg} \cdot \text{m/s}$  caused by the lamppost so much more painful than the change of  $+100 \text{ kg} \cdot \text{m/s}$  when you started walking?

▷ The situation is one-dimensional, so we can dispense with the vector notation. It probably takes you about 1 s to speed up initially, so the ground's force on you is  $F = \Delta p / \Delta t \approx 100 \text{ N}$ . Your impact with the lamppost, however, is over in the blink of an eye, say 1/10 s or less. Dividing by this much smaller  $\Delta t$  gives a much larger force, perhaps thousands of newtons. (The negative sign simply indicates that the force is in the opposite direction.)

This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long, the time required for your face to travel 20 or 30 cm. Without an airbag, your face would hit the dashboard, and the time interval would be the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of mechanical work has to be done on your head: enough to eliminate all its kinetic energy.

#### *Ion drive for spacecraft*

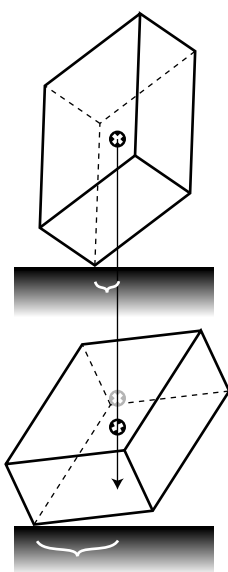
#### *example 14*

▷ The ion drive of the Deep Space 1 spacecraft, pictured on page 339 and discussed in example 2, produces a thrust of 90 mN (millinewtons). It carries about 80 kg of reaction mass, which it ejects at a speed of 30,000 m/s. For how long can the engine continue supplying this amount of thrust before running out of reaction mass to shove out the back?

▷ Solving the equation  $F = \Delta p / \Delta t$  for the unknown  $\Delta t$ , and treating force and momentum as scalars since the problem is one-



1 / The airbag increases  $\Delta t$  so as to reduce  $F = \Delta p / \Delta t$ .



m / Example 15.

dimensional, we find

$$\begin{aligned}
 \Delta t &= \frac{\Delta p}{F} \\
 &= \frac{m_{\text{exhaust}} \Delta v_{\text{exhaust}}}{F} \\
 &= \frac{(80 \text{ kg})(30,000 \text{ m/s})}{0.090 \text{ N}} \\
 &= 2.7 \times 10^7 \text{ s} \\
 &= 300 \text{ days}
 \end{aligned}$$

### A toppling box

### example 15

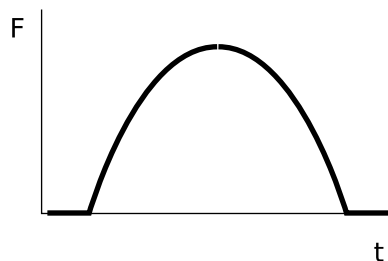
If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass moves in the same direction as its momentum vector points. There are two forces, a normal force and a gravitational force, both of which are vertical. (The gravitational force is actually many gravitational forces acting on all the atoms in the box.) The total force must be vertical, so the momentum vector must be purely vertical too, and the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

### The area under the force-time graph

Few real collisions involve a constant force. For example, when a tennis ball hits a racquet, the strings stretch and the ball flattens dramatically. They are both acting like springs that obey Hooke's law, which says that the force is proportional to the amount of stretching or flattening. The force is therefore small at first, ramps up to a maximum when the ball is about to reverse directions, and ramps back down again as the ball is on its way back out. The equation  $F = \Delta p / \Delta t$ , derived under the assumption of constant acceleration, does not apply here, and the force does not even have a single well-defined numerical value that could be plugged in to the equation.

As with similar-looking equations such as  $v = \Delta p / \Delta t$ , the equation  $F = \Delta p / \Delta t$  is correctly generalized by saying that the force is the slope of the  $p - t$  graph.

Conversely, if we wish to find  $\Delta p$  from a graph such as the one in figure n, one approach would be to divide the force by the mass of the ball, rescaling the  $F$  axis to create a graph of acceleration versus time. The area under the acceleration-versus-time graph gives the change in velocity, which can then be multiplied by the mass to find the change in momentum. An unnecessary complication was introduced, however, because we began by dividing by the mass and ended by multiplying by it. It would have made just as much



n / The  $F - t$  graph for a tennis racquet hitting a ball might look like this. The amount of momentum transferred equals the area under the curve.

sense to find the area under the original  $F - t$  graph, which would have given us the momentum change directly.

### Discussion question

**A** Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:

1. The collision is instantaneous.
2. The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.
3. The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?

## 14.5 Momentum in three dimensions

In this section we discuss how the concepts applied previously to one-dimensional situations can be used as well in three dimensions. Often vector addition is all that is needed to solve a problem:

*An explosion*

*example 16*

▷ Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquified rock, all having the same mass. If one fragment flies off with velocity components

$$\begin{aligned} v_{1x} &= 0 \\ v_{1y} &= 1.0 \times 10^4 \text{ km/hr} \end{aligned} \quad ,$$

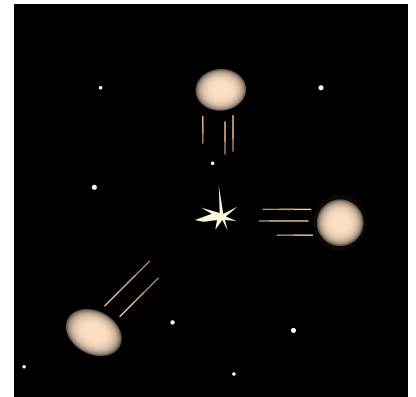
and the second with

$$\begin{aligned} v_{2x} &= 1.0 \times 10^4 \text{ km/hr} \\ v_{2y} &= 0 \end{aligned} \quad ,$$

(all in the center of mass frame) what is the magnitude of the third one's velocity?

▷ In the center of mass frame, the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

$$\begin{aligned} mv_{1x} + mv_{2x} + mv_{3x} &= 0 \quad , \quad \text{and} \\ mv_{1y} + mv_{2y} + mv_{3y} &= 0 \quad , \end{aligned}$$



o / Example 16.

where we have used the same symbol  $m$  for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by  $m$ , and we find

$$v_{3x} = -1.0 \times 10^4 \text{ km/hr}$$

$$v_{3y} = -1.0 \times 10^4 \text{ km/hr}$$

which gives a magnitude of

$$\begin{aligned} |\mathbf{v}_3| &= \sqrt{v_{3x}^2 + v_{3y}^2} \\ &= 1.4 \times 10^4 \text{ km/hr} \end{aligned}$$

### The center of mass

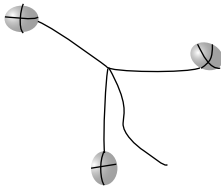
In three dimensions, we have the vector equations

$$\mathbf{F}_{total} = \frac{\Delta \mathbf{p}_{total}}{\Delta t}$$

and

$$\mathbf{p}_{total} = m_{total} \mathbf{v}_{cm} \quad .$$

The following is an example of their use.



p / Example 17.

#### *The bola* example 17

The bola, similar to the North American lasso, is used by South American gauchos to catch small animals by tangling up their legs in the three leather thongs. The motion of the whirling bola through the air is extremely complicated, and would be a challenge to analyze mathematically. The motion of its center of mass, however, is much simpler. The only forces on it are gravitational, so

$$\mathbf{F}_{total} = m_{total} \mathbf{g} \quad .$$

Using the equation  $\mathbf{F}_{total} = \Delta \mathbf{p}_{total} / \Delta t$ , we find

$$\Delta \mathbf{p}_{total} / \Delta t = m_{total} \mathbf{g} \quad ,$$

and since the mass is constant, the equation  $\mathbf{p}_{total} = m_{total} \mathbf{v}_{cm}$  allows us to change this to

$$m_{total} \Delta \mathbf{v}_{cm} / \Delta t = m_{total} \mathbf{g} \quad .$$

The mass cancels, and  $\Delta \mathbf{v}_{cm} / \Delta t$  is simply the acceleration of the center of mass, so

$$\mathbf{a}_{cm} = \mathbf{g} \quad .$$

In other words, the motion of the system is the same as if all its mass was concentrated at and moving with the center of mass. The bola has a constant downward acceleration equal to  $g$ , and flies along the same parabola as any other projectile thrown with the same initial center of mass velocity. Throwing a bola with the correct rotation is presumably a difficult skill, but making it hit its target is no harder than it is with a ball or a single rock.

[Based on an example by Kleppner and Kolenkow.]

## Counting equations and unknowns

Counting equations and unknowns is just as useful as in one dimension, but every object's momentum vector has three components, so an unknown momentum vector counts as three unknowns. Conservation of momentum is a single vector equation, but it says that all three components of the total momentum vector stay constant, so we count it as three equations. Of course if the motion happens to be confined to two dimensions, then we need only count vectors as having two components.

---

### *A two-car crash with sticking*

### *example 18*

Suppose two cars collide, stick together, and skid off together. If we know the cars' initial momentum vectors, we can count equations and unknowns as follows:

unknown #1:  $x$  component of cars' final, total momentum

unknown #2:  $y$  component of cars' final, total momentum

equation #1: conservation of the total  $p_x$

equation #2: conservation of the total  $p_y$

Since the number of equations equals the number of unknowns, there must be one unique solution for their total momentum vector after the crash. In other words, the speed and direction at which their common center of mass moves off together is unaffected by factors such as whether the cars collide center-to-center or catch each other a little off-center.

---

### *Shooting pool*

### *example 19*

Two pool balls collide, and as before we assume there is no decrease in the total kinetic energy, i.e., no energy converted from KE into other forms. As in the previous example, we assume we are given the initial velocities and want to find the final velocities. The equations and unknowns are:

unknown #1:  $x$  component of ball #1's final momentum

unknown #2:  $y$  component of ball #1's final momentum

unknown #3:  $x$  component of ball #2's final momentum

unknown #4:  $y$  component of ball #2's final momentum

equation #1: conservation of the total  $p_x$

equation #2: conservation of the total  $p_y$

equation #3: no decrease in total KE

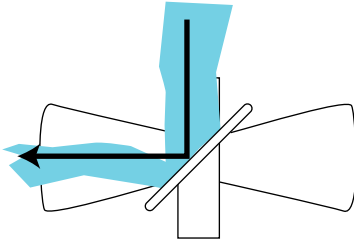
Note that we do not count the balls' final kinetic energies as unknowns, because knowing the momentum vector, one can always find the velocity and thus the kinetic energy. The number of equations is less than the number of unknowns, so no unique result is guaranteed. This is what makes pool an interesting game. By

aiming the cue ball to one side of the target ball you can have some control over the balls' speeds and directions of motion after the collision.

It is not possible, however, to choose any combination of final speeds and directions. For instance, a certain shot may give the correct direction of motion for the target ball, making it go into a pocket, but may also have the undesired side-effect of making the cue ball go in a pocket.

### Calculations with the momentum vector

The following example illustrates how a force is required to change the direction of the momentum vector, just as one would be required to change its magnitude.



q / Example 20.

#### A turbine

example 20

▷ In a hydroelectric plant, water flowing over a dam drives a turbine, which runs a generator to make electric power. The figure shows a simplified physical model of the water hitting the turbine, in which it is assumed that the stream of water comes in at a  $45^\circ$  angle with respect to the turbine blade, and bounces off at a  $90^\circ$  angle at nearly the same speed. The water flows at a rate  $R$ , in units of kg/s, and the speed of the water is  $v$ . What are the magnitude and direction of the water's force on the turbine?

▷ In a time interval  $\Delta t$ , the mass of water that strikes the blade is  $R\Delta t$ , and the magnitude of its initial momentum is  $mv = vR\Delta t$ . The water's final momentum vector is of the same magnitude, but in the perpendicular direction. By Newton's third law, the water's force on the blade is equal and opposite to the blade's force on the water. Since the force is constant, we can use the equation

$$F_{\text{blade on water}} = \frac{\Delta \mathbf{p}_{\text{water}}}{\Delta t}.$$

Choosing the  $x$  axis to be to the right and the  $y$  axis to be up, this can be broken down into components as

$$\begin{aligned} F_{\text{blade on water},x} &= \frac{\Delta p_{\text{water},x}}{\Delta t} \\ &= \frac{-vR\Delta t - 0}{\Delta t} \\ &= -vR \end{aligned}$$

and

$$\begin{aligned} F_{\text{blade on water},y} &= \frac{\Delta p_{\text{water},y}}{\Delta t} \\ &= \frac{0 - (-vR\Delta t)}{\Delta t} \\ &= vR. \end{aligned}$$

The water's force on the blade thus has components

$$\begin{aligned} F_{\text{water on blade},x} &= vR \\ F_{\text{water on blade},y} &= -vR \end{aligned} .$$

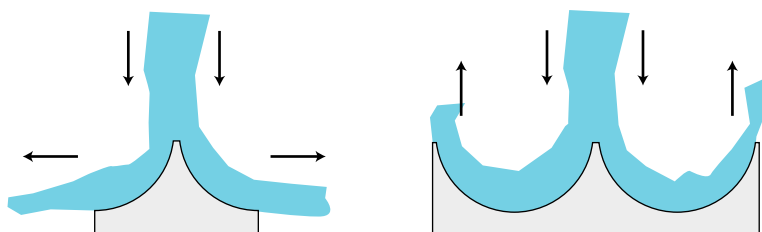
In situations like this, it is always a good idea to check that the result makes sense physically. The  $x$  component of the water's force on the blade is positive, which is correct since we know the blade will be pushed to the right. The  $y$  component is negative, which also makes sense because the water must push the blade down. The magnitude of the water's force on the blade is

$$|F_{\text{water on blade}}| = \sqrt{2}vR$$

and its direction is at a 45-degree angle down and to the right.

### Discussion questions

**A** The figures show a jet of water striking two different objects. How does the total downward force compare in the two cases? How could this fact be used to create a better waterwheel? (Such a waterwheel is known as a Pelton wheel.)



Discussion question A.

## 14.6 $\int$ Applications of calculus

By now you will have learned to recognize the circumlocutions I use in the sections without calculus in order to introduce calculus-like concepts without using the notation, terminology, or techniques of calculus. It will therefore come as no surprise to you that the rate of change of momentum can be represented with a derivative,

$$\mathbf{F}_{\text{total}} = \frac{d\mathbf{p}_{\text{total}}}{dt} .$$

And of course the business about the area under the  $F - t$  curve is really an integral,  $\Delta p_{\text{total}} = \int F_{\text{total}} dt$ , which can be made into an integral of a vector in the more general three-dimensional case:

$$\Delta \mathbf{p}_{\text{total}} = \int \mathbf{F}_{\text{total}} dt .$$

In the case of a material object that is neither losing nor picking up mass, these are just trivially rearranged versions of familiar equations, e.g.,  $F = mdv/dt$  rewritten as  $F = d(mv)/dt$ . The following is a less trivial example, where  $F = ma$  alone would not have been very easy to work with.



▷ If 1 kg/s of rain falls vertically into a 10-kg cart that is rolling without friction at an initial speed of 1.0 m/s, what is the effect on the speed of the cart when the rain first starts falling?

▷ The rain and the cart make horizontal forces on each other, but there is no external horizontal force on the rain-plus-cart system, so the horizontal motion obeys

$$F = \frac{d(mv)}{dt} = 0$$

We use the product rule to find

$$0 = \frac{dm}{dt}v + m\frac{dv}{dt} \quad .$$

We are trying to find how  $v$  changes, so we solve for  $dv/dt$ ,

$$\begin{aligned} \frac{dv}{dt} &= -\frac{v}{m} \frac{dm}{dt} \\ &= -\left(\frac{1 \text{ m/s}}{10 \text{ kg}}\right) (1 \text{ kg/s}) \\ &= -0.1 \text{ m/s}^2 \quad . \end{aligned}$$

(This is only at the moment when the rain starts to fall.)

Finally we note that there are cases where  $F = ma$  is not just less convenient than  $F = dp/dt$  but in fact  $F = ma$  is wrong and  $F = dp/dt$  is right. A good example is the formation of a comet's tail by sunlight. We cannot use  $F = ma$  to describe this process, since we are dealing with a collision of light with matter, whereas Newton's laws only apply to matter. The equation  $F = dp/dt$ , on the other hand, allows us to find the force experienced by an atom of gas in the comet's tail if we know the rate at which the momentum vectors of light rays are being turned around by reflection from the atom.

## Summary

### Selected vocabulary

momentum . . .	a measure of motion, equal to $mv$ for material objects
collision . . . . .	an interaction between moving objects that lasts for a certain time
center of mass . .	the balance point or average position of the mass in a system

### Notation

$\mathbf{p}$ . . . . .	the momentum vector
cm . . . . .	center of mass, as in $x_{cm}$ , $a_{cm}$ , etc.

### Other terminology and notation

impulse, $I$ , $J$ . .	the amount of momentum transferred, $\Delta p$
elastic collision .	one in which no KE is converted into other forms of energy
inelastic collision	one in which some KE is converted to other forms of energy

## Summary

If two objects interact via a force, Newton's third law guarantees that any change in one's velocity vector will be accompanied by a change in the other's which is in the opposite direction. Intuitively, this means that if the two objects are not acted on by any external force, they cannot cooperate to change their overall state of motion. This can be made quantitative by saying that the quantity  $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$  must remain constant as long as the only forces are the internal ones between the two objects. This is a conservation law, called the conservation of momentum, and like the conservation of energy, it has evolved over time to include more and more phenomena unknown at the time the concept was invented. The momentum of a material object is

$$\mathbf{p} = m\mathbf{v} \quad ,$$

but this is more like a standard for comparison of momenta rather than a definition. For instance, light has momentum, but has no mass, and the above equation is not the right equation for light. The law of conservation of momentum says that the total momentum of any closed system, i.e., the vector sum of the momentum vectors of all the things in the system, is a constant.

An important application of the momentum concept is to collisions, i.e., interactions between moving objects that last for a certain amount of time while the objects are in contact or near each other. Conservation of momentum tells us that certain outcomes of a collision are impossible, and in some cases may even be sufficient to predict the motion after the collision. In other cases, conservation of momentum does not provide enough equations to find all the unknowns. In some collisions, such as the collision of a superball with

the floor, very little kinetic energy is converted into other forms of energy, and this provides one more equation, which may suffice to predict the outcome.

The total momentum of a system can be related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm} \quad .$$

The center of mass, introduced on an intuitive basis in book 1 as the “balance point” of an object, can be generalized to any system containing any number of objects, and is defined mathematically as the weighted average of the positions of all the parts of all the objects,

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \dots}{m_1 + m_2 + \dots} \quad ,$$

with similar equations for the  $y$  and  $z$  coordinates.

The frame of reference moving with the center of mass of a closed system is always a valid inertial frame, and many problems can be greatly simplified by working them in the inertial frame. For example, any collision between two objects appears in the c.m. frame as a head-on one-dimensional collision.

When a system is not closed, the rate at which momentum is transferred in or out is simply the total force being exerted externally on the system. If the force is constant,

$$\mathbf{F}_{total} = \frac{\Delta\mathbf{p}_{total}}{\Delta t} \quad .$$

When the force is not constant, the force equals the slope of the tangent line on a graph of  $p$  versus  $t$ , and the change in momentum equals the area under the  $F - t$  graph.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass. ✓

**2** Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

**3** A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet's mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.) ✓

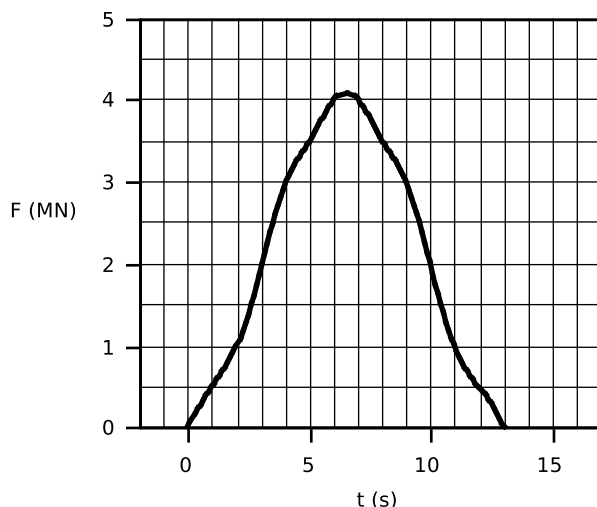
**4** A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.

(a) Find the bullet's final velocity. ✓

(b) Find the bullet's final momentum. ✓

(c) Find the momentum of the recoiling gun.

(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter. ✓



Problem 5

**5** The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket's mass is 4000 kg, at what speed is the rocket moving when the engine

stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.  $\checkmark$

**6** Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein's special theory of relativity. Don't worry about that, though.)

**7** Show that for a body made up of many *equal* masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

**8** The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed,  $v_i$ , before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.

▷ Solution, p. 528

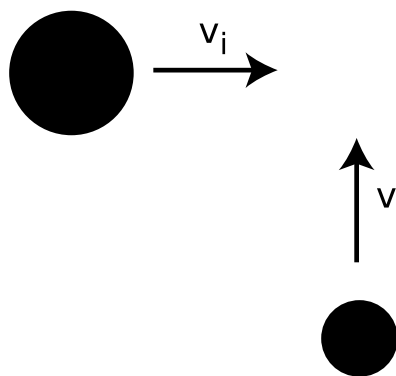
**9** A flexible rope of mass  $m$  and length  $L$  slides without friction over the edge of a table. Let  $x$  be the length of the rope that is hanging over the edge at a given moment in time.

(a) Show that  $x$  satisfies the equation of motion  $d^2x/dt^2 = gx/L$ . [Hint: Use  $F = dp/dt$ , which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]

(b) Give a physical explanation for the fact that a larger value of  $x$  on the right-hand side of the equation leads to a greater value of the acceleration on the left side.

(c) When we take the second derivative of the function  $x(t)$  we are supposed to get essentially the same function back again, except for a constant out in front. The function  $e^x$  has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form  $x = be^{ct}$ , where  $b$  and  $c$  are constants. Show that this does indeed provide a solution for two specific values of  $c$  (and for any value of  $b$ ).

(d) Show that the sum of any two solutions to the equation of motion



Problem 8

is also a solution.

(e) Find the solution for the case where the rope starts at rest at  $t = 0$  with some nonzero value of  $x$ .  $\int \star$

**10** A very massive object with velocity  $v$  collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity  $2v$ . [Hint: Use the center-of-mass frame of reference.]

**11** When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer *power* through its radiator coils, but not *force*?  $\triangleright$  Solution, p. 528

**12** A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

**13** A rocket ejects exhaust with an exhaust velocity  $u$ . The rate at which the exhaust mass is used (mass per unit time) is  $b$ . We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket's initial mass (fuel plus the body and payload) be  $m_i$ , and  $m_f$  be its final mass, after all the fuel is used up. (a) Find the rocket's final velocity,  $v$ , in terms of  $u$ ,  $m_i$ , and  $m_f$ . Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won't work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is fairly small compared to  $c$ , so it's not an unreasonable approximation to ignore relativity.)  $\checkmark \int \star$

**14** A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

$\triangleright$  Solution, p. 528

**15** Suppose a system consisting of pointlike particles has a total kinetic energy  $K_{cm}$  measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion.

(a) Prove that in a different frame of reference, moving with velocity  $\mathbf{u}$  relative to the center-of-mass frame, the total kinetic energy equals  $K_{cm} + M|\mathbf{u}|^2/2$ , where  $M$  is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.]  $\triangleright$  Solution, p. 529

(b) Use this to prove that if energy is conserved in one frame of

reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles' interactions with each other, which we assume depends only on the distance between particles. [For a simpler numerical example, see problem 13 on p. 290.] ★

**16** The big difference between the equations for momentum and kinetic energy is that one is proportional to  $v$  and one to  $v^2$ . Both, however, are proportional to  $m$ . Suppose someone tells you that there's a third quantity, funkosity, defined as  $f = m^2v$ , and that funkosity is conserved. How do you know your leg is being pulled?

▷ Solution, p. 529

**17** A mass  $m$  moving at velocity  $v$  collides with a stationary target having the same mass  $m$ . Find the maximum amount of energy that can be released as heat and sound. ✓



A tornado touches down in Spring Hill, Kansas, May 20, 1957.

## Chapter 15

# Conservation of angular momentum

“Sure, and maybe the sun won’t come up tomorrow.” Of course, the sun only appears to go up and down because the earth spins, so the cliché should really refer to the unlikelihood of the earth’s stopping its rotation abruptly during the night. Why can’t it stop? It wouldn’t violate conservation of momentum, because the earth’s rotation doesn’t add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth’s rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

Other examples along these lines are not hard to find. A hydrogen atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make



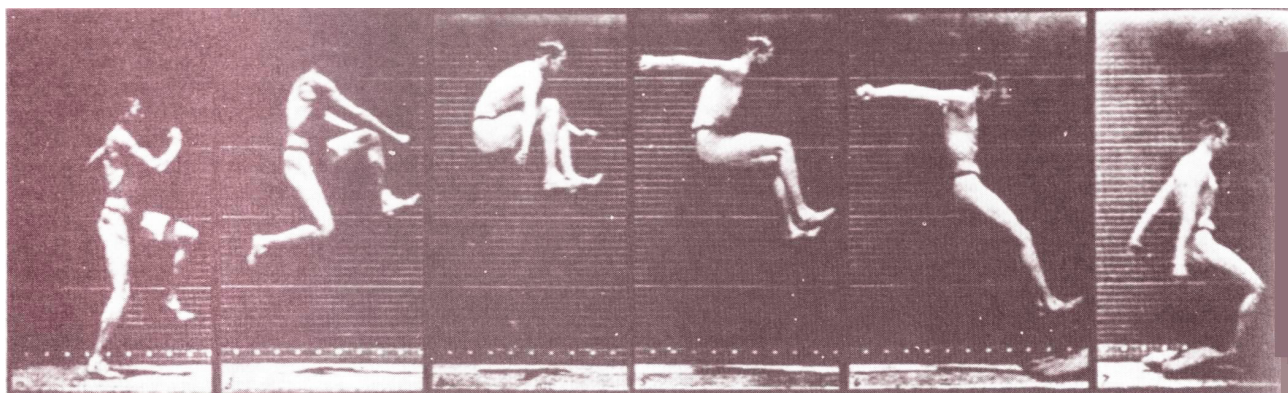
any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

**A closed system is involved.** Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

**Something remains unchanged.** There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.

**Something can be transferred back and forth without changing the total amount.** In figure a, the jumper wants to get his feet out in front of him so he can keep from doing a “face plant” when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn’t start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.



a / An early photograph of an old-fashioned long-jump.

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.’s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.

## 15.1 Conservation of angular momentum

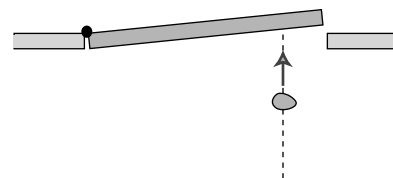
When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child's swinging leg, and a helicopter's spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid's total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

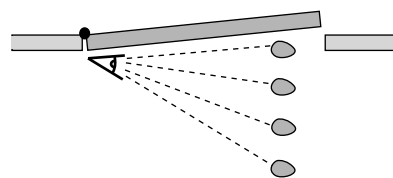
The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion doesn't repeat or even curve around. If you throw a piece of putty at a door, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some "rotation," which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge. For this reason, the conserved quantity we are investigating is called *angular* momentum. The symbol for angular momentum can't be  $a$  or  $m$ , since those are used for acceleration and mass, so the symbol  $L$  is arbitrarily chosen instead.

Imagine a 1-kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When



b / An overhead view of a piece of putty being thrown at a door. Even though the putty is neither spinning nor traveling along a curve, we must define it as having some kind of "rotation" because it is able to make the door rotate.



c / As seen by someone standing at the axis, the putty changes its angular position. We therefore define it as having angular momentum.

it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.<sup>1</sup>

Experiments show, not surprisingly, that a 2-kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

$$L \propto m \quad .$$

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

$$L \propto mv \quad .$$

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance,  $r$ , from the axis of rotation, so angular momentum must also be proportional to  $r$ ,

$$L \propto mvr \quad .$$

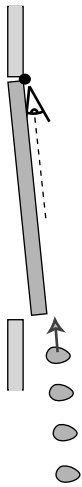
We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly inward toward the hinge will have no angular momentum to give to the door. After all, there would not even be any way to decide whether the ball's rotation was clockwise or counterclockwise in this situation. It is therefore only the component of the blob's velocity vector perpendicular to the door that should be counted in its angular momentum,

$$L = mv_{\perp}r \quad .$$

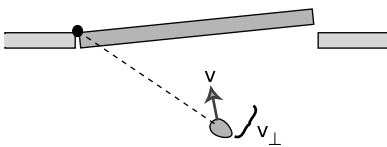
More generally,  $v_{\perp}$  should be thought of as the component of the object's velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are  $(\text{kg} \cdot \text{m/s}) \cdot \text{m}$ , i.e.,  $\text{kg} \cdot \text{m}^2/\text{s}$ . This gives us a way of calculating the angular momentum of any material object or any system consisting of material objects:

<sup>1</sup>We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.



d / A putty blob thrown directly at the axis has no angular motion, and therefore no angular momentum. It will not cause the door to rotate.



e / Only the component of the velocity vector perpendicular to the dashed line should be counted into the definition of angular momentum.

### angular momentum of a material object

The angular momentum of a moving particle is

$$L = mv_{\perp}r \quad ,$$

where  $m$  is its mass,  $v_{\perp}$  is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and  $r$  is its distance from the axis. Positive and negative signs are used to describe opposite directions of rotation.

The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum.

Note that  $r$  is not necessarily the radius of a circle. (As implied by the qualifiers, matter isn't the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the three most fundamental principles of physics, along with conservation of energy and momentum.

#### *A figure skater pulls her arms in*

#### *example 1*

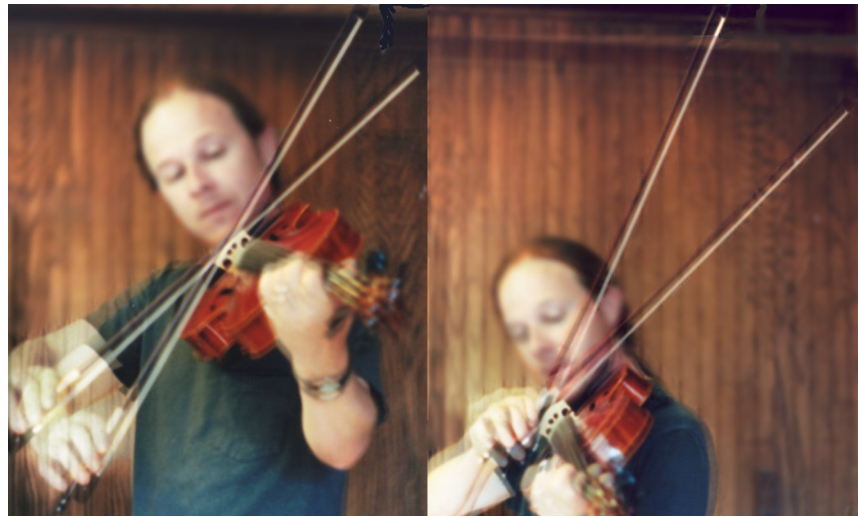
When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing  $r$  for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e., taking the same amount of time for each revolution, because her arms' contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in  $r$  for her arms must be compensated for by an overall increase in her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.



f / A figure skater pulls in her arms so that she can execute a spin more rapidly.



h / Example 3. A view of the earth-moon system from above the north pole. All distances have been highly distorted for legibility. The earth's rotation is counterclockwise from this point of view (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.



g / Example 2.

#### *Changing the axis*

#### *example 2*

An object's angular momentum can be different depending on the axis about which it rotates. Figure g shows two double-exposure photographs a viola player tipping the bow in order to cross from one string to another. Much more angular momentum is required when playing near the bow's handle, called the frog, as in the panel on the right; not only are most of the atoms in the bow at greater distances,  $r$ , from the axis of rotation, but the ones in the tip also have more momentum,  $p$ . It is difficult for the player to quickly transfer a large angular momentum into the bow, and then transfer it back out just as quickly. (In the language of section 15.4, large torques are required.) This is one of the reasons that string players tend to stay near the middle of the bow as much as possible.

#### *Earth's slowing rotation and the receding moon*

#### *example 3*

As noted in chapter 1, the earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon's greater value of  $r$  means that it has a greater

angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth's tidal bulges. The effect is described qualitatively in the caption of the figure. The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

### Restriction to rotation in a plane

Is angular momentum a vector, or a scalar? It does have a direction in space, but it's a direction of rotation, not a straight-line direction like the directions of vectors such as velocity or force. It turns out (see problem 29) that there is a way of defining angular momentum as a vector, but in this book the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum.

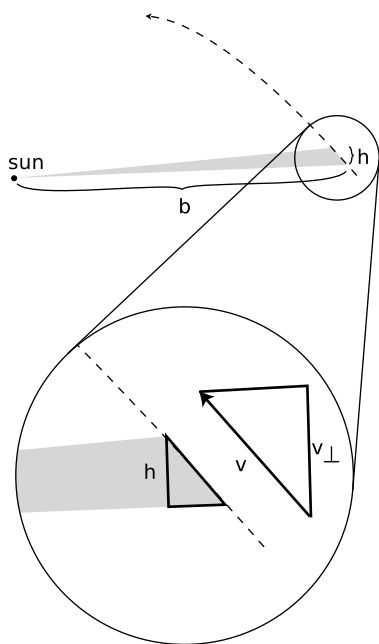
### Discussion question

**A** Conservation of plain old momentum,  $p$ , can be thought of as the greatly expanded and modified descendant of Galileo's original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force *is* needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum,  $L$ , also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from "running down."

## 15.2 Angular momentum in planetary motion

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler's law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler's time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing



i / The planet's angular momentum is related to the rate at which it sweeps out area.

and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet's angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure i, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

$$\text{area} = \frac{1}{2}bh \quad .$$

We wish to relate this to angular momentum, which contains the variables  $r$  and  $v_{\perp}$ . If we consider the sun to be the axis of rotation, then the variable  $r$  is identical to the base of the triangle,  $r = b$ . Referring to the magnified portion of the figure,  $v_{\perp}$  can be related to  $h$ , because the two right triangles are similar:

$$\frac{h}{\text{distance traveled}} = \frac{v_{\perp}}{|\mathbf{v}|}$$

The area can thus be rewritten as

$$\text{area} = \frac{1}{2}r \frac{v_{\perp}(\text{distance traveled})}{|\mathbf{v}|} \quad .$$

The distance traveled equals  $|\mathbf{v}|\Delta t$ , so this simplifies to

$$\text{area} = \frac{1}{2}rv_{\perp}\Delta t \quad .$$

We have found the following relationship between angular momentum and the rate at which area is swept out:

$$L = 2m \frac{\text{area}}{\Delta t} \quad .$$

The factor of 2 in front is simply a matter of convention, since any conserved quantity would be an equally valid conserved quantity if you multiplied it by a constant. The factor of  $m$  was not relevant to Kepler, who did not know the planets' masses, and who was only describing the motion of one planet at a time.

We thus find that Kepler's equal-area law is equivalent to a statement that the planet's angular momentum remains constant. But wait, why should it remain constant? — the planet is not a closed system, since it is being acted on by the sun's gravitational force. There are two valid answers. The first is that it is actually the total angular momentum of the sun plus the planet that is conserved. The sun, however, is millions of times more massive than the typical planet, so it accelerates very little in response to the planet's gravitational force. It is thus a good approximation to say that the sun

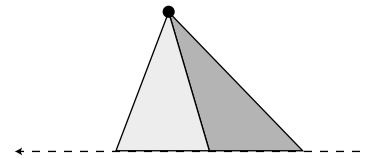


doesn't move at all, so that no angular momentum is transferred between it and the planet.

The second answer is that to change the planet's angular momentum requires not just a force but a force applied in a certain way. In section 15.4 we discuss the transfer of angular momentum by a force, but the basic idea here is that a force directly in toward the axis does not change the angular momentum.

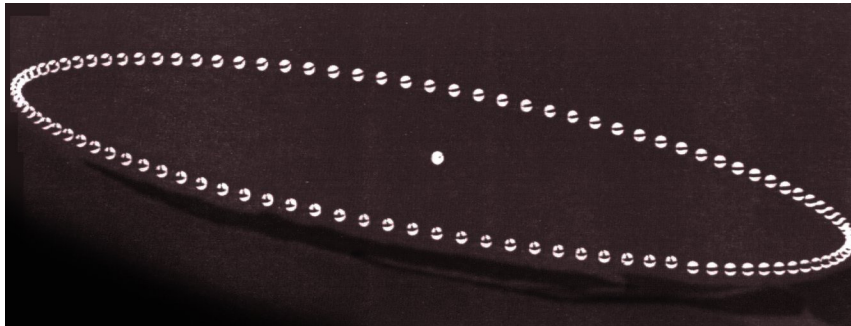
### Discussion questions

**A** Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate? Would it matter if we chose a different axis?



Discussion question A.

**B** The figure is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can't be seen in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e., the position the bob would have if it hung straight down at us. Does the bob's angular momentum appear to remain constant if we consider the center to be the axis of rotation? What if we choose a different axis?



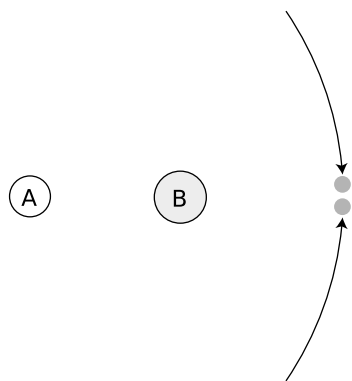
Discussion question B.

## 15.3 Two theorems about angular momentum

With plain old momentum,  $p$ , we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the frames were not at rest with respect to each other. Conservation of momentum, however, would be true in either frame. As long as we employed a single frame consistently throughout a calculation, everything would work.

The same is true for angular momentum, and in addition there is an ambiguity that arises from the definition of an axis of rotation. For a wheel, the natural choice of an axis of rotation is obviously the axle, but what about an egg rotating on its side? The egg

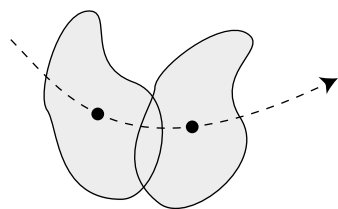




j / Example 4.



k / Everyone has a strong tendency to think of the diver as rotating about his own center of mass. However, he is flying in an arc, and he also has angular momentum because of this motion.



l / This rigid object has angular momentum both because it is spinning about its center of mass and because it is moving through space.

has an asymmetric shape, and thus no clearly defined geometric center. A similar issue arises for a cyclone, which does not even have a sharply defined shape, or for a complicated machine with many gears. The following theorem, the first of two presented in this section without proof, explains how to deal with this issue. Although I have put descriptive titles above both theorems, they have no generally accepted names.

### the choice of axis theorem

It is entirely arbitrary what point one defines as the axis for purposes of calculating angular momentum. If a closed system's angular momentum is conserved when calculated with one choice of axis, then it will also be conserved for any other choice. Likewise, any inertial frame of reference may be used.

### *Colliding asteroids described with different axes* example 4

Observers on planets A and B both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet B. The asteroids collide and stick. For simplicity, assume planets A and B are both at rest.

With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that  $r$  is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult to calculate the earth's angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass's circular motion around the sun. This is a special case of the following general theorem:

### the spin theorem

An object's angular momentum with respect to some outside axis A can be found by adding up two parts:

- (1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e., the angular

momentum the object has because it is spinning.

(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

---

*A system with its center of mass at rest* *example 5*

In the special case of an object whose center of mass is at rest, the spin theorem implies that the object's angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)

---

*Angular momentum of a rigid object* *example 6*

▷ A motorcycle wheel has almost all its mass concentrated at the outside. If the wheel has mass  $m$  and radius  $r$ , and the time required for one revolution is  $T$ , what is the spin part of its angular momentum?

▷ This is an example of the commonly encountered special case of rigid motion, as opposed to the rotation of a system like a hurricane in which the different parts take different amounts of time to go around. We don't really have to go through a laborious process of adding up contributions from all the many parts of a wheel, because they are all at about the same distance from the axis, and are all moving around the axis at about the same speed. The velocity is all perpendicular to the spokes,

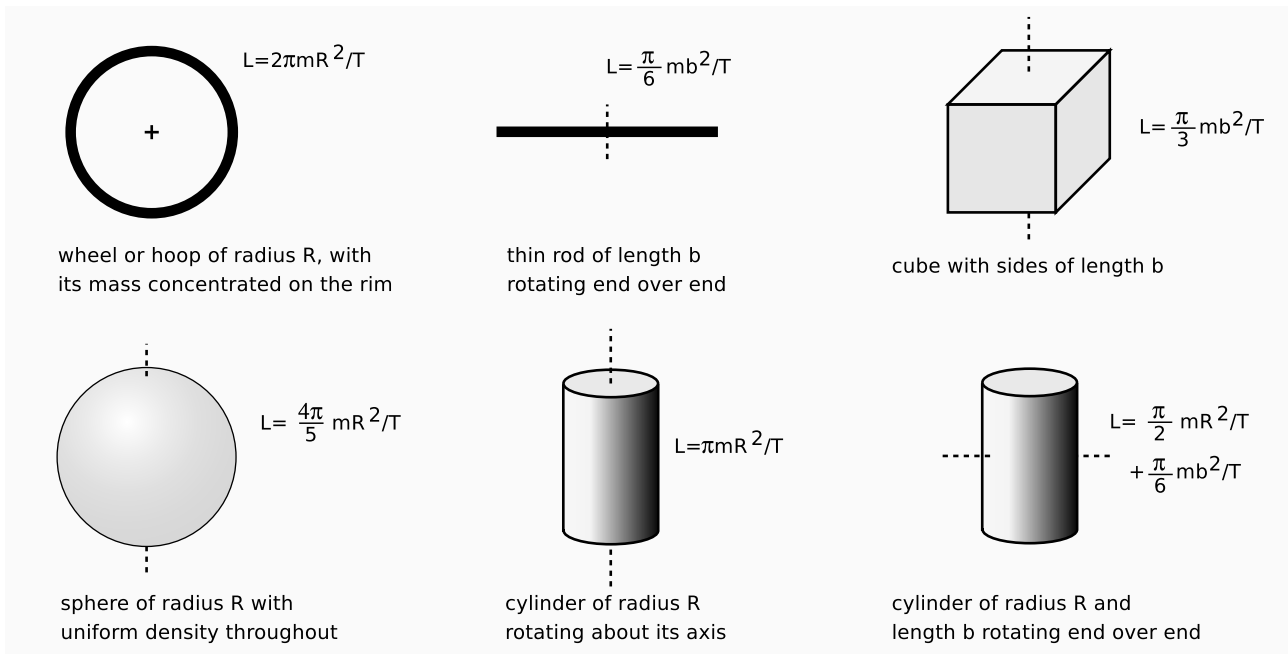
$$\begin{aligned}v_{\perp} &= v \\&= (\text{circumference})/T \\&= 2\pi r/T \quad ,\end{aligned}$$

and the angular momentum of the wheel about its center is

$$\begin{aligned}L &= mv_{\perp}r \\&= m(2\pi r/T)r \\&= 2\pi mr^2/T \quad .\end{aligned}$$

Note that although the factors of  $2\pi$  in this expression is peculiar to a wheel with its mass concentrated on the rim, the proportionality to  $m/T$  would have been the same for any other rigidly rotating object. Although an object with a noncircular shape does not have a radius, it is also true in general that angular momentum is proportional to the square of the object's size for fixed values of  $m$  and  $T$ . For instance doubling an object's size doubles both the  $v_{\perp}$  and  $r$  factors in the contribution of each of its parts to the total angular momentum, resulting in an overall factor of four increase.

The figure shows some examples of angular momenta of various shapes rotating about their centers of mass. The equations for their angular momenta were derived using calculus, as described in my calculus-based book *Simple Nature*. Do not memorize these equations!



### *The hammer throw*

### *example 7*

▷ In the men's Olympic hammer throw, a steel ball of radius 6.1 cm is swung on the end of a wire of length 1.22 m. What fraction of the ball's angular momentum comes from its rotation, as opposed to its motion through space?

▷ It's always important to solve problems symbolically first, and plug in numbers only at the end, so let the radius of the ball be  $b$ , and the length of the wire  $\ell$ . If the time the ball takes to go once around the circle is  $T$ , then this is also the time it takes to revolve once around its own axis. Its speed is  $v = 2\pi\ell / T$ , so its angular momentum due to its motion through space is  $mv\ell = 2\pi m\ell^2 / T$ . Its angular momentum due to its rotation around its own center is  $(4\pi/5)mb^2 / T$ . The ratio of these two angular momenta is  $(2/5)(b/\ell)^2 = 1.0 \times 10^{-3}$ . The angular momentum due to the ball's spin is extremely small.

### Toppling a rod

### example 8

▷ A rod of length  $b$  and mass  $m$  stands upright. We want to strike the rod at the bottom, causing it to fall and land flat. Find the momentum,  $p$ , that should be delivered, in terms of  $m$ ,  $b$ , and  $g$ . Can this really be done without having the rod scrape on the floor?

▷ This is a nice example of a question that can very nearly be answered based only on units. Since the three variables,  $m$ ,  $b$ , and  $g$ , all have different units, they can't be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form

$$p = Am^j b^k g^l \quad ,$$

where  $A$ ,  $j$ ,  $k$ , and  $l$  are unitless constants. The result has to have units of  $\text{kg} \cdot \text{m/s}$ . To get kilograms to the first power, we need

$$j = 1 \quad ,$$

meters to the first power requires

$$k + l = 1 \quad ,$$

and seconds to the power  $-1$  implies

$$l = 1/2 \quad .$$

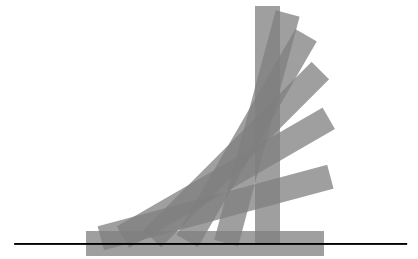
We find  $j = 1$ ,  $k = 1/2$ , and  $l = 1/2$ , so the solution must be of the form

$$p = Am\sqrt{bg} \quad .$$

Note that no physics was required!

Consideration of units, however, won't help us to find the unitless constant  $A$ . Let  $t$  be the time the rod takes to fall, so that  $(1/2)gt^2 = b/2$ . If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be  $1/4$ , or  $3/4$ , or  $5/4$ ,  $\dots$ , but all the possibilities greater than  $1/4$  would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time  $T = 4t$ , and it has angular momentum  $L = (\pi/6)mb^2/T$ .

The momentum lost by the object striking the rod is  $p$ , and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum  $bp/2$ , which is also transferred to the rod. Equating this to the expression above for  $L$ , we find  $p = (\pi/12)m\sqrt{bg}$ .



m / Example 8.

Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The figure shows that the answer is no for this rod of finite width, but it appears that the answer would be yes for a sufficiently thin rod. This is analyzed further in homework problem 28 on page 399.

### Discussion question

**A** In the example of the colliding asteroids, suppose planet A was moving toward the top of the page, at the same speed as the bottom asteroid. How would planet A's astronomers describe the angular momenta of the asteroids? Would angular momentum still be conserved?

## 15.4 Torque: the rate of transfer of angular momentum

Force can be interpreted as the rate of transfer of momentum. The equivalent in the case of angular momentum is called *torque* (rhymes with “fork”). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on it. Torque is represented by the Greek letter tau,  $\tau$ , and the rate of change of an object's angular momentum equals the total torque acting on it,

$$\tau_{\text{total}} = \frac{\Delta L}{\Delta t} \quad .$$

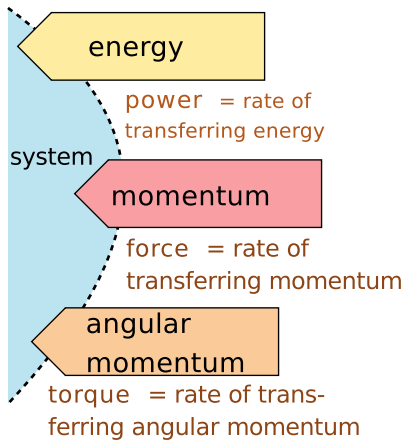
(If the angular momentum does not change at a constant rate, the total torque equals the slope of the tangent line on a graph of  $L$  versus  $t$ .)

As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines' torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.

### Torque distinguished from force

Of course a force is necessary in order to create a torque — you can't twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counterclockwise, not a linear direction.

The other difference between torque and force is a matter of leverage. A given force applied at a door's knob will change the door's angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.



n / Energy, momentum, and angular momentum can be transferred. The rates of transfer are called power, force, and torque.

It is possible to have a zero total torque with a nonzero total force. An airplane with four jet engines, o, would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counterclockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque. A merry-go-round's engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there was not zero total force on it, its center of mass would accelerate!

### Relationship between force and torque

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. We'll derive an equation relating torque to force for a particular very simple situation, and state without proof that the equation actually applies to all situations.

Consider a pointlike object which is initially at rest at a distance  $r$  from the axis we have chosen for defining angular momentum. We first observe that a force directly inward or outward, along the line connecting the axis to the object, does not impart any angular momentum to the object.

A force perpendicular to the line connecting the axis and the object does, however, make the object pick up angular momentum. Newton's second law gives

$$a = \frac{F}{m} \quad ,$$

and assuming for simplicity that the force is constant, the constant acceleration equation  $a = \Delta v / \Delta t$  allows us to find the velocity the object acquires after a time  $\Delta t$ ,

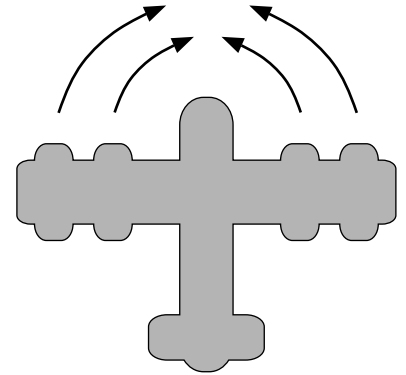
$$\Delta v = \frac{F \Delta t}{m} \quad .$$

We are trying to relate force to a change in angular momentum, so we multiply both sides of the equation by  $mr$  to give

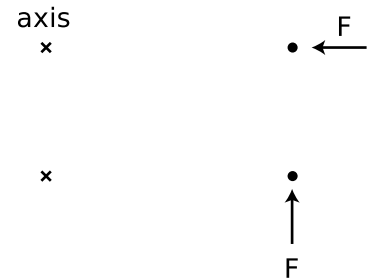
$$\begin{aligned} m \Delta v r &= F \Delta t r \\ \Delta L &= F \Delta t r \quad . \end{aligned}$$

Dividing by  $\Delta t$  gives the torque:

$$\begin{aligned} \frac{\Delta L}{\Delta t} &= F r \\ \tau &= F r \quad . \end{aligned}$$



o / The plane's four engines produce zero total torque but not zero total force.



p / The simple physical situation we use to derive an equation for torque. A force that points directly in at or out away from the axis produces neither clockwise nor counterclockwise angular momentum. A force in the perpendicular direction does transfer angular momentum.

If a force acts at an angle other than 0 or 90° with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

$$\tau = F_{\perp} r$$

Although this result was proved under a simplified set of circumstances, it is more generally valid:

### relationship between force and torque

The rate at which a force transfers angular momentum to an object, i.e., the torque produced by the force, is given by

$$|\tau| = r|F_{\perp}|$$

where  $r$  is the distance from the axis to the point of application of the force, and  $F_{\perp}$  is the component of the force that is perpendicular to the line joining the axis to the point of application.

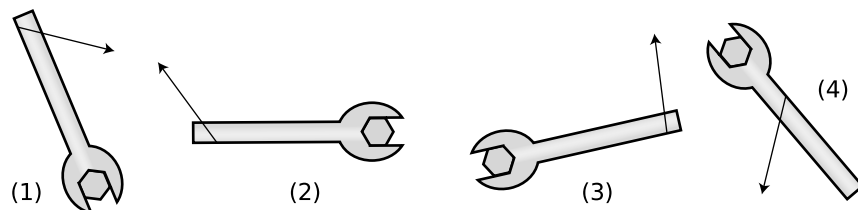
The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N·m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

### self-check A

Compare the magnitudes and signs of the four torques shown in the figure.

▷ Answer, p. 534

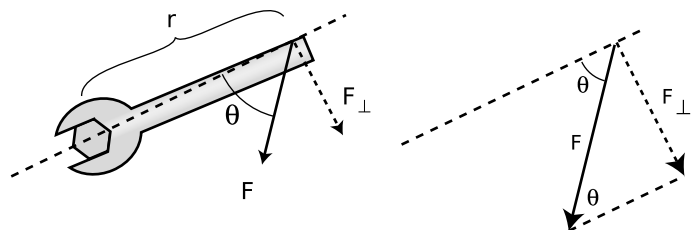


q / The geometric relationships referred to in the relationship between force and torque.

*How torque depends on the direction of the force*      *example 9*

▷ How can the torque applied to the wrench in the figure be expressed in terms of  $r$ ,  $|F|$ , and the angle  $\theta$  between these two vectors?

▷ The force vector and its  $F_{\perp}$  component form the hypotenuse and one leg of a right triangle,



and the interior angle opposite to  $F_{\perp}$  equals  $\theta$ . The absolute value of  $F_{\perp}$  can thus be expressed as

$$F_{\perp} = |F| \sin \theta \quad ,$$

leading to

$$|\tau| = r|F| \sin \theta \quad .$$

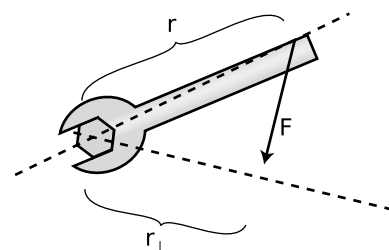
Sometimes torque can be more neatly visualized in terms of the quantity  $r_{\perp}$  shown in figure r, which gives us a third way of expressing the relationship between torque and force:

$$|\tau| = r_{\perp}|F| \quad .$$

Of course you would not want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

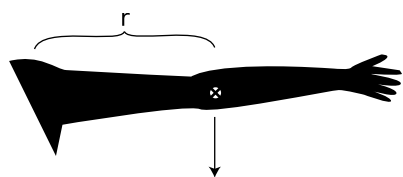
### The torque due to gravity

Up until now we've been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand's torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing



r / The quantity  $r_{\perp}$ .





s / Example 10.

the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object's center of mass.

*Gravitational torque on an outstretched arm* *example 10*

▷ Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

▷ The total gravitational force acting on your arm is

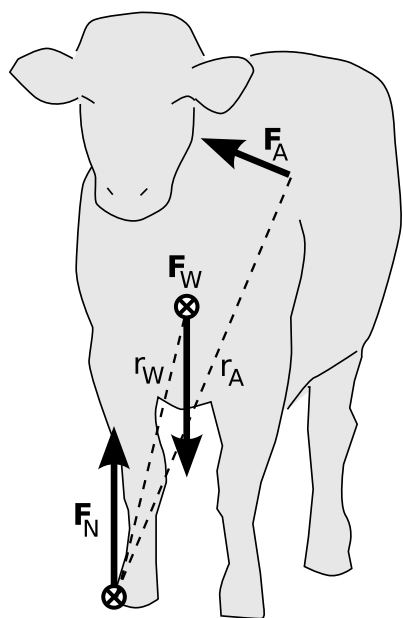
$$|F| = (3.0 \text{ kg})(9.8 \text{ m/s}^2) = 29 \text{ N} \quad .$$

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm's center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

$$F_{\perp} = |F| = 29 \text{ N} \quad .$$

Continuing to pretend that the force acts at the center of the arm,  $r$  equals 30 cm = 0.30 m, so the torque is

$$\tau = rF_{\perp} = 9 \text{ N}\cdot\text{m} \quad .$$



t / Example 11.

*Cow tipping* *example 11*

In 2005, Dr. Margo Lillie and her graduate student Tracy Boechler published a study claiming to debunk cow tipping. Their claim was based on an analysis of the torques that would be required to tip a cow, which showed that one person wouldn't be able to make enough torque to do it. A lively discussion ensued on the popular web site slashdot.org ("news for nerds, stuff that matters") concerning the validity of the study. Personally, I had always assumed that cow-tipping was a group sport anyway, but as a physicist, I also had some quibbles with their calculation. Here's my own analysis.

There are three forces on the cow: the force of gravity  $F_W$ , the ground's normal force  $F_N$ , and the tippers' force  $F_A$ .

As soon as the cow's left hooves (on the right from our point of view) break contact with the ground, the ground's force is being applied only to hooves on the other side. We don't know the ground's force, and we don't want to find it. Therefore we take the axis to be at its point of application, so that its torque is zero.

For the purpose of computing torques, we can pretend that gravity acts at the cow's center of mass, which I've placed a little lower than the center of its torso, since its legs and head also have some mass, and the legs are more massive than the head and

stick out farther, so they lower the c.m. more than the head raises it. The angle  $\theta_W$  between the vertical gravitational force and the line  $r_W$  is about  $14^\circ$ . (An estimate by Matt Semke at the University of Nebraska-Lincoln gives  $20^\circ$ , which is in the same ballpark.)

To generate the maximum possible torque with the least possible force, the tippers want to push at a point as far as possible from the axis, which will be the shoulder on the other side, and they want to push at a  $90^\circ$  angle with respect to the radius line  $r_A$ .

When the tippers are just barely applying enough force to raise the cow's hooves on one side, the total torque has to be just slightly more than zero. (In reality, they want to push a lot harder than this — hard enough to impart a lot of angular momentum to the cow fair in a short time, before it gets mad and hurts them. We're just trying to calculate the bare minimum force they can possibly use, which is the question that science can answer.) Setting the total torque equal to zero,

$$\tau_N + \tau_W + \tau_A = 0 \quad ,$$

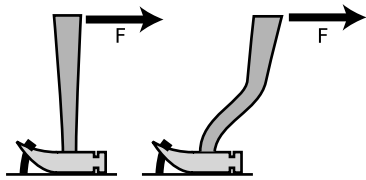
and letting counterclockwise torques be positive, we have

$$0 - mgr_W \sin \theta_W + F_A r_A \sin 90^\circ = 0$$

$$\begin{aligned} F_A &= \frac{r_W}{r_A} mg \sin \theta_W \\ &\approx \frac{1}{1.5} (680 \text{ kg})(9.8 \text{ m/s}^2) \sin 14^\circ \\ &= 1100 \text{ N} \quad . \end{aligned}$$

The 680 kg figure for the typical mass of a cow is due to Lillie and Boechler, who are veterinarians, so I assume it's fairly accurate. My estimate of 1100 N comes out significantly lower than their 1400 N figure, mainly because their incorrect placement of the center of mass gives  $\theta_W = 24^\circ$ . I don't think 1100 N is an impossible amount of force to require of one big, strong person (it's equivalent to lifting about 110 kg, or 240 pounds), but given that the tippers need to impart a large angular momentum fairly quickly, it's probably true that several people would be required.

The main practical issue with cow tipping is that cows generally sleep lying down. Falling on its side can also seriously injure a cow.



Discussion question B.

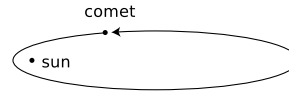


Discussion question E.

## Discussion questions

**A** This series of discussion questions deals with past students' incorrect reasoning about the following problem.

Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun's gravitational force.



Throughout the question, define all torques and angular momenta using the sun as the axis.

- (1) Is the sun producing a nonzero torque on the comet? Explain.
- (2) Is the comet's angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong. (a) Incorrect answer to part (1): "Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun."

(b) Incorrect answer to part (1): "No, because the torques cancel out."

(c) Incorrect answer to part (2): "Increasing, because the comet is speeding up."

**B** Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?

**C** You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock's angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

**D** A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?

**E** The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?

## 15.5 Statics

### Equilibrium

There are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine's torque is being canceled by the torque due to friction.

When an object has constant momentum and constant angular momentum, we say that it is in equilibrium. This is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Trees and bridges have been designed by evolution and engineers to stay at rest, and to do so they must have not just zero total force acting on them but zero total torque. It is not enough that they don't fall down, they also must not tip over. Statics is the branch of physics concerned with problems such as these.

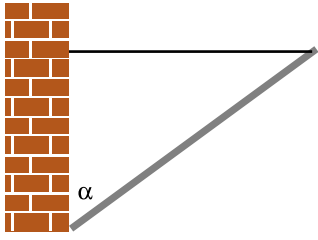
Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you'll encounter will not be this complicated. In the following example, only the equation for zero total torque is required in order to get an answer.



u / The windmills are not closed systems, but angular momentum is being transferred out of them at the same rate it is transferred in, resulting in constant angular momentum. To get an idea of the huge scale of the modern windmill farm, note the sizes of the trucks and trailers.



v / Example 12.

### A flagpole

example 12

▷ A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle  $\alpha$  be without breaking the cable?

▷ All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don't want to find. We could study the forces and torques on the cable, but that doesn't let us use the given information about the pole. The object we need to analyze is the pole.

The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable's force, and (3) the wall's force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since by that definition the wall's force on the pole is applied at  $r = 0$  and thus makes no torque on the pole. This is good, because we don't know what the wall's force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation  $|\tau| = r|F| \sin \theta$ , we have

$$r_{\text{cable}}|F_{\text{cable}}| \sin \theta_{\text{cable}} - r_{\text{grav}}|F_{\text{grav}}| \sin \theta_{\text{grav}} = 0 \quad .$$

A little geometry gives  $\theta_{\text{cable}} = 90^\circ - \alpha$  and  $\theta_{\text{grav}} = \alpha$ , so

$$r_{\text{cable}}|F_{\text{cable}}| \sin(90^\circ - \alpha) - r_{\text{grav}}|F_{\text{grav}}| \sin \alpha = 0 \quad .$$

The gravitational force can be considered as acting at the pole's center of mass, i.e., at its geometrical center, so  $r_{\text{cable}}$  is twice  $r_{\text{grav}}$ , and we can simplify the equation to read

$$2|F_{\text{cable}}| \sin(90^\circ - \alpha) - |F_{\text{grav}}| \sin \alpha = 0 \quad .$$

These are all quantities we were given, except for  $\alpha$ , which is the angle we want to find. To solve for  $\alpha$  we need to use the trig identity  $\sin(90^\circ - x) = \cos x$ ,

$$2|F_{\text{cable}}| \cos \alpha - |F_{\text{grav}}| \sin \alpha = 0 \quad ,$$

which allows us to find

$$\begin{aligned}\tan \alpha &= 2 \frac{|\mathbf{F}_{cable}|}{|\mathbf{F}_{grav}|} \\ \alpha &= \tan^{-1} \left( 2 \frac{|\mathbf{F}_{cable}|}{|\mathbf{F}_{grav}|} \right) \\ &= \tan^{-1} \left( 2 \times \frac{70 \text{ N}}{98 \text{ N}} \right) \\ &= 55^\circ \quad .\end{aligned}$$

**Art!**

*example 13*

▷ The abstract sculpture shown in figure w contains a cube of mass  $m$  and sides of length  $b$ . The cube rests on top of a cylinder, which is off-center by a distance  $a$ . Find the tension in the cable.

▷ There are four forces on the cube: a gravitational force  $mg$ , the force  $F_T$  from the cable, the upward normal force from the cylinder,  $F_N$ , and the horizontal static frictional force from the cylinder,  $F_s$ .

The total force on the cube in the vertical direction is zero:

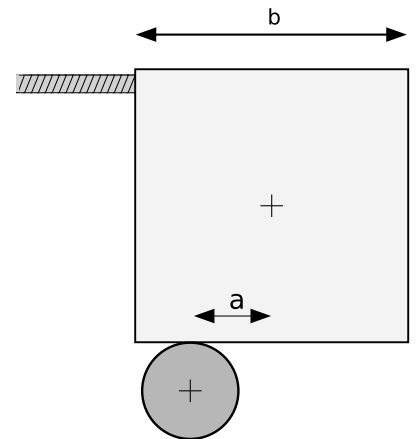
$$F_N - mg = 0 \quad .$$

As our axis for defining torques, it's convenient to choose the point of contact between the cube and the cylinder, because then neither  $F_s$  nor  $F_N$  makes any torque. The cable's torque is counter-clockwise, the torque due to gravity is clockwise. Letting counter-clockwise torques be positive, and using the convenient equation  $\tau = r_\perp F$ , we find the equation for the total torque:

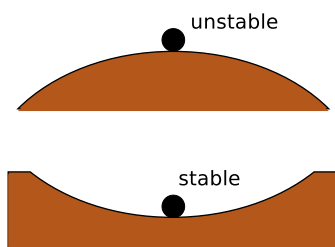
$$bF_T - mga = 0 \quad .$$

We could also write down the equation saying that the total horizontal force is zero, but that would bring in the cylinder's frictional force on the cube, which we don't know and don't need to find. We already have two equations in the two unknowns  $F_T$  and  $F_N$ , so there's no need to make it into three equations in three unknowns. Solving the first equation for  $F_N = mg$ , we then substitute into the second equation to eliminate  $F_N$ , and solve for  $F_T = (a/b)mg$ .

As a check, our result makes sense when  $a = 0$ ; the cube is balanced on the cylinder, so the cable goes slack.



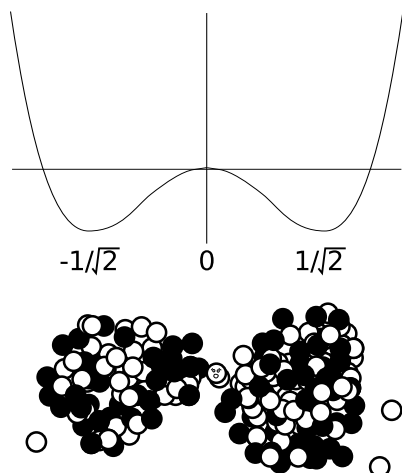
w / Example 13.



x / Stable and unstable equilibria.



y / The dancer's equilibrium is unstable. If she didn't constantly make tiny adjustments, she would tip over.



z / Example 14.

## Stable and unstable equilibria

A pencil balanced upright on its tip could theoretically be in equilibrium, but even if it was initially perfectly balanced, it would topple in response to the first air current or vibration from a passing truck. The pencil can be put in equilibrium, but not in stable equilibrium. The things around us that we really do see staying still are all in stable equilibrium.

Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, your head responds with a gentle force to the right, which keeps your nose from flying off of your face. If you push your nose a centimeter to the left, your face's force on your nose becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

This idea can be rephrased in terms of energy. The difference between the stable and unstable equilibria shown in figure x is that in the stable equilibrium, the potential energy is at a minimum, and moving to either side of equilibrium will increase it, whereas the unstable equilibrium represents a maximum.

Note that we are using the term “stable” in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by *any* force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.

### *An application of calculus*

### *example 14*

▷ Nancy Neutron is living in a uranium nucleus that is undergoing fission. Nancy's potential energy as a function of position can be approximated by  $PE = x^4 - x^2$ , where all the units and numerical constants have been suppressed for simplicity. Use calculus to locate the equilibrium points, and determine whether they are stable or unstable.

▷ The equilibrium points occur where the PE is at a minimum or maximum, and minima and maxima occur where the derivative

(which equals minus the force on Nancy) is zero. This derivative is  $dPE/dx = 4x^3 - 2x$ , and setting it equal to zero, we have  $x = 0, \pm 1/\sqrt{2}$ . Minima occur where the second derivative is positive, and maxima where it is negative. The second derivative is  $12x^2 - 2$ , which is negative at  $x = 0$  (unstable) and positive at  $x = \pm 1/\sqrt{2}$  (stable). Interpretation: the graph of the PE is shaped like a rounded letter 'W,' with the two troughs representing the two halves of the splitting nucleus. Nancy is going to have to decide which half she wants to go with.

## 15.6 Simple machines: the lever

Although we have discussed some simple machines such as the pulley, without the concept of torque we were not yet ready to address the lever, which is the machine nature used in designing living things, almost to the exclusion of all others. (We can speculate what life on our planet might have been like if living things had evolved wheels, gears, pulleys, and screws.) The figures show two examples of levers within your arm. Different muscles are used to flex and extend the arm, because muscles work only by contraction.

Analyzing example aa physically, there are two forces that do work. When we lift a load with our biceps muscle, the muscle does positive work, because it brings the bone in the forearm in the direction it is moving. The load's force on the arm does negative work, because the arm moves in the direction opposite to the load's force. This makes sense, because we expect our arm to do positive work on the load, so the load must do an equal amount of negative work on the arm. (If the biceps was lowering a load, the signs of the works would be reversed. Any muscle is capable of doing either positive or negative work.)

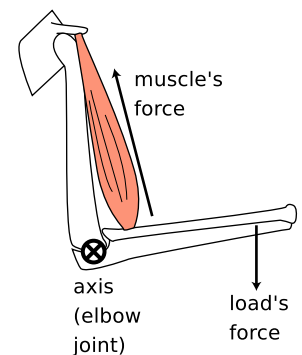
There is also a third force on the forearm: the force of the upper arm's bone exerted on the forearm at the elbow joint (not shown with an arrow in the figure). This force does no work, because the elbow joint is not moving.

Because the elbow joint is motionless, it is natural to define our torques using the joint as the axis. The situation now becomes quite simple, because the upper arm bone's force exerted at the elbow neither does work nor creates a torque. We can ignore it completely. In any lever there is such a point, called the fulcrum.

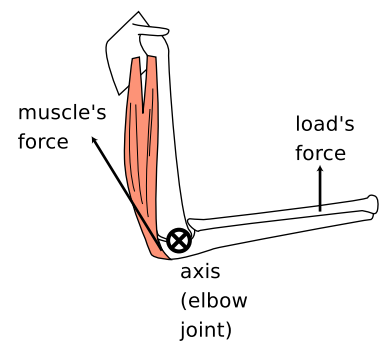
If we restrict ourselves to the case in which the forearm rotates with constant angular momentum, then we know that the total torque on the forearm is zero,

$$\tau_{\text{muscle}} + \tau_{\text{load}} = 0 \quad .$$

If we choose to represent counterclockwise torques as positive, then the muscle's torque is positive, and the load's is negative. In terms



aa / The biceps muscle flexes the arm.



ab / The triceps extends the arm.



of their absolute values,

$$|\tau_{muscle}| = |\tau_{load}| \quad .$$

Assuming for simplicity that both forces act at angles of  $90^\circ$  with respect to the lines connecting the axis to the points at which they act, the absolute values of the torques are

$$r_{muscle}F_{muscle} = r_{load}F_{arm} \quad ,$$

where  $r_{muscle}$ , the distance from the elbow joint to the biceps' point of insertion on the forearm, is only a few cm, while  $r_{load}$  might be 30 cm or so. The force exerted by the muscle must therefore be about ten times the force exerted by the load. We thus see that this lever is a force reducer. In general, a lever may be used either to increase or to reduce a force.

Why did our arms evolve so as to reduce force? In general, your body is built for compactness and maximum speed of motion rather than maximum force. This is the main anatomical difference between us and the Neanderthals (their brains covered the same range of sizes as those of modern humans), and it seems to have worked for us.

As with all machines, the lever is incapable of changing the amount of mechanical work we can do. A lever that increases force will always reduce motion, and vice versa, leaving the amount of work unchanged.

It is worth noting how simple and yet how powerful this analysis was. It was simple because we were well prepared with the concepts of torque and mechanical work. In anatomy textbooks, whose readers are assumed not to know physics, there is usually a long and complicated discussion of the different types of levers. For example, the biceps lever, aa, would be classified as a class III lever, since it has the fulcrum and load on the ends and the muscle's force acting in the middle. The triceps, ab, is called a class I lever, because the load and muscle's force are on the ends and the fulcrum is in the middle. How tiresome! With a firm grasp of the concept of torque, we realize that all such examples can be analyzed in much the same way. Physics is at its best when it lets us understand many apparently complicated phenomena in terms of a few simple yet powerful concepts.

## 15.7 ★ Proof of Kepler's elliptical orbit law

Kepler determined purely empirically that the planets' orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton's proof of this fact based on his laws of motion and law of gravity was considered his crowning achievement both by him and by his contemporaries, because it showed that the same physical laws could be used to analyze both the heavens and the earth. Newton's proof was very lengthy, but by applying the more recent concepts of conservation of energy and angular momentum we can carry out the proof quite simply and succinctly, and without calculus.

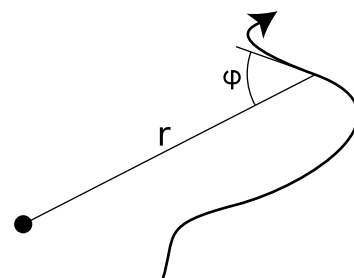
The basic idea of the proof is that we want to describe the shape of the planet's orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton's original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time from his equations, leaving only space variables. But conservation laws tell us that certain things don't change over time, so they have already had time eliminated from them.

There are many ways of representing a curve by an equation, of which the most familiar is  $y = ax + b$  for a line in two dimensions. It would be perfectly possible to describe a planet's orbit using an  $x - y$  equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis,  $r$ , and the angle between the velocity vector and the  $r$  vector, which we will call  $\phi$ . The planet will have  $\phi = 90^\circ$  when it is moving perpendicular to the  $r$  vector, i.e., at the moments when it is at its smallest or greatest distances from the sun. When  $\phi$  is less than  $90^\circ$  the planet is approaching the sun, and when it is greater than  $90^\circ$  it is receding from it. Describing a curve with an  $r - \phi$  equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.

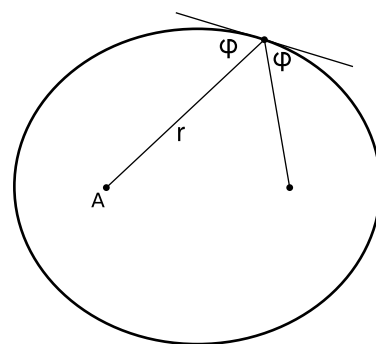
The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of figure ad; you will not be missing much if you merely absorb the result without reading the proof.

(1) If we use one of the two foci of an ellipse as an axis for defining the variables  $r$  and  $\phi$ , then the angle between the tangent line and the line drawn to the other focus is the same as  $\phi$ , i.e., the two angles labeled  $\phi$  in figure ad are in fact equal.

The other two parts form the meat of our proof. We state the



ac / The  $r - \phi$  representation of a curve.



ad / Proof that the two angles labeled  $\phi$  are in fact equal: The definition of an ellipse is that the sum of the distances from the two foci stays constant. If we move a small distance  $\ell$  along the ellipse, then one distance shrinks by an amount  $\ell \cos \phi_1$ , while the other grows by  $\ell \cos \phi_2$ . These are equal, so  $\phi_1 = \phi_2$ .

results first and then prove them.

(2) A planet, moving under the influence of the sun's gravity with less than the energy required to escape, obeys an equation of the form

$$\sin \phi = \frac{1}{\sqrt{-pr^2 + qr}} \quad ,$$

where  $p$  and  $q$  are positive constants that depend on the planet's energy and angular momentum.

(3) A curve is an ellipse if and only if its  $r - \phi$  equation is of the form

$$\sin \phi = \frac{1}{\sqrt{-pr^2 + qr}} \quad ,$$

where  $p$  and  $q$  are constants that depend on the size and shape of the ellipse and  $p$  is greater than zero.

### Proof of part (2)

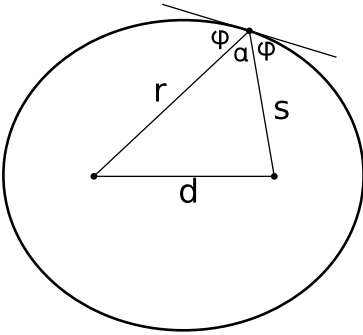
The component of the planet's velocity vector that is perpendicular to the  $\mathbf{r}$  vector is  $v_{\perp} = v \sin \phi$ , so conservation of angular momentum tells us that  $L = mrv \sin \phi$  is a constant. Since the planet's mass is a constant, this is the same as the condition

$$rv \sin \phi = \text{constant} \quad .$$

Conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant} \quad .$$

We solve the first equation for  $v$  and plug into the second equation to eliminate  $v$ . Straightforward algebra then leads to the equation claimed above, with the constant  $p$  being positive because of our assumption that the planet's energy is insufficient to escape from the sun, i.e., its total energy is negative.



ae / Proof of part (3).

### Proof of part (3)

We define the quantities  $\alpha$ ,  $d$ , and  $s$  as shown in the figure. The law of cosines gives

$$d^2 = r^2 + s^2 - 2rs \cos \alpha \quad .$$

Using  $\alpha = 180^\circ - 2\phi$  and the trigonometric identities  $\cos(180^\circ - x) = -\cos x$  and  $\cos 2x = 1 - 2\sin^2 x$ , we can rewrite this as

$$d^2 = r^2 + s^2 - 2rs (2\sin^2 \phi - 1) \quad .$$

Straightforward algebra transforms this into

$$\sin \phi = \sqrt{\frac{(r+s)^2 - d^2}{4rs}} \quad .$$

Since  $r + s$  is constant, the top of the fraction is constant, and the denominator can be rewritten as  $4rs = 4r(\text{constant} - r)$ , which is equivalent to the desired form.

## Summary

### Selected vocabulary

angular momentum . . . . .	a measure of rotational motion; a conserved quantity for a closed system
axis . . . . .	An arbitrarily chosen point used in the definition of angular momentum. Any object whose direction changes relative to the axis is considered to have angular momentum. No matter what axis is chosen, the angular momentum of a closed system is conserved.
torque . . . . .	the rate of change of angular momentum; a numerical measure of a force's ability to twist on an object
equilibrium . . . . .	a state in which an object's momentum and angular momentum are constant
stable equilibrium	one in which a force always acts to bring the object back to a certain point
unstable equilibrium . . . . .	one in which any deviation of the object from its equilibrium position results in a force pushing it even farther away

### Notation

$L$ . . . . .	angular momentum
$t$ . . . . .	torque
$T$ . . . . .	the time required for a rigidly rotating body to complete one rotation

### Other terminology and notation

period . . . . .	a name for the variable $T$ defined above
moment of inertia, $I$ . . . . .	the proportionality constant in the equation $L = 2\pi I/T$

### Summary

Angular momentum is a measure of rotational motion which is conserved for a closed system. This book only discusses angular momentum for rotation of material objects in two dimensions. Not all rotation is rigid like that of a wheel or a spinning top. An example of nonrigid rotation is a cyclone, in which the inner parts take less time to complete a revolution than the outer parts. In order to define a measure of rotational motion general enough to include nonrigid rotation, we define the angular momentum of a system by dividing it up into small parts, and adding up all the angular momenta of the small parts, which we think of as tiny particles. We arbitrarily choose some point in space, the *axis*, and we say that anything that changes its direction relative to that point possesses angular momentum. The angular momentum of a single particle is

$$L = mv_{\perp}r \quad ,$$

where  $v_{\perp}$  is the component of its velocity perpendicular to the line

joining it to the axis, and  $r$  is its distance from the axis. Positive and negative signs of angular momentum are used to indicate clockwise and counterclockwise rotation.

The *choice of axis theorem* states that any axis may be used for defining angular momentum. If a system's angular momentum is constant for one choice of axis, then it is also constant for any other choice of axis.

The *spin theorem* states that an object's angular momentum with respect to some outside axis A can be found by adding up two parts:

(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.

(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Torque is the rate of change of angular momentum. The torque a force can produce is a measure of its ability to twist on an object. The relationship between force and torque is

$$|\boldsymbol{\tau}| = r|F_{\perp}| \quad ,$$

where  $r$  is the distance from the axis to the point where the force is applied, and  $F_{\perp}$  is the component of the force perpendicular to the line connecting the axis to the point of application. Statics problems can be solved by setting the total force and total torque on an object equal to zero and solving for the unknowns.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt? ✓

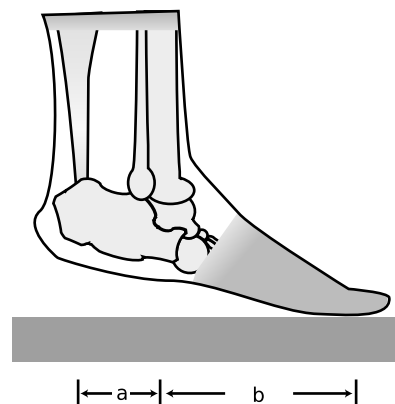
**2** A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.) ✓

**3** An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

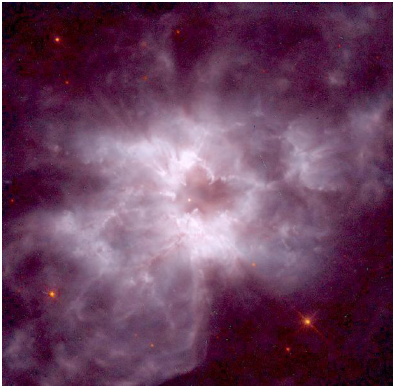
**4** An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

**5** A person of weight  $W$  stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of  $W$ ,  $a$ , and  $b$ . For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.

**6** Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]



Problem 5.



Problem 7.

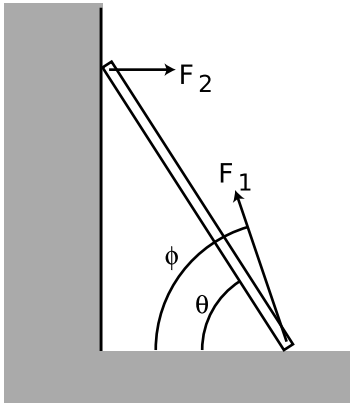
**7** The sun turns on its axis once every 26.0 days. Its mass is  $2.0 \times 10^{30}$  kg and its radius is  $7.0 \times 10^8$  m. Assume it is a rigid sphere of uniform density.

(a) What is the sun's angular momentum? ✓

In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes  $5.8 \times 10^6$  m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)

(b) What will its angular momentum be?

(c) How long will it take to turn once on its axis? ✓



Problems 8 and 9.

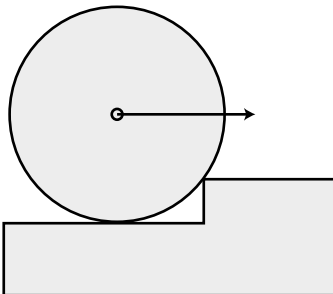
**8** A uniform ladder of mass  $m$  and length  $L$  leans against a smooth wall, making an angle  $\theta$  with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude  $F_1$  at an angle  $\phi$  with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be  $F_2$ .

(a) Explain why  $\phi$  must be greater than  $\theta$ . No math is needed.

(b) Choose any numerical values you like for  $m$  and  $L$ , and show that the ladder can be in equilibrium (zero torque and zero total force vector) for  $\theta = 45.00^\circ$  and  $\phi = 63.43^\circ$ .

**9** Continuing the previous problem, find an equation for  $\phi$  in terms of  $\theta$ , and show that  $m$  and  $L$  do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity  $\sin(a - b) = \sin a \cos b - \sin b \cos a$ . (As a numerical check on your result, you may wish to check that the angles given in part *b* of the previous problem satisfy your equation.)

✓ ★



Problem 10.

**10** (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]

(b) Under what circumstances does your result become infinite? Give a physical interpretation.

**11** A yo-yo of total mass  $m$  consists of two solid cylinders of radius  $R$ , connected by a small spindle of negligible mass and radius  $r$ . The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is  $g/(1 + R^2/2r^2)$ . [Hint: The acceleration and the tension in the string are unknown. Use  $\tau = \Delta L/\Delta t$  and  $F = ma$  to determine these two unknowns.] ★

**12** A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

**13** In the 1950's, serious articles began appearing in magazines like *Life* predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy  $E = KE + PE$  and angular momentum  $L$ . Assume that the energy is negative, i.e., the object is moving at less than escape velocity. Show that it can never reach a radius less than

$$r_{min} = \frac{GMm}{2E} \left( -1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \right) \quad .$$

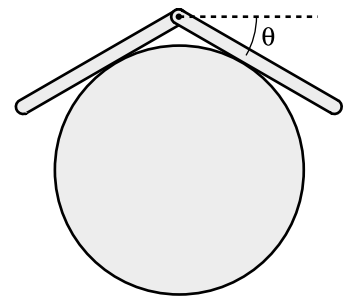
[Note that both factors are negative, giving a positive result.]

**14** [Problem 14 has been deleted.]

**15** [Problem 15 has been deleted.] ★

**16** Two bars of length  $L$  are connected with a hinge and placed on a frictionless cylinder of radius  $r$ . (a) Show that the angle  $\theta$  shown in the figure is related to the unitless ratio  $r/L$  by the equation

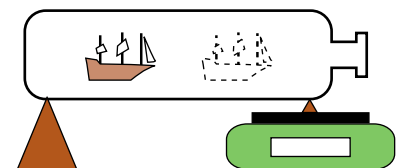
$$\frac{r}{L} = \frac{\cos^2 \theta}{2 \tan \theta} \quad .$$



Problem 16.

(b) Discuss the physical behavior of this equation for very large and very small values of  $r/L$ . ★

**17** You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can't determine the position of



Problem 17.



the ship's center of mass just by looking at it, and likewise for the bottle. In particular, you can't just say, "position the ship right on top of the fulcrum" or "position it right on top of the balance."

**18** Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the potential energy is the Lennard-Jones formula,

$$PE(r) = k \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^6 \right],$$

where  $r$  is the center-to-center distance between the atoms and  $k$  is a positive constant. Show that (a) there is an equilibrium point at  $r = a$ ,

(b) the equilibrium is stable, and

(c) the energy required to bring the atoms from their equilibrium separation to infinity is  $k$ . ▷ Hint, p. 516 ∫

**19** Suppose that we lived in a universe in which Newton's law of gravity gave forces proportional to  $r^{-7}$  rather than  $r^{-2}$ . Which, if any, of Kepler's laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?

**20** The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person's hand. ▷ Solution, p. 529

**21** Show that a sphere of radius  $R$  that is rolling without slipping has angular momentum and momentum in the ratio  $L/p = (2/5)R$ .

**22** Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals 5/7 of its initial velocity. [Hint: You'll need the result of problem 21.]

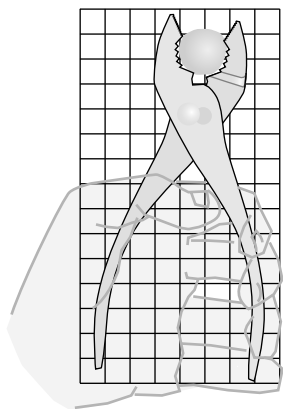
**23** The rod in the figure is supported by the finger and the string.

(a) Find the tension,  $T$ , in the string, and the force,  $F$ , from the finger, in terms of  $m$ ,  $b$ ,  $L$ , and  $g$ . ✓

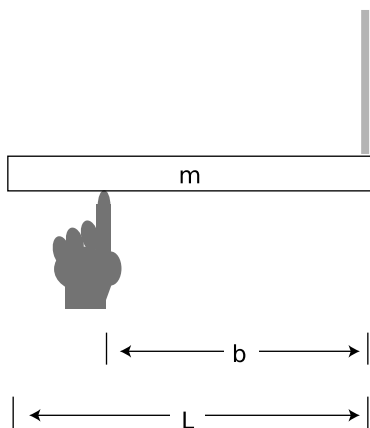
(b) Comment on the cases  $b = L$  and  $b = L/2$ .

(c) Are any values of  $b$  unphysical?

**24** Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]



Problem 20.



Problem 23.

**25** (a) Alice says Cathy's body has zero momentum, but Bob says Cathy's momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.

(b) Alice and Bob agree that Dong's body has nonzero momentum, but disagree about Dong's angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

**26** Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux's angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.

**27** Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

**28** In example 8 on page 377, prove that if the rod is sufficiently thin, it can be toppled without scraping on the floor.

▷ Solution, p. 529 ★

**29** A massless rod of length  $\ell$  has weights, each of mass  $m$ , attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance  $b$  from the rod's center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

**30** A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

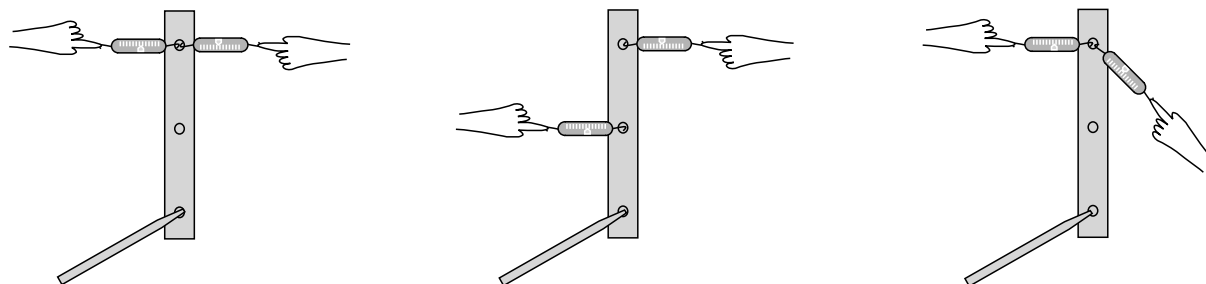


Problem 27.

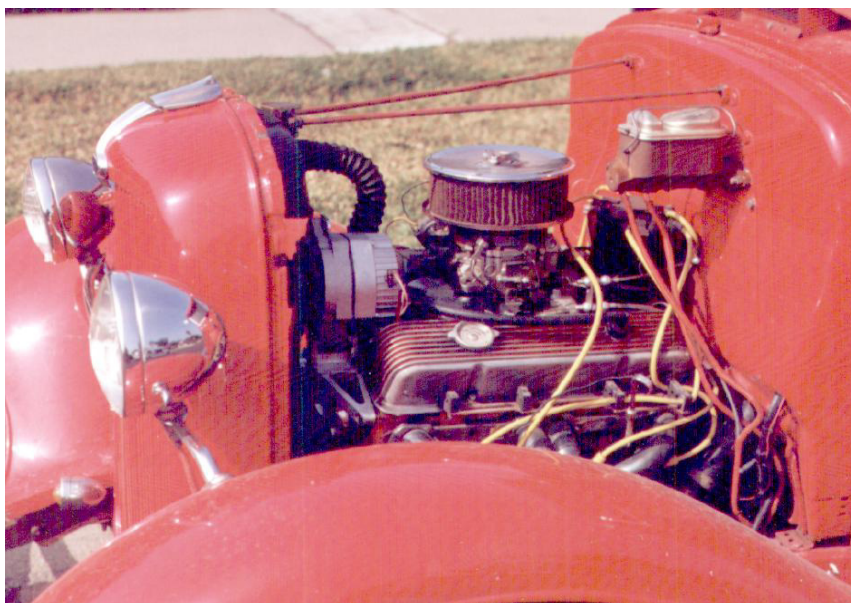
## Exercise 15: Torque

Equipment:

- rulers with holes in them
- spring scales (two per group)



While one person holds the pencil which forms the axle for the ruler, the other members of the group pull on the scale and take readings. In each case, calculate the total torque on the ruler, and find out whether it equals zero to roughly within the accuracy of the experiment. Finish the calculations for each part before moving on to the next one.



## Chapter 16

# Thermodynamics

*This chapter is optional, and should probably be omitted from a two-semester survey course. It can be covered at any time after chapter 13.*

In a developing country like China, a refrigerator is the mark of a family that has arrived in the middle class, and a car is the ultimate symbol of wealth. Both of these are *heat engines*: devices for converting between heat and other forms of energy. Unfortunately for the Chinese, neither is a very efficient device. Burning fossil fuels has made China's big cities the most polluted on the planet, and the country's total energy supply isn't sufficient to support American levels of energy consumption by more than a small fraction of China's population. Could we somehow manipulate energy in a more efficient way?

Conservation of energy is a statement that the total amount of energy is constant at all times, which encourages us to believe that any energy transformation can be undone — indeed, the laws of physics you've learned so far don't even distinguish the past from the future. If you get in a car and drive around the block, the net effect is to consume some of the energy you paid for at the gas station, using it to heat the neighborhood. There would not seem to be any fundamental physical principle to prevent you from

recapturing all that heat and using it again the next time you want to go for a drive. More modestly, why don't engineers design a car engine so that it recaptures the heat energy that would otherwise be wasted via the radiator and the exhaust?

Hard experience, however, has shown that designers of more and more efficient engines run into a brick wall at a certain point. The generators that the electric company uses to produce energy at an oil-fueled plant are indeed much more efficient than a car engine, but even if one is willing to accept a device that is very large, expensive, and complex, it turns out to be impossible to make a perfectly efficient heat engine — not just impossible with present-day technology, but impossible due to a set of fundamental physical principles known as the science of *thermodynamics*. And thermodynamics isn't just a pesky set of constraints on heat engines. Without thermodynamics, there is no way to explain the direction of time's arrow — why we can remember the past but not the future, and why it's easier to break Humpty Dumpty than to put him back together again.

## 16.1 Pressure and temperature

When we heat an object, we speed up the mind-bogglingly complex random motion of its molecules. One method for taming complexity is the conservation laws, since they tell us that certain things must remain constant regardless of what process is going on. Indeed, the law of conservation of energy is also known as the first law of thermodynamics.

But as alluded to in the introduction to this chapter, conservation of energy by itself is not powerful enough to explain certain empirical facts about heat. A second way to sidestep the complexity of heat is to ignore heat's atomic nature and concentrate on quantities like temperature and pressure that tell us about a system's properties as a whole. This approach is called macroscopic in contrast to the microscopic method of attack. Pressure and temperature were fairly well understood in the age of Newton and Galileo, hundreds of years before there was any firm evidence that atoms and molecules even existed.

Unlike the conserved quantities such as mass, energy, momentum, and angular momentum, neither pressure nor temperature is additive. Two cups of coffee have twice the heat energy of a single cup, but they do not have twice the temperature. Likewise, the painful pressure on your eardrums at the bottom of a pool is not affected if you insert or remove a partition between the two halves of the pool.

### Pressure

We restrict ourselves to a discussion of pressure in fluids at rest and in equilibrium. In physics, the term “fluid” is used to mean

either a gas or a liquid. The important feature of a fluid can be demonstrated by comparing with a cube of jello on a plate. The jello is a solid. If you shake the plate from side to side, the jello will respond by shearing, i.e., by slanting its sides, but it will tend to spring back into its original shape. A solid can sustain shear forces, but a fluid cannot. A fluid does not resist a change in shape unless it involves a change in volume.

If you're at the bottom of a pool, you can't relieve the pain in your ears by turning your head. The water's force on your eardrum is always the same, and is always perpendicular to the surface where the eardrum contacts the water. If your ear is on the east side of your head, the water's force is to the west. If you keep your head in the same spot while turning around so your ear is on the north, the force will still be the same in magnitude, and it will change its direction so that it is still perpendicular to the eardrum: south. This shows that pressure has no direction in space, i.e., it is a scalar. The direction of the force is determined by the orientation of the surface on which the pressure acts, not by the pressure itself. A fluid flowing over a surface can also exert frictional forces, which are parallel to the surface, but the present discussion is restricted to fluids at rest.

Experiments also show that a fluid's force on a surface is proportional to the surface area. The vast force of the water behind a dam, for example, is in proportion to the dam's great surface area. (The bottom of the dam experiences a higher proportion of its force.)

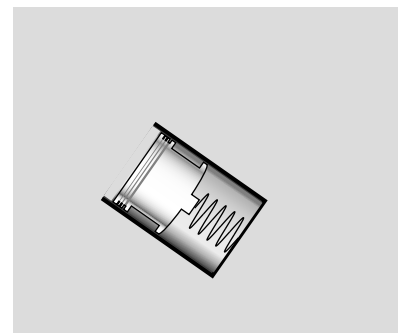
Based on these experimental results, it appears that the useful way to define pressure is as follows. The pressure of a fluid at a given point is defined as  $F_{\perp}/A$ , where  $A$  is the area of a small surface inserted in the fluid at that point, and  $F_{\perp}$  is the component of the fluid's force on the surface which is perpendicular to the surface.

This is essentially how a pressure gauge works. The reason that the surface must be small is so that there will not be any significant different in pressure between one part of it and another part. The SI units of pressure are evidently  $\text{N/m}^2$ , and this combination can be abbreviated as the pascal,  $1 \text{ Pa} = 1 \text{ N/m}^2$ . The pascal turns out to be an inconveniently small unit, so car tires, for example, have recommended pressures imprinted on them in units of kilopascals.

#### *Pressure in U.S. units*

#### *example 1*

In U.S. units, the unit of force is the pound, and the unit of distance is the inch. The unit of pressure is therefore pounds per square inch, or p.s.i. (Note that the pound is not a unit of mass.)



a / A simple pressure gauge consists of a cylinder open at one end, with a piston and a spring inside. The depth to which the spring is depressed is a measure of the pressure. To determine the absolute pressure, the air needs to be pumped out of the interior of the gauge, so that there is no air pressure acting outward on the piston. In many practical gauges, the back of the piston is open to the atmosphere, so the pressure the gauge registers equals the pressure of the fluid minus the pressure of the atmosphere.

*Atmospheric pressure in U.S. and metric units* *example 2*

▷ A figure that many people in the U.S. remember is that atmospheric pressure is about 15 pounds per square inch. What is this in metric units?

▷

$$\begin{aligned}\frac{15 \text{ lb}}{1 \text{ in}^2} &= \frac{68 \text{ N}}{(0.0254 \text{ m})^2} &&= 1.0 \times 10^5 \text{ N/m}^2 \\ &= 100 \text{ kPa}\end{aligned}$$

*Only pressure differences are normally significant.*

If you spend enough time on an airplane, the pain in your ears subsides. This is because your body has gradually been able to admit more air into the cavity behind the eardrum. Once the pressure inside is equalized with the pressure outside, the inward and outward forces on your eardrums cancel out, and there is no physical sensation to tell you that anything unusual is going on. For this reason, it is normally only pressure differences that have any physical significance. Thus deep-sea fish are perfectly healthy in their habitat because their bodies have enough internal pressure to cancel the pressure from the water in which they live; if they are caught in a net and brought to the surface rapidly, they explode because their internal pressure is so much greater than the low pressure outside.

*Getting killed by a pool pump* *example 3*

▷ My house has a pool, which I maintain myself. A pool always needs to have its water circulated through a filter for several hours a day in order to keep it clean. The filter is a large barrel with a strong clamp that holds the top and bottom halves together. My filter has a prominent warning label that warns me not to try to open the clamps while the pump is on, and it shows a cartoon of a person being struck by the top half of the pump. The cross-sectional area of the filter barrel is  $0.25 \text{ m}^2$ . Like most pressure gauges, the one on my pool pump actually reads the difference in pressure between the pressure inside the pump and atmospheric pressure. The gauge reads 90 kPa. What is the force that is trying to pop open the filter?

▷ If the gauge told us the absolute pressure of the water inside, we'd have to find the force of the water pushing outward and the force of the air pushing inward, and subtract in order to find the total force. Since air surrounds us all the time, we would have to do such a subtraction every time we wanted to calculate anything useful based on the gauge's reading. The manufacturers of the gauge decided to save us from all this work by making it read the difference in pressure between inside and outside, so all we have to do is multiply the gauge reading by the cross-sectional area of

the filter:

$$\begin{aligned}
 F &= PA \\
 &= (90 \times 10^3 \text{ N/m}^2)(0.25 \text{ m}^2) \\
 &= 22000 \text{ N}
 \end{aligned}$$

That's a lot of force!

The word “suction” and other related words contain a hidden misunderstanding related to this point about pressure differences. When you suck water up through a straw, there is nothing in your mouth that is attracting the water upward. The force that lifts the water is from the pressure of the water in the cup. By creating a partial vacuum in your mouth, you decreased the air's downward force on the water so that it no longer exactly canceled the upward force.

#### Variation of pressure with depth

The pressure within a fluid in equilibrium can only depend on depth, due to gravity. If the pressure could vary from side to side, then a piece of the fluid in between, b, would be subject to unequal forces from the parts of the fluid on its two sides. But fluids do not exhibit shear forces, so there would be no other force that could keep this piece of fluid from accelerating. This contradicts the assumption that the fluid was in equilibrium.

#### self-check A

How does this proof fail for solids?

▷ Answer, p. 534

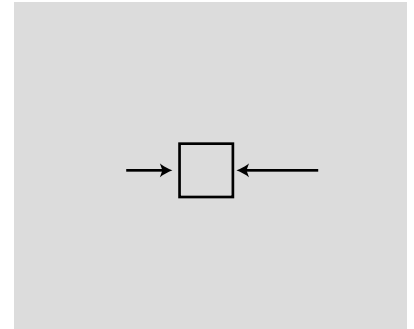
To find the variation with depth, we consider the vertical forces acting on a tiny, imaginary cube of the fluid having height  $\Delta y$  and areas  $dA$  on the top and bottom. Using positive numbers for upward forces, we have

$$P_{\text{bottom}}\Delta A - P_{\text{top}}\Delta A - F_g = 0 \quad .$$

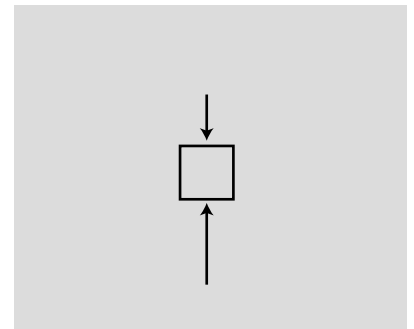
The weight of the fluid is  $F_g = mg = \rho Vg = \rho \Delta A \Delta y g$ , where  $\rho$  is the density of the fluid, so the difference in pressure is

$$\Delta P = -\rho g \Delta y \quad . \quad \left[ \begin{array}{l} \text{variation in pressure with depth for} \\ \text{a fluid of density } \rho \text{ in equilibrium;} \\ \text{positive } y \text{ is up.} \end{array} \right]$$

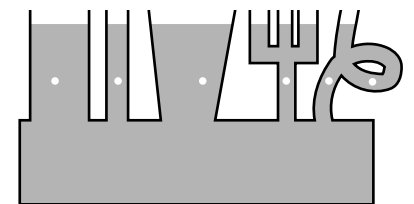
The factor of  $\rho$  explains why we notice the difference in pressure when diving 3 m down in a pool, but not when going down 3 m of stairs. Note also that the equation only tells us the difference in pressure, not the absolute pressure. The pressure at the surface of a swimming pool equals the atmospheric pressure, not zero, even though the depth is zero at the surface. The blood in your body does not even have an upper surface.



b / This doesn't happen. If pressure could vary horizontally in equilibrium, the cube of water would accelerate horizontally. This is a contradiction, since we assumed the fluid was in equilibrium.



c / This does happen. The sum of the forces from the surrounding parts of the fluid is upward, canceling the downward force of gravity.



d / The pressure is the same at all the points marked with dots.



---

*Pressure of lava underneath a volcano**example 4*

▷ A volcano has just finished erupting, and a pool of molten lava is lying at rest in the crater. The lava has come up through an opening inside the volcano that connects to the earth's molten mantle. The density of the lava is  $4.1 \text{ g/cm}^3$ . What is the pressure in the lava underneath the base of the volcano, 3000 m below the surface of the pool?

▷

$$\begin{aligned}\Delta P &= \rho g \Delta y \\ &= (4.1 \text{ g/cm}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \\ &= (4.1 \times 10^6 \text{ g/m}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \\ &= (4.1 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \\ &= 1.2 \times 10^8 \text{ N/m}^2 \\ &= 1.2 \times 10^8 \text{ Pa}\end{aligned}$$

This is the difference between the pressure we want to find and atmospheric pressure at the surface. The latter, however, is tiny compared to the  $\Delta P$  we just calculated, so what we've found is essentially the pressure,  $P$ .

---

*Atmospheric pressure**example 5*

This example uses calculus.

Gases, unlike liquids, are quite compressible, and at a given temperature, the density of a gas is approximately proportional to the pressure. The proportionality constant is discussed in section 16.2, but for now let's just call it  $k$ ,  $\rho = kP$ . Using this fact, we can find the variation of atmospheric pressure with altitude, assuming constant temperature:

$$dP = -\rho g dy$$

$$dP = -kPg dy$$

$$\frac{dP}{P} = -kg dy$$

$$\ln P = -kgy + \text{constant} \quad [\text{integrating both sides}]$$

$$P = (\text{constant})e^{-kgy} \quad [\text{exponentiating both sides}]$$

Pressure falls off exponentially with height. There is no sharp cutoff to the atmosphere, but the exponential gets extremely small by the time you're ten or a hundred miles up.

## Temperature

### *Thermal equilibrium*

We use the term temperature casually, but what is it exactly? Roughly speaking, temperature is a measure of how concentrated the heat energy is in an object. A large, massive object with very little heat energy in it has a low temperature.



e / We have to wait for the thermometer to equilibrate its temperature with the temperature of Irene's armpit.

But physics deals with operational definitions, i.e., definitions of how to measure the thing in question. How do we measure temperature? One common feature of all temperature-measuring devices is that they must be left for a while in contact with the thing whose temperature is being measured. When you take your temperature with a fever thermometer, you wait for the mercury inside to come up to the same temperature as your body. The thermometer actually tells you the temperature of its own working fluid (in this case the mercury). In general, the idea of temperature depends on the concept of thermal equilibrium. When you mix cold eggs from the refrigerator with flour that has been at room temperature, they rapidly reach a compromise temperature. What determines this compromise temperature is conservation of energy, and the amount of energy required to heat or cool each substance by one degree. But without even having constructed a temperature scale, we can see that the important point is the phenomenon of thermal equilibrium itself: two objects left in contact will approach the same temperature. We also assume that if object A is at the same temperature as object B, and B is at the same temperature as C, then A is at the same temperature as C. This statement is sometimes known as the zeroth law of thermodynamics, so called because after the first, second, and third laws had been developed, it was realized that there was another law that was even more fundamental.

### *Thermal expansion*

The familiar mercury thermometer operates on the principle that the mercury, its working fluid, expands when heated and contracts when cooled. In general, all substances expand and contract with changes in temperature. The zeroth law of thermodynamics guarantees that we can construct a comparative scale of temperatures that is independent of what type of thermometer we use. If a thermometer gives a certain reading when it's in thermal equilibrium with object A, and also gives the same reading for object B, then A and B must be the same temperature, regardless of the details of how the thermometers works.

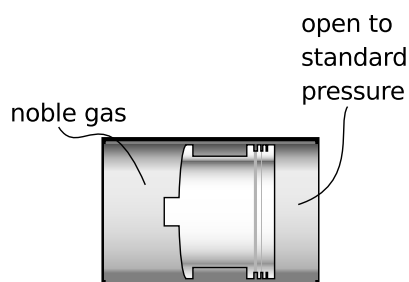
What about constructing a temperature scale in which every degree represents an equal step in temperature? The Celsius scale has 0 as the freezing point of water and 100 as its boiling point. The hidden assumption behind all this is that since two points define a line, any two thermometers that agree at two points must agree at all other points. In reality if we calibrate a mercury thermometer and an alcohol thermometer in this way, we will find that a graph of one thermometer's reading versus the other is not a perfectly straight  $y = x$  line. The subtle inconsistency becomes a drastic one when we try to extend the temperature scale through the points where mercury and alcohol boil or freeze. Gases, however, are much more consistent among themselves in their thermal expansion than



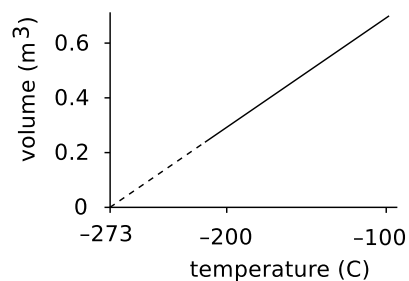
f / Thermal equilibrium can be prevented. Otters have a coat of fur that traps air bubbles for insulation. If a swimming otter was in thermal equilibrium with cold water, it would be dead. Heat is still conducted from the otter's body to the water, but much more slowly than it would be in a warm-blooded animal that didn't have this special adaptation.



g / A hot air balloon is inflated. Because of thermal expansion, the hot air is less dense than the surrounding cold air, and therefore floats as the cold air drops underneath it and pushes it up out of the way.



h / A simplified version of an ideal gas thermometer. The whole instrument is allowed to come into thermal equilibrium with the substance whose temperature is to be measured, and the mouth of the cylinder is left open to standard pressure. The volume of the noble gas gives an indication of temperature.



i / The volume of 1 kg of neon gas as a function of temperature (at standard pressure). Although neon would actually condense into a liquid at some point, extrapolating the graph to zero volume gives the same temperature as for any other gas: absolute zero.

solids or liquids, and the noble gases like helium and neon are more consistent with each other than gases in general. Continuing to search for consistency, we find that noble gases are more consistent with each other when their pressure is very low.

As an idealization, we imagine a gas in which the atoms interact only with the sides of the container, not with each other. Such a gas is perfectly nonreactive (as the noble gases very nearly are), and never condenses to a liquid (as the noble gases do only at extremely low temperatures). Its atoms take up a negligible fraction of the available volume. Any gas can be made to behave very much like this if the pressure is extremely low, so that the atoms hardly ever encounter each other. Such a gas is called an ideal gas, and we define the Celsius scale in terms of the volume of the gas in a thermometer whose working substance is an ideal gas maintained at a fixed (very low) pressure, and which is calibrated at 0 and 100 degrees according to the melting and boiling points of water. The Celsius scale is not just a comparative scale but an additive one as well: every step in temperature is equal, and it makes sense to say that the difference in temperature between 18 and 28°C is the same as the difference between 48 and 58.

#### *Absolute zero and the kelvin scale*

We find that if we extrapolate a graph of volume versus temperature, the volume becomes zero at nearly the same temperature for all gases: -273°C. Real gases will all condense into liquids at some temperature above this, but an ideal gas would achieve zero volume at this temperature, known as absolute zero. The most useful temperature scale in scientific work is one whose zero is defined by absolute zero, rather than by some arbitrary standard like the melting point of water. The ideal temperature scale for scientific work, called the Kelvin scale, is the same as the Celsius scale, but shifted by 273 degrees to make its zero coincide with absolute zero. Scientists use the Celsius scale only for comparisons or when a change in temperature is all that is required for a calculation. Only on the Kelvin scale does it make sense to discuss ratios of temperatures, e.g., to say that one temperature is twice as hot as another.

#### *Which temperature scale to use* example 6

▷ You open an astronomy book and encounter the equation

$$(\text{light emitted}) = (\text{constant}) \times T^4$$

for the light emitted by a star as a function of its surface temperature. What temperature scale is implied?

▷ The equation tells us that doubling the temperature results in the emission of 16 times as much light. Such a ratio only makes sense if the Kelvin scale is used.

## 16.2 Microscopic description of an ideal gas

### Evidence for the kinetic theory

Why does matter have the thermal properties it does? The basic answer must come from the fact that matter is made of atoms. How, then, do the atoms give rise to the bulk properties we observe? Gases, whose thermal properties are so simple, offer the best chance for us to construct a simple connection between the microscopic and macroscopic worlds.

A crucial observation is that although solids and liquids are nearly incompressible, gases can be compressed, as when we increase the amount of air in a car's tire while hardly increasing its volume at all. This makes us suspect that the atoms in a solid are packed shoulder to shoulder, while a gas is mostly vacuum, with large spaces between molecules. Most liquids and solids have densities about 1000 times greater than most gases, so evidently each molecule in a gas is separated from its nearest neighbors by a space something like 10 times the size of the molecules themselves.

If gas molecules have nothing but empty space between them, why don't the molecules in the room around you just fall to the floor? The only possible answer is that they are in rapid motion, continually rebounding from the walls, floor and ceiling. In chapter 12, we have already seen some of the evidence for the kinetic theory of heat, which states that heat is the kinetic energy of randomly moving molecules. This theory was proposed by Daniel Bernoulli in 1738, and met with considerable opposition, because there was no precedent for this kind of perpetual motion. No rubber ball, however elastic, rebounds from a wall with exactly as much energy as it originally had, nor do we ever observe a collision between balls in which none of the kinetic energy at all is converted to heat and sound. The analogy is a false one, however. A rubber ball consists of atoms, and when it is heated in a collision, the heat is a form of motion of those atoms. An individual molecule, however, cannot possess heat. Likewise sound is a form of bulk motion of molecules, so colliding molecules in a gas cannot convert their kinetic energy to sound. Molecules can indeed induce vibrations such as sound waves when they strike the walls of a container, but the vibrations of the walls are just as likely to impart energy to a gas molecule as to take energy from it. Indeed, this kind of exchange of energy is the mechanism by which the temperatures of the gas and its container become equilibrated.

### Pressure, volume, and temperature

A gas exerts pressure on the walls of its container, and in the kinetic theory we interpret this apparently constant pressure as the averaged-out result of vast numbers of collisions occurring every second between the gas molecules and the walls. The empirical

facts about gases can be summarized by the relation

$$PV \propto nT, \quad [\text{ideal gas}]$$

which really only holds exactly for an ideal gas. Here  $n$  is the number of molecules in the sample of gas.

---

*Volume related to temperature* *example 7*

The proportionality of volume to temperature at fixed pressure was the basis for our definition of temperature.

---

*Pressure related to temperature* *example 8*

Pressure is proportional to temperature when volume is held constant. An example is the increase in pressure in a car's tires when the car has been driven on the freeway for a while and the tires and air have become hot.

We now connect these empirical facts to the kinetic theory of a classical ideal gas. For simplicity, we assume that the gas is monoatomic (i.e., each molecule has only one atom), and that it is confined to a cubical box of volume  $V$ , with  $L$  being the length of each edge and  $A$  the area of any wall. An atom whose velocity has an  $x$  component  $v_x$  will collide regularly with the left-hand wall, traveling a distance  $2L$  parallel to the  $x$  axis between collisions with that wall. The time between collisions is  $\Delta t = 2L/v_x$ , and in each collision the  $x$  component of the atom's momentum is reversed from  $-mv_x$  to  $mv_x$ . The total force on the wall is

$$F = \frac{\Delta p_{x,1}}{\Delta t_1} + \frac{\Delta p_{x,2}}{\Delta t_2} + \dots \quad [\text{monoatomic ideal gas}] \quad ,$$

where the indices 1, 2, ... refer to the individual atoms. Substituting  $\Delta p_{x,i} = 2mv_{x,i}$  and  $\Delta t_i = 2L/v_{x,i}$ , we have

$$F = \frac{mv_{x,1}^2}{L} + \frac{mv_{x,2}^2}{L} + \dots \quad [\text{monoatomic ideal gas}] \quad .$$

The quantity  $mv_{x,i}^2$  is twice the contribution to the kinetic energy from the part of the atom's center of mass motion that is parallel to the  $x$  axis. Since we're assuming a monoatomic gas, center of mass motion is the only type of motion that gives rise to kinetic energy. (A more complex molecule could rotate and vibrate as well.) If the quantity inside the sum included the  $y$  and  $z$  components, it would be twice the total kinetic energy of all the molecules. By symmetry, it must therefore equal 2/3 of the total kinetic energy, so

$$F = \frac{2KE_{total}}{3L} \quad [\text{monoatomic ideal gas}] \quad .$$

Dividing by  $A$  and using  $AL = V$ , we have

$$P = \frac{2KE_{total}}{3V} \quad [\text{monoatomic ideal gas}] \quad .$$

This can be connected to the empirical relation  $PV \propto nT$  if we multiply by  $V$  on both sides and rewrite  $KE_{total}$  as  $nKE_{av}$ , where  $KE_{av}$  is the average kinetic energy per molecule:

$$PV = \frac{2}{3}nKE_{av} \quad [\text{monoatomic ideal gas}] \quad .$$

For the first time we have an interpretation for the temperature based on a microscopic description of matter: in a monoatomic ideal gas, the temperature is a measure of the average kinetic energy per molecule. The proportionality between the two is  $KE_{av} = (3/2)kT$ , where the constant of proportionality  $k$ , known as Boltzmann's constant, has a numerical value of  $1.38 \times 10^{-23}$  J/K. In terms of Boltzmann's constant, the relationship among the bulk quantities for an ideal gas becomes

$$PV = nkT \quad , \quad [\text{ideal gas}]$$

which is known as the ideal gas law. Although I won't prove it here, this equation applies to all ideal gases, even though the derivation assumed a monoatomic ideal gas in a cubical box. (You may have seen it written elsewhere as  $PV = NRT$ , where  $N = n/N_A$  is the number of moles of atoms,  $R = kN_A$ , and  $N_A = 6.0 \times 10^{23}$ , called Avogadro's number, is essentially the number of hydrogen atoms in 1 g of hydrogen.)

---

*Pressure in a car tire* *example 9*

▷ After driving on the freeway for a while, the air in your car's tires heats up from 10°C to 35°C. How much does the pressure increase?

▷ The tires may expand a little, but we assume this effect is small, so the volume is nearly constant. From the ideal gas law, the ratio of the pressures is the same as the ratio of the absolute temperatures,

$$\begin{aligned} P_2/P_1 &= T_2/T_1 \\ &= (308 \text{ K})/(283 \text{ K}) \\ &= 1.09 \quad , \end{aligned}$$

or a 9% increase.

---

*Earth's senescence* *example 10*

Microbes were the only life on Earth up until the relatively recent advent of multicellular life, and are arguably still the dominant form of life on our planet. Furthermore, the sun has been gradually heating up ever since it first formed, and this continuing process will soon ("soon" in the sense of geological time) eliminate multicellular life again. Heat-induced decreases in the atmosphere's  $\text{CO}_2$  content will kill off all complex plants within about

500 million years, and although some animals may be able to live by eating algae, it will only be another few hundred million years at most until the planet is completely heat-sterilized.

Why is the sun getting brighter? The only thing that keeps a star like our sun from collapsing due to its own gravity is the pressure of its gases. The sun's energy comes from nuclear reactions at its core, and the net result of these reactions is to fuse hydrogen atoms into helium atoms. It takes four hydrogens to make one helium, so the number of atoms in the sun is continuously decreasing. Since  $PV = nkT$ , this causes a decrease in pressure, which makes the core contract. As the core contracts, collisions between hydrogen atoms become more frequent, and the rate of fusion reactions increases.

---

*A piston, a refrigerator, and a space suit* *example 11*

Both sides of the equation  $PV = nkT$  have units of energy. Suppose the pressure in a cylinder of gas pushes a piston out, as in the power stroke of an automobile engine. Let the cross-sectional area of the piston and cylinder be  $A$ , and let the piston travel a small distance  $\Delta x$ . Then the gas's force on the piston  $F = PA$  does an amount of mechanical work  $W = F\Delta x = PA\Delta x = P\Delta V$ , where  $\Delta V$  is the change in volume. This energy has to come from somewhere; it comes from cooling the gas. In a car, what this means is that we're harvesting the energy released by burning the gasoline.

In a refrigerator, we use the same process to cool the gas, which then cools the food.

In a space suit, the quantity  $P\Delta V$  represents the work the astronaut has to do because bending her limbs changes the volume of the suit. The suit inflates under pressure like a balloon, and doesn't want to bend. This makes it very tiring to work for any significant period of time.



j / A space suit (example 11).



## 16.3 Entropy

### Efficiency and grades of energy

Some forms of energy are more convenient than others in certain situations. You can't run a spring-powered mechanical clock on a battery, and you can't run a battery-powered clock with mechanical energy. However, there is no fundamental physical principle that prevents you from converting 100% of the electrical energy in a battery into mechanical energy or vice-versa. More efficient motors and generators are being designed every year. In general, the laws of physics permit perfectly efficient conversion within a broad class of forms of energy.

Heat is different. Friction tends to convert other forms of energy into heat even in the best lubricated machines. When we slide a book on a table, friction brings it to a stop and converts all its kinetic energy into heat, but we never observe the opposite process, in which a book spontaneously converts heat energy into mechanical energy and starts moving! Roughly speaking, heat is different because it is disorganized. Scrambling an egg is easy. Unscrambling it is harder.

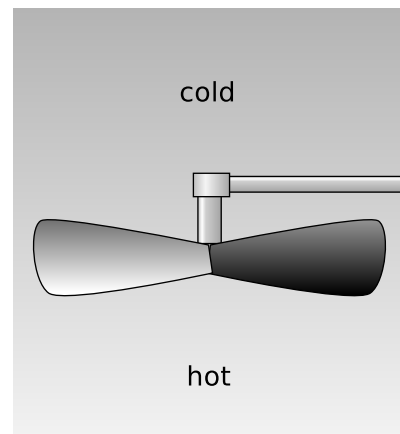
We summarize these observations by saying that heat is a lower grade of energy than other forms such as mechanical energy.

Of course it is possible to convert heat into other forms of energy such as mechanical energy, and that is what a car engine does with the heat created by exploding the air-gasoline mixture. But a car engine is a tremendously inefficient device, and a great deal of the heat is simply wasted through the radiator and the exhaust. Engineers have never succeeded in creating a perfectly efficient device for converting heat energy into mechanical energy, and we now know that this is because of a deeper physical principle that is far more basic than the design of an engine.

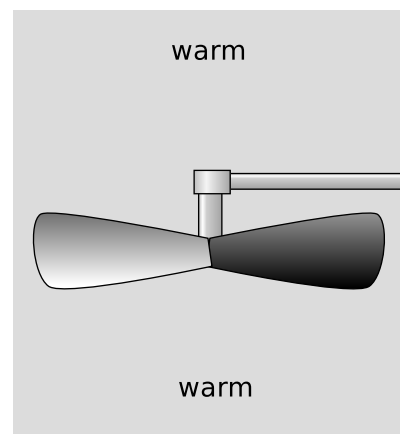
### Heat engines

Heat may be more useful in some forms than in other, i.e., there are different grades of heat energy. In figure k, the difference in temperature can be used to extract mechanical work with a fan blade. This principle is used in power plants, where steam is heated by burning oil or by nuclear reactions, and then allowed to expand through a turbine which has cooler steam on the other side. On a smaller scale, there is a Christmas toy that consists of a small propeller spun by the hot air rising from a set of candles, very much like the setup shown in the figure.

In figure l, however, no mechanical work can be extracted because there is no difference in temperature. Although the air in l has the same total amount of energy as the air in k, the heat in l is a lower grade of energy, since none of it is accessible for doing

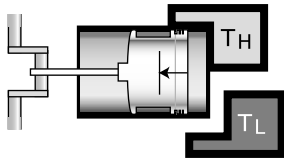


k / The temperature difference between the hot and cold parts of the air can be used to extract mechanical energy, for example with a fan blade that spins because of the rising hot air currents.

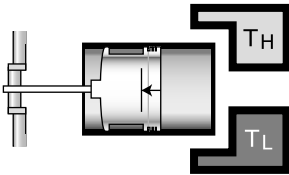


l / If the temperature of the air is first allowed to become uniform, then no mechanical energy can be extracted. The same amount of heat energy is present, but it is no longer accessible for doing mechanical work.

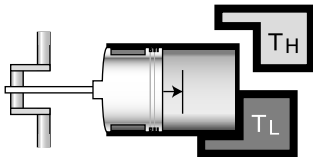




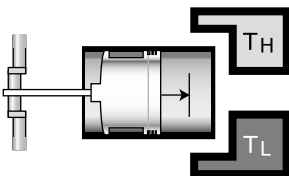
m / The beginning of the first expansion stroke, in which the working gas is kept in thermal equilibrium with the hot reservoir.



n / The beginning of the second expansion stroke, in which the working gas is thermally insulated. The working gas cools because it is doing work on the piston and thus losing energy.



o / The beginning of the first compression stroke. The working gas begins the stroke at the same temperature as the cold reservoir, and remains in thermal contact with it the whole time. The engine does negative work.



p / The beginning of the second compression stroke, in which mechanical work is absorbed, heating the working gas back up to  $T_H$ .

mechanical work.

In general, we define a heat engine as any device that takes heat from a reservoir of hot matter, extracts some of the heat energy to do mechanical work, and expels a lesser amount of heat into a reservoir of cold matter. The efficiency of a heat engine equals the amount of useful work extracted,  $W$ , divided by the amount of energy we had to pay for in order to heat the hot reservoir. This latter amount of heat is the same as the amount of heat the engine extracts from the high-temperature reservoir,  $Q_H$ . (The letter  $Q$  is the standard notation for a transfer of heat.) By conservation of energy, we have  $Q_H = W + Q_L$ , where  $Q_L$  is the amount of heat expelled into the low-temperature reservoir, so the efficiency of a heat engine,  $W/Q_H$ , can be rewritten as

$$\text{efficiency} = 1 - \frac{Q_L}{Q_H} \quad . \quad [\text{efficiency of any heat engine}]$$

It turns out that there is a particular type of heat engine, the Carnot engine, which, although not 100% efficient, is more efficient than any other. The grade of heat energy in a system can thus be unambiguously defined in terms of the amount of heat energy in it that *cannot* be extracted, even by a Carnot engine.

How can we build the most efficient possible engine? Let's start with an unnecessarily inefficient engine like a car engine and see how it could be improved. The radiator and exhaust expel hot gases, which is a waste of heat energy. These gases are cooler than the exploded air-gas mixture inside the cylinder, but hotter than the air that surrounds the car. We could thus improve the engine's efficiency by adding an auxiliary heat engine to it, which would operate with the first engine's exhaust as its hot reservoir and the air as its cold reservoir. In general, any heat engine that expels heat at an intermediate temperature can be made more efficient by changing it so that it expels heat only at the temperature of the cold reservoir.

Similarly, any heat engine that absorbs some energy at an intermediate temperature can be made more efficient by adding an auxiliary heat engine to it which will operate between the hot reservoir and this intermediate temperature.

Based on these arguments, we define a Carnot engine as a heat engine that absorbs heat only from the hot reservoir and expels it only into the cold reservoir. Figures m-p show a realization of a Carnot engine using a piston in a cylinder filled with a monoatomic ideal gas. This gas, known as the working fluid, is separate from, but exchanges energy with, the hot and cold reservoirs. It turns out that this particular Carnot engine has an efficiency given by

$$\text{efficiency} = 1 - \frac{T_L}{T_H} \quad , \quad [\text{efficiency of a Carnot engine}]$$

where  $T_L$  is the temperature of the cold reservoir and  $T_H$  is the temperature of the hot reservoir. (A proof of this fact is given in my book *Simple Nature*, which you can download for free.)

Even if you do not wish to dig into the details of the proof, the basic reason for the temperature dependence is not so hard to understand. Useful mechanical work is done on strokes m and n, in which the gas expands. The motion of the piston is in the same direction as the gas's force on the piston, so positive work is done on the piston. In strokes o and p, however, the gas does negative work on the piston. We would like to avoid this negative work, but we must design the engine to perform a complete cycle. Luckily the pressures during the compression strokes are lower than the ones during the expansion strokes, so the engine doesn't undo all its work with every cycle. The ratios of the pressures are in proportion to the ratios of the temperatures, so if  $T_L$  is 20% of  $T_H$ , the engine is 80% efficient.

We have already proved that any engine that is not a Carnot engine is less than optimally efficient, and it is also true that all Carnot engines operating between a given pair of temperatures  $T_H$  and  $T_L$  have the same efficiency. Thus a Carnot engine is the most efficient possible heat engine.

## Entropy

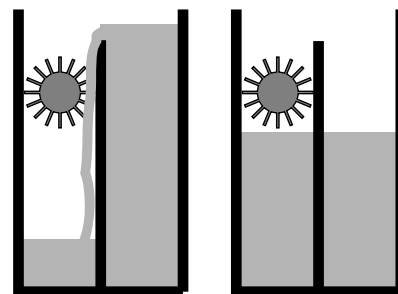
We would like to have some numerical way of measuring the grade of energy in a system. We want this quantity, called entropy, to have the following two properties:

(1) Entropy is additive. When we combine two systems and consider them as one, the entropy of the combined system equals the sum of the entropies of the two original systems. (Quantities like mass and energy also have this property.)

(2) The entropy of a system is not changed by operating a Carnot engine within it.

It turns out to be simpler and more useful to define changes in entropy than absolute entropies. Suppose as an example that a system contains some hot matter and some cold matter. It has a relatively high grade of energy because a heat engine could be used to extract mechanical work from it. But if we allow the hot and cold parts to equilibrate at some lukewarm temperature, the grade of energy has gotten worse. Thus putting heat into a hotter area is more useful than putting it into a cold area. Motivated by these considerations, we define a change in entropy as follows:

$$\Delta S = \frac{Q}{T} \quad \begin{array}{l} \text{[change in entropy when adding} \\ \text{heat } Q \text{ to matter at temperature } T; \\ \Delta S \text{ is negative if heat is taken out]} \end{array}$$



q / Entropy can be understood using the metaphor of a water wheel. Letting the water levels equalize is like letting the entropy maximize. Taking water from the high side and putting it into the low side increases the entropy. Water levels in this metaphor correspond to temperatures in the actual definition of entropy.

A system with a higher grade of energy has a lower entropy.

---

*Entropy is additive.* *example 12*

Since changes in entropy are defined by an additive quantity (heat) divided by a non-additive one (temperature), entropy is additive.

---

*Entropy isn't changed by a Carnot engine.* *example 13*

The efficiency of a heat engine is defined by

$$\text{efficiency} = 1 - Q_L/Q_H \quad ,$$

and the efficiency of a Carnot engine is

$$\text{efficiency} = 1 - T_L/T_H \quad ,$$

so for a Carnot engine we have  $Q_L/Q_H = T_L/T_H$ , which can be rewritten as  $Q_L/T_L = Q_H/T_H$ . The entropy lost by the hot reservoir is therefore the same as the entropy gained by the cold one.

---

*Entropy increases in heat conduction.* *example 14*

When a hot object gives up energy to a cold one, conservation of energy tells us that the amount of heat lost by the hot object is the same as the amount of heat gained by the cold one. The change in entropy is  $-Q/T_H + Q/T_L$ , which is positive because  $T_L < T_H$ .

---

*Entropy is increased by a non-Carnot engine.* *example 15*

The efficiency of a non-Carnot engine is less than  $1 - T_L/T_H$ , so  $Q_L/Q_H > T_L/T_H$  and  $Q_L/T_L > Q_H/T_H$ . This means that the entropy increase in the cold reservoir is greater than the entropy decrease in the hot reservoir.

---

*A book sliding to a stop* *example 16*

A book slides across a table and comes to a stop. Once it stops, all its kinetic energy has been transformed into heat. As the book and table heat up, their entropies both increase, so the total entropy increases as well.

Examples 14-16 involved closed systems, and in all of them the total entropy either increased or stayed the same. It never decreased. Here are two examples of schemes for decreasing the entropy of a closed system, with explanations of why they don't work.

---

*Using a refrigerator to decrease entropy?* *example 17*

▷ A refrigerator takes heat from a cold area and dumps it into a hot area. (1) Does this lead to a net decrease in the entropy of a closed system? (2) Could you make a Carnot engine more efficient by running a refrigerator to cool its low-temperature reservoir and eject heat into its high-temperature reservoir?

▷ (1) No. The heat that comes off of the radiator coils on the back of your kitchen fridge is a great deal more than the heat the fridge removes from inside; the difference is what it costs to run your fridge. The heat radiated from the coils is so much more

than the heat removed from the inside that the increase in the entropy of the air in the room is greater than the decrease of the entropy inside the fridge. The most efficient refrigerator is actually a Carnot engine running in reverse, which leads to neither an increase nor a decrease in entropy.

(2) No. The most efficient refrigerator is a reversed Carnot engine. You will not achieve anything by running one Carnot engine in reverse and another forward. They will just cancel each other out.

---

*Maxwell's daemon*

*example 18*

▷ Physicist James Clerk Maxwell imagined pair of neighboring rooms, their air being initially in thermal equilibrium, having a partition across the middle with a tiny door. A miniscule daemon is posted at the door with a little ping-pong paddle, and his duty is to try to build up faster-moving air molecules in room B and slower ones in room A. For instance, when a fast molecule is headed through the door, going from A to B, he lets it by, but when a slower than average molecule tries the same thing, he hits it back into room A. Would this decrease the total entropy of the pair of rooms?

▷ No. The daemon needs to eat, and we can think of his body as a little heat engine. His metabolism is less efficient than a Carnot engine, so he ends up increasing the entropy rather than decreasing it.

Observation such as these lead to the following hypothesis, known as the second law of thermodynamics:

The entropy of a closed system always increases, or at best stays the same:  $\Delta S \geq 0$ .

At present my arguments to support this statement may seem less than convincing, since they have so much to do with obscure facts about heat engines. A more satisfying and fundamental explanation for the continual increase in entropy was achieved by Ludwig Boltzmann, and you may wish to learn more about Boltzmann's ideas from my book *Simple Nature*, which you can download for free. Briefly, Boltzmann realized that entropy was a measure of randomness or disorder at the atomic level, and disorder doesn't spontaneously change into order.

To emphasize the fundamental and universal nature of the second law, here are a few examples.

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*Entropy and evolution*

*example 19*

A favorite argument of many creationists who don't believe in evolution is that evolution would violate the second law of thermodynamics: the death and decay of a living thing releases heat (as

when a compost heap gets hot) and lessens the amount of energy available for doing useful work, while the reverse process, the emergence of life from nonliving matter, would require a decrease in entropy. Their argument is faulty, since the second law only applies to closed systems, and the earth is not a closed system. The earth is continuously receiving energy from the sun.

---

*The heat death of the universe* *example 20*

Victorian philosophers realized that living things had low entropy, as discussed in example 19, and spent a lot of time worrying about the heat death of the universe: eventually the universe would have to become a high-entropy, lukewarm soup, with no life or organized motion of any kind. Fortunately (?), we now know a great many other things that will make the universe inhospitable to life long before its entropy is maximized. Life on earth, for instance, will end when the sun evolves into a giant star and vaporizes our planet.

---

*Hawking radiation* *example 21*

Any process that could destroy heat (or convert it into nothing but mechanical work) would lead to a reduction in entropy. Black holes are supermassive stars whose gravity is so strong that nothing, not even light, can escape from them once it gets within a boundary known as the event horizon. Black holes are commonly observed to suck hot gas into them. Does this lead to a reduction in the entropy of the universe? Of course one could argue that the entropy is still there inside the black hole, but being able to “hide” entropy there amounts to the same thing as being able to destroy entropy.

The physicist Steven Hawking was bothered by this question, and finally realized that although the actual stuff that enters a black hole is lost forever, the black hole will gradually lose energy in the form of light emitted from just outside the event horizon. This light ends up reintroducing the original entropy back into the universe at large.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** (a) Show that under conditions of standard pressure and temperature, the volume of a sample of an ideal gas depends only on the number of molecules in it.

(b) One mole is defined as  $6.0 \times 10^{23}$  atoms. Find the volume of one mole of an ideal gas, in units of liters, at standard temperature and pressure ( $0^\circ\text{C}$  and  $101\text{ kPa}$ ). ✓

**2** A gas in a cylinder expands its volume by an amount  $\Delta V$ , pushing out a piston. Show that the work done by the gas on the piston is given by  $\Delta W = P\Delta V$ .

**3** (a) A helium atom contains 2 protons, 2 electrons, and 2 neutrons. Find the mass of a helium atom. ✓

(b) Find the number of atoms in 1 kg of helium. ✓

(c) Helium gas is monoatomic. Find the amount of heat needed to raise the temperature of 1 kg of helium by 1 degree C. (This is known as helium's heat capacity at constant volume.) ✓

**4** Refrigerators, air conditioners, and heat pumps are heat engines that work in reverse. You put in mechanical work, and it the effect is to take heat out of a cooler reservoir and deposit heat in a warmer one:  $Q_L + W = Q_H$ . As with the heat engines discussed previously, the efficiency is defined as the energy transfer you want ( $Q_L$  for a refrigerator or air conditioner,  $Q_H$  for a heat pump) divided by the energy transfer you pay for ( $W$ ).

Efficiencies are supposed to be unitless, but the efficiency of an air conditioner is normally given in terms of an EER rating (or a more complex version called an SEER). The EER is defined as  $Q_L/W$ , but expressed in the barbaric units of of Btu/watt-hour. A typical EER rating for a residential air conditioner is about 10 Btu/watt-hour, corresponding to an efficiency of about 3. The standard temperatures used for testing an air conditioner's efficiency are  $80^\circ\text{F}$  ( $27^\circ\text{C}$ ) inside and  $95^\circ\text{F}$  ( $35^\circ\text{C}$ ) outside.

(a) What would be the EER rating of a reversed Carnot engine used as an air conditioner? ✓

(b) If you ran a 3-kW residential air conditioner, with an efficiency of 3, for one hour, what would be the effect on the total entropy of the universe? Is your answer consistent with the second law of thermodynamics? ✓

**5** (a) Estimate the pressure at the center of the Earth, assuming it is of constant density throughout. Use the technique of example 5 on page 406. Note that  $g$  is not constant with respect to depth

— it equals  $Gmr/b^3$  for  $r$ , the distance from the center, less than  $b$ , the earth's radius.<sup>1</sup> State your result in terms of  $G$ ,  $m$ , and  $b$ .

(b) Show that your answer from part a has the right units for pressure.

(c) Evaluate the result numerically. ✓

(d) Given that the earth's atmosphere is on the order of one thousandth the thickness of the earth's radius, and that the density of the earth is several thousand times greater than the density of the lower atmosphere, check that your result is of a reasonable order of magnitude.  $\int$

**6** (a) Determine the ratio between the escape velocities from the surfaces of the earth and the moon. ✓

(b) The temperature during the lunar daytime gets up to about  $130^\circ\text{C}$ . In the extremely thin (almost nonexistent) lunar atmosphere, estimate how the typical velocity of a molecule would compare with that of the same type of molecule in the earth's atmosphere. Assume that the earth's atmosphere has a temperature of  $0^\circ\text{C}$ . ✓

(c) Suppose you were to go to the moon and release some fluorocarbon gas, with molecular formula  $\text{C}_n\text{F}_{2n+2}$ . Estimate what is the smallest fluorocarbon molecule (lowest  $n$ ) whose typical velocity would be lower than that of an  $\text{N}_2$  molecule on earth in proportion to the moon's lower escape velocity. The moon would be able to retain an atmosphere made of these molecules. ✓

**7** Most of the atoms in the universe are in the form of gas that is not part of any star or galaxy: the intergalactic medium (IGM). The IGM consists of about  $10^{-5}$  atoms per cubic centimeter, with a typical temperature of about  $10^3$  K. These are, in some sense, the density and temperature of the universe (not counting light, or the exotic particles known as “dark matter”). Calculate the pressure of the universe (or, speaking more carefully, the typical pressure due to the IGM). ✓

**8** A sample of gas is enclosed in a sealed chamber. The gas consists of molecules, which are then split in half through some process such as exposure to ultraviolet light, or passing an electric spark through the gas. The gas returns to thermal equilibrium with the surrounding room. How does its pressure now compare with its pressure before the molecules were split?

**9** The figure shows a demonstration performed by Otto von Guericke for Emperor Ferdinand III, in which two teams of horses failed to pull apart a pair of hemispheres from which the air had been evacuated. (a) What object makes the force that holds the

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<sup>1</sup>Derivation: The shell theorem tells us that the gravitational field at  $r$  is the same as if all the mass existing at greater depths was concentrated at the earth's center. Since volume scales like the third power of distance, this constitutes a fraction  $(r/b)^3$  of the earth's mass, so the field is  $(Gm/r^2)(r/b)^3 = Gmr/b^3$ .



Problem 9.

hemispheres together? (b) The hemispheres are in a museum in Berlin, and have a diameter of 65 cm. What is the amount of force holding them together? (Hint: The answer would be the same if they were cylinders or pie plates rather than hemispheres.)

**10** Even when resting, the human body needs to do a certain amount of mechanical work to keep the heart beating. This quantity is difficult to define and measure with high precision, and also depends on the individual and her level of activity, but it's estimated to be about 1 to 5 watts. Suppose we consider the human body as nothing more than a pump. A person who is just lying in bed all day needs about 1000 kcal/day worth of food to stay alive. (a) Estimate the person's thermodynamic efficiency as a pump, and (b) compare with the maximum possible efficiency imposed by the laws of thermodynamics for a heat engine operating across the difference between a body temperature of  $37^{\circ}\text{C}$  and an ambient temperature of  $22^{\circ}\text{C}$ . (c) Interpret your answer. ▷ Answer, p. 535



# **Vibrations and waves**







The vibrations of this electric bass string are converted to electrical vibrations, then to sound vibrations, and finally to vibrations of our eardrums.

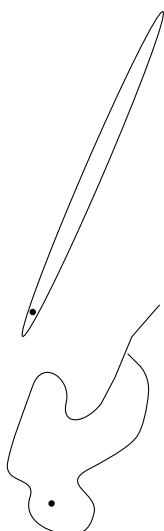
## Chapter 17

# Vibrations

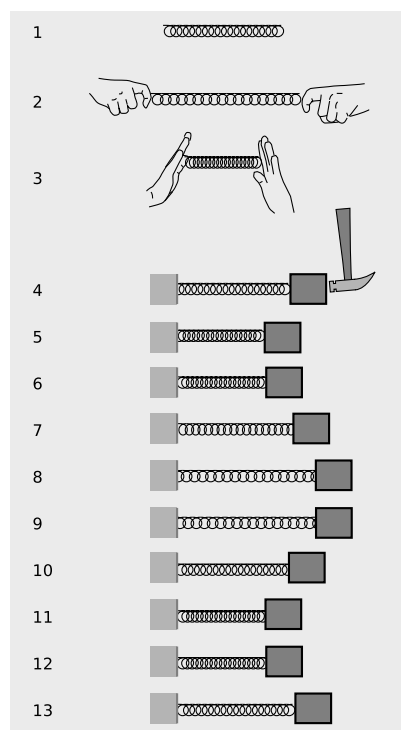
Dandelion. Cello. Read those two words, and your brain instantly conjures a stream of associations, the most prominent of which have to do with vibrations. Our mental category of “dandelion-ness” is strongly linked to the color of light waves that vibrate about half a million billion times a second: yellow. The velvety throb of a cello has as its most obvious characteristic a relatively low musical pitch — the note you are spontaneously imagining right now might be one whose sound vibrations repeat at a rate of a hundred times a second.

Evolution has designed our two most important senses around the assumption that not only will our environment be drenched with information-bearing vibrations, but in addition those vibrations will often be repetitive, so that we can judge colors and pitches by the rate of repetition. Granting that we do sometimes encounter non-repeating waves such as the consonant “sh,” which has no recognizable pitch, why was Nature’s assumption of repetition nevertheless so right in general?

Repeating phenomena occur throughout nature, from the orbits of electrons in atoms to the reappearance of Halley’s Comet every 75 years. Ancient cultures tended to attribute repetitious phenomena



a / If we try to draw a non-repeating orbit for Halley's Comet, it will inevitably end up crossing itself.



b / A spring has an equilibrium length, 1, and can be stretched, 2, or compressed, 3. A mass attached to the spring can be set into motion initially, 4, and will then vibrate, 4-13.

like the seasons to the cyclical nature of time itself, but we now have a less mystical explanation. Suppose that instead of Halley's Comet's true, repeating elliptical orbit that closes seamlessly upon itself with each revolution, we decide to take a pen and draw a whimsical alternative path that never repeats. We will not be able to draw for very long without having the path cross itself. But at such a crossing point, the comet has returned to a place it visited once before, and since its potential energy is the same as it was on the last visit, conservation of energy proves that it must again have the same kinetic energy and therefore the same speed. Not only that, but the comet's direction of motion cannot be randomly chosen, because angular momentum must be conserved as well. Although this falls short of being an ironclad proof that the comet's orbit must repeat, it no longer seems surprising that it does.

Conservation laws, then, provide us with a good reason why repetitive motion is so prevalent in the universe. But it goes deeper than that. Up to this point in your study of physics, I have been indoctrinating you with a mechanistic vision of the universe as a giant piece of clockwork. Breaking the clockwork down into smaller and smaller bits, we end up at the atomic level, where the electrons circling the nucleus resemble — well, little clocks! From this point of view, particles of matter are the fundamental building blocks of everything, and vibrations and waves are just a couple of the tricks that groups of particles can do. But at the beginning of the 20th century, the tables were turned. A chain of discoveries initiated by Albert Einstein led to the realization that the so-called subatomic “particles” were in fact waves. In this new world-view, it is vibrations and waves that are fundamental, and the formation of matter is just one of the tricks that waves can do.

## 17.1 Period, frequency, and amplitude

Figure b shows our most basic example of a vibration. With no forces on it, the spring assumes its equilibrium length,  $b/1$ . It can be stretched, 2, or compressed, 3. We attach the spring to a wall on the left and to a mass on the right. If we now hit the mass with a hammer, 4, it oscillates as shown in the series of snapshots, 4-13. If we assume that the mass slides back and forth without friction and that the motion is one-dimensional, then conservation of energy proves that the motion must be repetitive. When the block comes back to its initial position again, 7, its potential energy is the same again, so it must have the same kinetic energy again. The motion is in the opposite direction, however. Finally, at 10, it returns to its initial position with the same kinetic energy and the same direction of motion. The motion has gone through one complete cycle, and will now repeat forever in the absence of friction.

The usual physics terminology for motion that repeats itself over

and over is periodic motion, and the time required for one repetition is called the period,  $T$ . (The symbol  $P$  is not used because of the possible confusion with momentum.) One complete repetition of the motion is called a cycle.

We are used to referring to short-period sound vibrations as “high” in pitch, and it sounds odd to have to say that high pitches have low periods. It is therefore more common to discuss the rapidity of a vibration in terms of the number of vibrations per second, a quantity called the frequency,  $f$ . Since the period is the number of seconds per cycle and the frequency is the number of cycles per second, they are reciprocals of each other,

$$f = 1/T \quad .$$

#### *A carnival game*

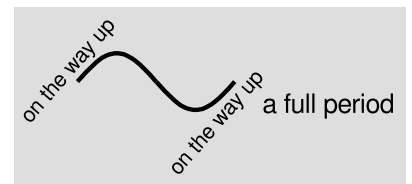
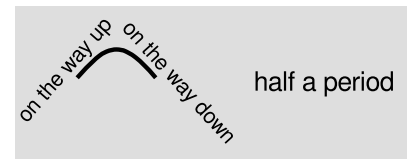
#### *example 1*

In the carnival game shown in figure d, the rube is supposed to push the bowling ball on the track just hard enough so that it goes over the hump and into the valley, but does not come back out again. If the only types of energy involved are kinetic and potential, this is impossible. Suppose you expect the ball to come back to a point such as the one shown with the dashed outline, then stop and turn around. It would already have passed through this point once before, going to the left on its way into the valley. It was moving then, so conservation of energy tells us that it cannot be at rest when it comes back to the same point. The motion that the customer hopes for is physically impossible. There is a physically possible periodic motion in which the ball rolls back and forth, staying confined within the valley, but there is no way to get the ball into that motion beginning from the place where we start. There is a way to beat the game, though. If you put enough spin on the ball, you can create enough kinetic friction so that a significant amount of heat is generated. Conservation of energy then allows the ball to be at rest when it comes back to a point like the outlined one, because kinetic energy has been converted into heat.

#### *Period and frequency of a fly's wing-beats*

#### *example 2*

A Victorian parlor trick was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and announce how many times the fly's wings had flapped in one second. If the fly's wings flap, say, 200 times in one second, then the frequency of their motion is  $f = 200/1 \text{ s} = 200 \text{ s}^{-1}$ . The period is one 200th of a second,  $T = 1/f = (1/200) \text{ s} = 0.005 \text{ s}$ .



c / Position-versus-time graphs for half a period and a full period.



d / Example 1.

Units of inverse second,  $s^{-1}$ , are awkward in speech, so an abbreviation has been created. One Hertz, named in honor of a pioneer of radio technology, is one cycle per second. In abbreviated form,  $1 \text{ Hz} = 1 \text{ s}^{-1}$ . This is the familiar unit used for the frequencies on the radio dial.

### Frequency of a radio station

example 3

▷ KKJZ's frequency is 88.1 MHz. What does this mean, and what period does this correspond to?

▷ The metric prefix M- is mega-, i.e., millions. The radio waves emitted by KKJZ's transmitting antenna vibrate 88.1 million times per second. This corresponds to a period of

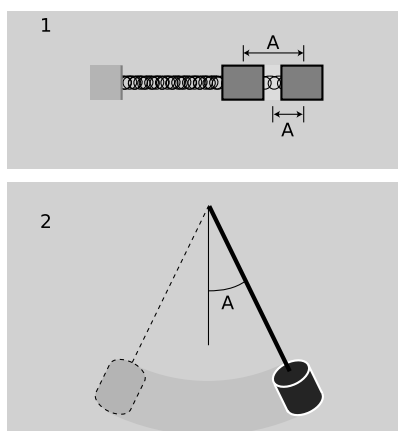
$$T = 1/f = 1.14 \times 10^{-8} \text{ s} \quad .$$

This example shows a second reason why we normally speak in terms of frequency rather than period: it would be painful to have to refer to such small time intervals routinely. I could abbreviate by telling people that KKJZ's period was 11.4 nanoseconds, but most people are more familiar with the big metric prefixes than with the small ones.

Units of frequency are also commonly used to specify the speeds of computers. The idea is that all the little circuits on a computer chip are synchronized by the very fast ticks of an electronic clock, so that the circuits can all cooperate on a task without getting ahead or behind. Adding two numbers might require, say, 30 clock cycles. Microcomputers these days operate at clock frequencies of about a gigahertz.

We have discussed how to measure how fast something vibrates, but not how big the vibrations are. The general term for this is amplitude,  $A$ . The definition of amplitude depends on the system being discussed, and two people discussing the same system may not even use the same definition. In the example of the block on the end of the spring, e/1, the amplitude will be measured in distance units such as cm. One could work in terms of the distance traveled by the block from the extreme left to the extreme right, but it would be somewhat more common in physics to use the distance from the center to one extreme. The former is usually referred to as the peak-to-peak amplitude, since the extremes of the motion look like mountain peaks or upside-down mountain peaks on a graph of position versus time.

In other situations we would not even use the same units for amplitude. The amplitude of a child on a swing, or a pendulum, e/2, would most conveniently be measured as an angle, not a distance, since her feet will move a greater distance than her head. The electrical vibrations in a radio receiver would be measured in electrical units such as volts or amperes.



e/1. The amplitude of the vibrations of the mass on a spring could be defined in two different ways. It would have units of distance. 2. The amplitude of a swinging pendulum would more naturally be defined as an angle.

## 17.2 Simple harmonic motion

### Why are sine-wave vibrations so common?

If we actually construct the mass-on-a-spring system discussed in the previous section and measure its motion accurately, we will find that its  $x-t$  graph is nearly a perfect sine-wave shape, as shown in figure f/1. (We call it a “sine wave” or “sinusoidal” even if it is a cosine, or a sine or cosine shifted by some arbitrary horizontal amount.) It may not be surprising that it is a wiggle of this general sort, but why is it a specific mathematically perfect shape? Why is it not a sawtooth shape like 2 or some other shape like 3? The mystery deepens as we find that a vast number of apparently unrelated vibrating systems show the same mathematical feature. A tuning fork, a sapling pulled to one side and released, a car bouncing on its shock absorbers, all these systems will exhibit sine-wave motion under one condition: the amplitude of the motion must be small.

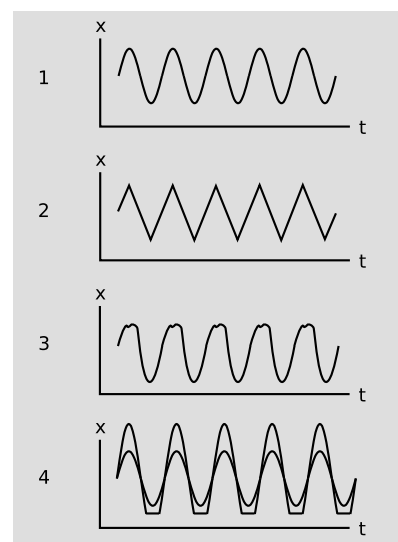
It is not hard to see intuitively why extremes of amplitude would act differently. For example, a car that is bouncing lightly on its shock absorbers may behave smoothly, but if we try to double the amplitude of the vibrations the bottom of the car may begin hitting the ground, f/4. (Although we are assuming for simplicity in this chapter that energy is never dissipated, this is clearly not a very realistic assumption in this example. Each time the car hits the ground it will convert quite a bit of its potential and kinetic energy into heat and sound, so the vibrations would actually die out quite quickly, rather than repeating for many cycles as shown in the figure.)

The key to understanding how an object vibrates is to know how the force on the object depends on the object’s position. If an object is vibrating to the right and left, then it must have a leftward force on it when it is on the right side, and a rightward force when it is on the left side. In one dimension, we can represent the direction of the force using a positive or negative sign, and since the force changes from positive to negative there must be a point in the middle where the force is zero. This is the equilibrium point, where the object would stay at rest if it was released at rest. For convenience of notation throughout this chapter, we will define the origin of our coordinate system so that  $x$  equals zero at equilibrium.

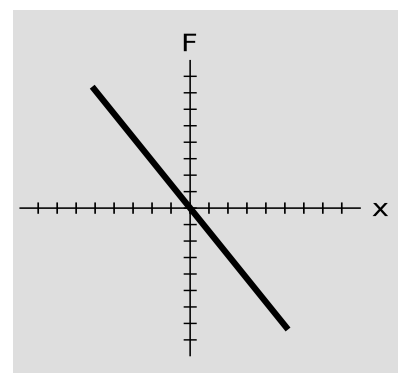
The simplest example is the mass on a spring, for which the force on the mass is given by Hooke’s law,

$$F = -kx \quad .$$

We can visualize the behavior of this force using a graph of  $F$  versus  $x$ , as shown in figure g. The graph is a line, and the spring constant,  $k$ , is equal to minus its slope. A stiffer spring has a larger value of  $k$  and a steeper slope. Hooke’s law is only an approximation, but



f / Sinusoidal and non-sinusoidal vibrations.



g / The force exerted by an ideal spring, which behaves exactly according to Hooke’s law.



it works very well for most springs in real life, as long as the spring isn't compressed or stretched so much that it is permanently bent or damaged.

The following important theorem, whose proof is given in optional section 17.3, relates the motion graph to the force graph.

**Theorem:** A linear force graph makes a sinusoidal motion graph.

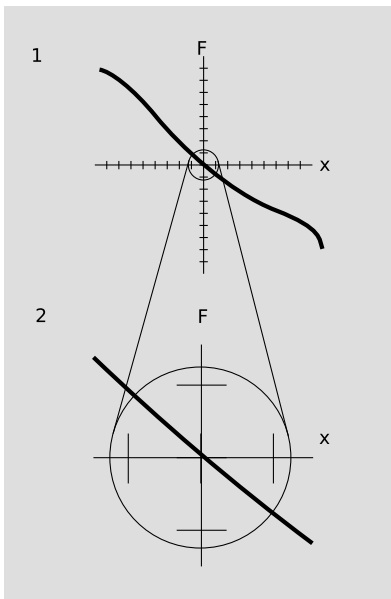
If the total force on a vibrating object depends only on the object's position, and is related to the object's displacement from equilibrium by an equation of the form  $F = -kx$ , then the object's motion displays a sinusoidal graph with period  $T = 2\pi\sqrt{m/k}$ .

Even if you do not read the proof, it is not too hard to understand why the equation for the period makes sense. A greater mass causes a greater period, since the force will not be able to whip a massive object back and forth very rapidly. A larger value of  $k$  causes a shorter period, because a stronger force can whip the object back and forth more rapidly.

This may seem like only an obscure theorem about the mass-on-a-spring system, but figure h shows it to be far more general than that. Figure h/1 depicts a force curve that is not a straight line. A system with this  $F - x$  curve would have large-amplitude vibrations that were complex and not sinusoidal. But the same system would exhibit sinusoidal small-amplitude vibrations. This is because any curve looks linear from very close up. If we magnify the  $F - x$  graph as shown in figure h/2, it becomes very difficult to tell that the graph is not a straight line. If the vibrations were confined to the region shown in h/2, they would be very nearly sinusoidal. This is the reason why sinusoidal vibrations are a universal feature of all vibrating systems, if we restrict ourselves to small amplitudes. The theorem is therefore of great general significance. It applies throughout the universe, to objects ranging from vibrating stars to vibrating nuclei. A sinusoidal vibration is known as simple harmonic motion.

**Period is approximately independent of amplitude, if the amplitude is small.**

Until now we have not even mentioned the most counterintuitive aspect of the equation  $T = 2\pi\sqrt{m/k}$ : it does not depend on amplitude at all. Intuitively, most people would expect the mass-on-a-spring system to take longer to complete a cycle if the amplitude was larger. (We are comparing amplitudes that are different from each other, but both small enough that the theorem applies.) In fact the larger-amplitude vibrations take the same amount of time as the small-amplitude ones. This is because at large amplitudes, the force is greater, and therefore accelerates the object to higher



h / Seen from close up, any  $F - x$  curve looks like a line.

speeds.

Legend has it that this fact was first noticed by Galileo during what was apparently a less than enthralling church service. A gust of wind would now and then start one of the chandeliers in the cathedral swaying back and forth, and he noticed that regardless of the amplitude of the vibrations, the period of oscillation seemed to be the same. Up until that time, he had been carrying out his physics experiments with such crude time-measuring techniques as feeling his own pulse or singing a tune to keep a musical beat. But after going home and testing a pendulum, he convinced himself that he had found a superior method of measuring time. Even without a fancy system of pulleys to keep the pendulum's vibrations from dying down, he could get very accurate time measurements, because the gradual decrease in amplitude due to friction would have no effect on the pendulum's period. (Galileo never produced a modern-style pendulum clock with pulleys, a minute hand, and a second hand, but within a generation the device had taken on the form that persisted for hundreds of years after.)

#### The pendulum

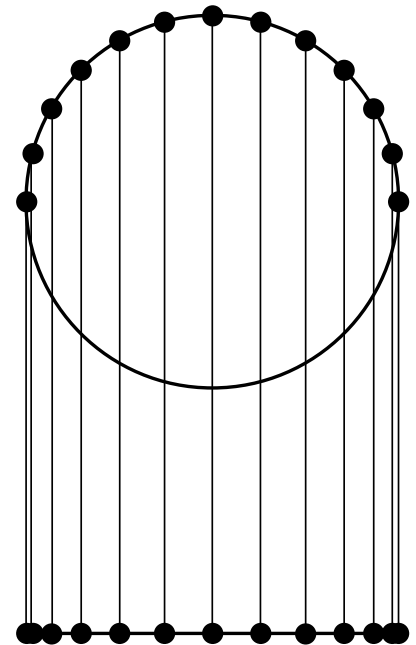
example 4

- ▷ Compare the periods of pendula having bobs with different masses.
- ▷ From the equation  $T = 2\pi\sqrt{m/k}$ , we might expect that a larger mass would lead to a longer period. However, increasing the mass also increases the forces that act on the pendulum: gravity and the tension in the string. This increases  $k$  as well as  $m$ , so the period of a pendulum is independent of  $m$ .

## 17.3 ★ Proofs

In this section we prove (1) that a linear  $F - x$  graph gives sinusoidal motion, (2) that the period of the motion is  $2\pi\sqrt{m/k}$ , and (3) that the period is independent of the amplitude. You may omit this section without losing the continuity of the chapter.

The basic idea of the proof can be understood by imagining that you are watching a child on a merry-go-round from far away. Because you are in the same horizontal plane as her motion, she appears to be moving from side to side along a line. Circular motion viewed edge-on doesn't just look like any kind of back-and-forth motion, it looks like motion with a sinusoidal  $x - t$  graph, because the sine and cosine functions can be defined as the  $x$  and  $y$  coordinates of a point at angle  $\theta$  on the unit circle. The idea of the proof, then, is to show that an object acted on by a force that varies as  $F = -kx$  has motion that is identical to circular motion projected down to



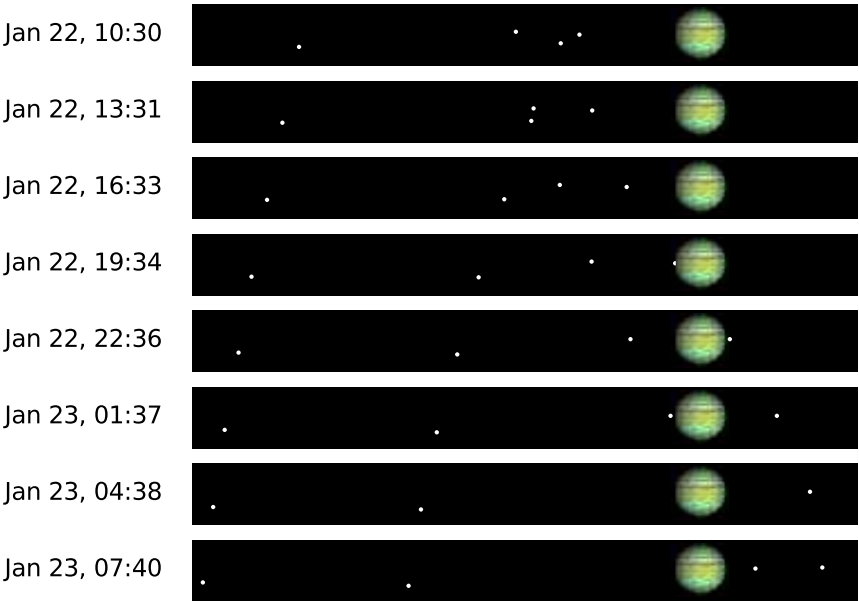
i / The object moves along the circle at constant speed, but even though its overall speed is constant, the  $x$  and  $y$  components of its velocity are continuously changing, as shown by the unequal spacing of the points when projected onto the line below. Projected onto the line, its motion is the same as that of an object experiencing a force  $F = -kx$ .

one dimension. The  $v^2/r$  expression will also fall out at the end.

*The moons of Jupiter.* example 5

Before moving on to the proof, we illustrate the concept using the moons of Jupiter. Their discovery by Galileo was an epochal event in astronomy, because it proved that not everything in the universe had to revolve around the earth as had been believed. Galileo's telescope was of poor quality by modern standards, but figure j shows a simulation of how Jupiter and its moons might appear at intervals of three hours through a large present-day instrument. Because we see the moons' circular orbits edge-on, they appear to perform sinusoidal vibrations. Over this time period, the innermost moon, Io, completes half a cycle.

j / Example 5.



For an object performing uniform circular motion, we have

$$|\mathbf{a}| = \frac{v^2}{r} \quad .$$

The  $x$  component of the acceleration is therefore

$$a_x = \frac{v^2}{r} \cos \theta \quad ,$$

where  $\theta$  is the angle measured counterclockwise from the  $x$  axis. Applying Newton's second law,

$$\begin{aligned} \frac{F_x}{m} &= -\frac{v^2}{r} \cos \theta \quad , \quad \text{so} \\ F_x &= -m \frac{v^2}{r} \cos \theta \quad . \end{aligned}$$

Since our goal is an equation involving the period, it is natural to eliminate the variable  $v = \text{circumference}/T = 2\pi r/T$ , giving

$$F_x = -\frac{4\pi^2 mr}{T^2} \cos \theta \quad .$$

The quantity  $r \cos \theta$  is the same as  $x$ , so we have

$$F_x = -\frac{4\pi^2 m}{T^2} x \quad .$$

Since everything is constant in this equation except for  $x$ , we have proved that motion with force proportional to  $x$  is the same as circular motion projected onto a line, and therefore that a force proportional to  $x$  gives sinusoidal motion. Finally, we identify the constant factor of  $4\pi^2 m/T^2$  with  $k$ , and solving for  $T$  gives the desired equation for the period,

$$T = 2\pi \sqrt{\frac{m}{k}} \quad .$$

Since this equation is independent of  $r$ ,  $T$  is independent of the amplitude, subject to the initial assumption of perfect  $F = -kx$  behavior, which in reality will only hold approximately for small  $x$ .

# Summary

## Selected vocabulary

periodic motion . . . . .	motion that repeats itself over and over
period . . . . .	the time required for one cycle of a periodic motion
frequency . . . . .	the number of cycles per second, the inverse of the period
amplitude . . . . .	the amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration
simple harmonic motion . . . . .	motion whose $x - t$ graph is a sine wave

## Notation

$T$ . . . . .	period
$f$ . . . . .	frequency
$A$ . . . . .	amplitude
$k$ . . . . .	the slope of the graph of $F$ versus $x$ , where $F$ is the total force acting on an object and $x$ is the object's position; for a spring, this is known as the spring constant.

## Other terminology and notation

$\nu$ . . . . .	The Greek letter $\nu$ , nu, is used in many books for frequency.
$\omega$ . . . . .	The Greek letter $\omega$ , omega, is often used as an abbreviation for $2\pi f$ .

## Summary

Periodic motion is common in the world around us because of conservation laws. An important example is one-dimensional motion in which the only two forms of energy involved are potential and kinetic; in such a situation, conservation of energy requires that an object repeat its motion, because otherwise when it came back to the same point, it would have to have a different kinetic energy and therefore a different total energy.

Not only are periodic vibrations very common, but small-amplitude vibrations are always sinusoidal as well. That is, the  $x - t$  graph is a sine wave. This is because the graph of force versus position will always look like a straight line on a sufficiently small scale. This type of vibration is called simple harmonic motion. In simple harmonic motion, the period is independent of the amplitude, and is given by

$$T = 2\pi\sqrt{m/k} \quad .$$

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

1 Find an equation for the frequency of simple harmonic motion in terms of  $k$  and  $m$ . ✓

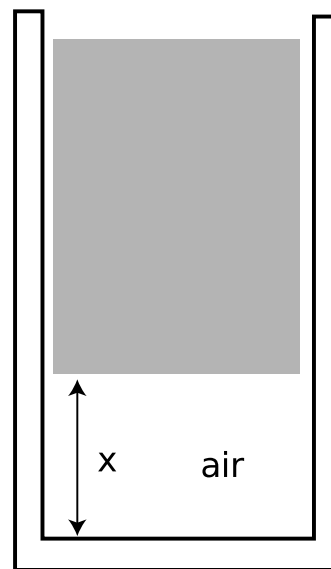
2 Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail's vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond?

3 (a) Pendulum 2 has a string twice as long as pendulum 1. If we define  $x$  as the distance traveled by the bob along a circle away from the bottom, how does the  $k$  of pendulum 2 compare with the  $k$  of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of  $x$ .]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

4 A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by  $F_{air} = ax^{-1.4}$ , where  $a$  is a constant with funny units of  $\text{N} \cdot \text{m}^{1.4}$ . For simplicity, assume the air only supports the weight,  $F_W$ , of the piston itself, although in practice this device is used to support some other object. The equilibrium position,  $x_0$ , is where  $F_W$  equals  $-F_{air}$ . (Note that in the main text I have assumed the equilibrium position to be at  $x = 0$ , but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Let  $a = 1.0 \text{ N} \cdot \text{m}^{1.4}$ ,  $x_0 = 1.00 \text{ m}$ , and  $F_W = -1.00 \text{ N}$ . The piston is released from  $x = 1.01 \text{ m}$ . Draw a neat, accurate graph of the total force,  $F$ , as a function of  $x$ , on graph paper, covering the range from  $x = 0.98 \text{ m}$  to  $1.02 \text{ m}$ . Over this small range, you will find that the force is very nearly proportional to  $x - x_0$ . Approximate the curve with a straight line, find its slope, and derive the approximate period of oscillation. ✓

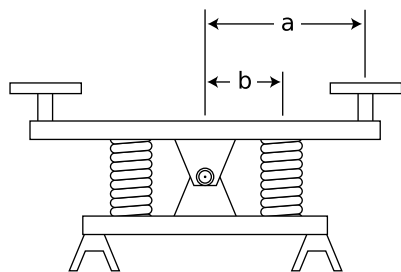
5 Consider the same pneumatic piston described in problem 4, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of  $x$  vs  $t$ , showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the



Problem 4.

curvature of the  $x - t$  graph, so if the force is greater, the graph should curve around more quickly.]

**6** Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new "chunk" of water. (a) Show that a cube of mass  $m$  with edges of length  $b$  floating upright (not tilted) in a fluid of density  $\rho$  will have a draft (depth to which it sinks below the waterline)  $h$  given at equilibrium by  $h_0 = m/b^2\rho$ . (b) Find the total force on the cube when its draft is  $h$ , and verify that plugging in  $h - h_0$  gives a total force of zero. (c) Find the cube's period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of  $g$  only.  $\checkmark$



Problem 7.

**7** The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant  $k$ , and a kid of mass  $m$  sits on each seat. (a) Find the period of vibration in terms of the variables  $k$ ,  $m$ ,  $a$ , and  $b$ . (b) Discuss the special case where  $a = b$ , rather than  $a > b$  as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of  $b = 0$ .  $\checkmark \star$

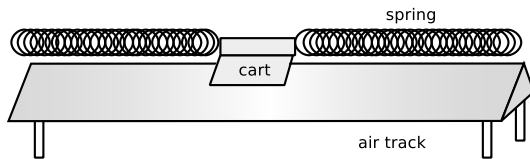
**8** Show that the equation  $T = 2\pi\sqrt{m/k}$  has units that make sense.

**9** A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826  $\text{m/s}^2$ . This change, amounting to 0.046  $\text{m/s}^2$ , is greater than the 0.022  $\text{m/s}^2$  effect to be expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator?

## Exercise 17: Vibrations

Equipment:

- air track and carts of two different masses
- springs
- spring scales

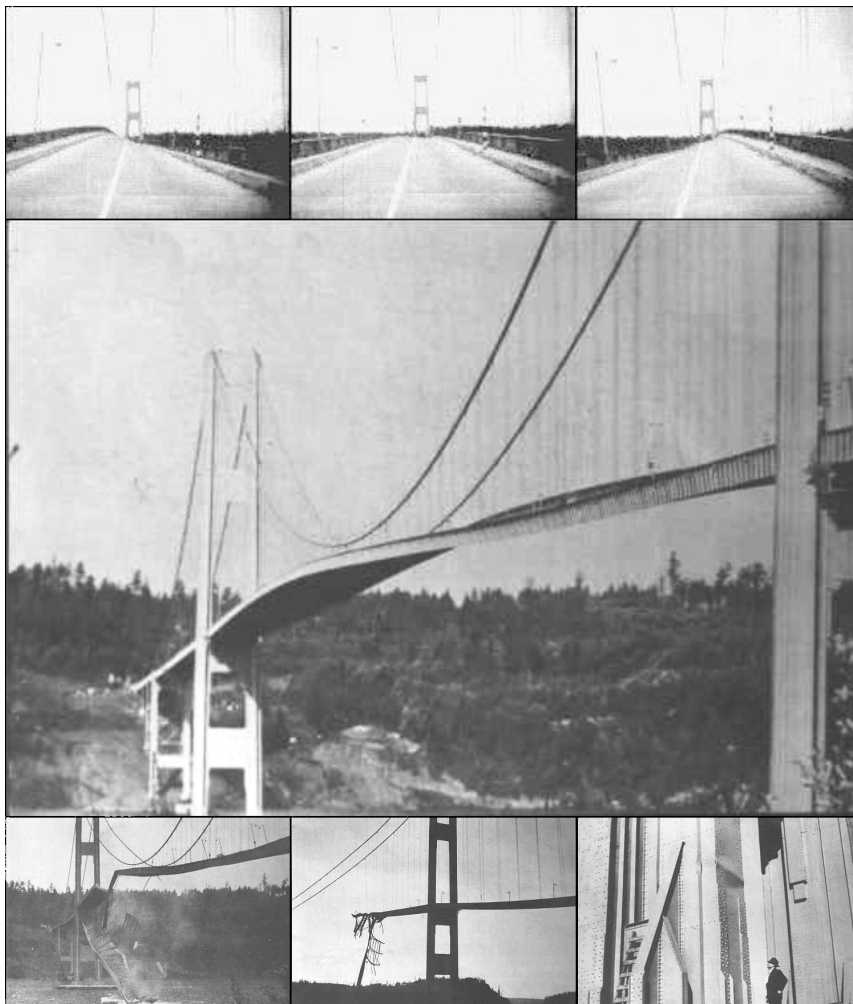


Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least one moderate amplitude, for which the springs do not go slack, at least one amplitude that is large enough so that they do go slack, and one amplitude that's the very smallest you can possibly observe.
2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation  $T = 2\pi\sqrt{m/k}$ ?
3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?
4. Test the equation  $T = 2\pi\sqrt{m/k}$  numerically.







*Top:* A series of images from a film of the Tacoma Narrows Bridge vibrating on the day it was to collapse. *Middle:* The bridge immediately before the collapse, with the sides vibrating 8.5 meters (28 feet) up and down. Note that the bridge is over a mile long. *Bottom:* During and after the final collapse. The right-hand picture gives a sense of the massive scale of the construction.

## Chapter 18

# Resonance

Soon after the mile-long Tacoma Narrows Bridge opened in July 1940, motorists began to notice its tendency to vibrate frighteningly in even a moderate wind. Nicknamed “Galloping Gertie,” the bridge collapsed in a steady 42-mile-per-hour wind on November 7 of the same year. The following is an eyewitness report from a newspaper editor who found himself on the bridge as the vibrations approached the breaking point.

“Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car... I jammed on the brakes and

got out, only to be thrown onto my face against the curb.

“Around me I could hear concrete cracking. I started to get my dog Tubby, but was thrown again before I could reach the car. The car itself began to slide from side to side of the roadway.

“On hands and knees most of the time, I crawled 500 yards or more to the towers... My breath was coming in gasps; my knees were raw and bleeding, my hands bruised and swollen from gripping the concrete curb... Toward the last, I risked rising to my feet and running a few yards at a time... Safely back at the toll plaza, I saw the bridge in its final collapse and saw my car plunge into the Narrows.”

The ruins of the bridge formed an artificial reef, one of the world’s largest. It was not replaced for ten years. The reason for its collapse was not substandard materials or construction, nor was the bridge under-designed: the piers were hundred-foot blocks of concrete, the girders massive and made of carbon steel. The bridge was destroyed because of the physical phenomenon of resonance, the same effect that allows an opera singer to break a wine glass with her voice and that lets you tune in the radio station you want. The replacement bridge, which has lasted half a century so far, was built smarter, not stronger. The engineers learned their lesson and simply included some slight modifications to avoid the resonance phenomenon that spelled the doom of the first one.

## 18.1 Energy in vibrations

One way of describing the collapse of the bridge is that the bridge kept taking energy from the steadily blowing wind and building up more and more energetic vibrations. In this section, we discuss the energy contained in a vibration, and in the subsequent sections we will move on to the loss of energy and the adding of energy to a vibrating system, all with the goal of understanding the important phenomenon of resonance.

Going back to our standard example of a mass on a spring, we find that there are two forms of energy involved: the potential energy stored in the spring and the kinetic energy of the moving mass. We may start the system in motion either by hitting the mass to put in kinetic energy by pulling it to one side to put in potential energy. Either way, the subsequent behavior of the system is identical. It trades energy back and forth between kinetic and potential energy. (We are still assuming there is no friction, so that no energy is converted to heat, and the system never runs down.)

The most important thing to understand about the energy content of vibrations is that the total energy is proportional to the

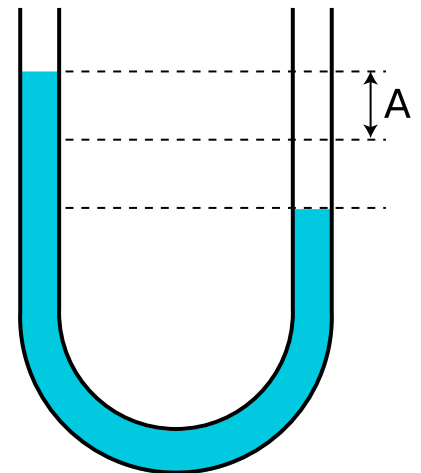
square of the amplitude. Although the total energy is constant, it is instructive to consider two specific moments in the motion of the mass on a spring as examples. When the mass is all the way to one side, at rest and ready to reverse directions, all its energy is potential. We have already seen that the potential energy stored in a spring equals  $(1/2)kx^2$ , so the energy is proportional to the square of the amplitude. Now consider the moment when the mass is passing through the equilibrium point at  $x = 0$ . At this point it has no potential energy, but it does have kinetic energy. The velocity is proportional to the amplitude of the motion, and the kinetic energy,  $(1/2)mv^2$ , is proportional to the square of the velocity, so again we find that the energy is proportional to the square of the amplitude. The reason for singling out these two points is merely instructive; proving that energy is proportional to  $A^2$  at any point would suffice to prove that energy is proportional to  $A^2$  in general, since the energy is constant.

Are these conclusions restricted to the mass-on-a-spring example? No. We have already seen that  $F = -kx$  is a valid approximation for any vibrating object, as long as the amplitude is small. We are thus left with a very general conclusion: the energy of any vibration is approximately proportional to the square of the amplitude, provided that the amplitude is small.

#### *Water in a U-tube*

#### *example 1*

If water is poured into a U-shaped tube as shown in the figure, it can undergo vibrations about equilibrium. The energy of such a vibration is most easily calculated by considering the “turnaround point” when the water has stopped and is about to reverse directions. At this point, it has only potential energy and no kinetic energy, so by calculating its potential energy we can find the energy of the vibration. This potential energy is the same as the work that would have to be done to take the water out of the right-hand side down to a depth  $A$  below the equilibrium level, raise it through a height  $A$ , and place it in the left-hand side. The weight of this chunk of water is proportional to  $A$ , and so is the height through which it must be lifted, so the energy is proportional to  $A^2$ .



a / Example 1.

#### *The range of energies of sound waves*

#### *example 2*

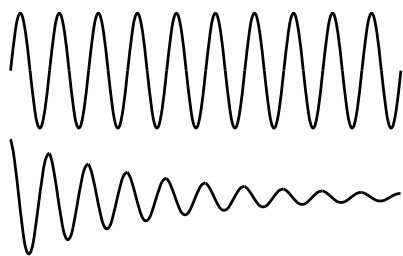
- ▷ The amplitude of vibration of your eardrum at the threshold of pain is about  $10^6$  times greater than the amplitude with which it vibrates in response to the softest sound you can hear. How many times greater is the energy with which your ear has to cope for the painfully loud sound, compared to the soft sound?
- ▷ The amplitude is  $10^6$  times greater, and energy is proportional to the square of the amplitude, so the energy is greater by a factor

of  $10^{12}$ . This is a phenomenally large factor!

We are only studying vibrations right now, not waves, so we are not yet concerned with how a sound wave works, or how the energy gets to us through the air. Note that because of the huge range of energies that our ear can sense, it would not be reasonable to have a sense of loudness that was additive. Consider, for instance, the following three levels of sound:

barely audible wind	
quiet conversation . . . .	$10^5$ times more energy than the wind
heavy metal concert . .	$10^{12}$ times more energy than the wind

In terms of addition and subtraction, the difference between the wind and the quiet conversation is nothing compared to the difference between the quiet conversation and the heavy metal concert. Evolution wanted our sense of hearing to be able to encompass all these sounds without collapsing the bottom of the scale so that anything softer than the crack of doom would sound the same. So rather than making our sense of loudness additive, mother nature made it multiplicative. We sense the difference between the wind and the quiet conversation as spanning a range of about  $5/12$  as much as the whole range from the wind to the heavy metal concert. Although a detailed discussion of the decibel scale is not relevant here, the basic point to note about the decibel scale is that it is logarithmic. The zero of the decibel scale is close to the lower limit of human hearing, and adding 1 unit to the decibel measurement corresponds to *multiplying* the energy level (or actually the power per unit area) by a certain factor.



b / Friction has the effect of pinching the  $x - t$  graph of a vibrating object.

## 18.2 Energy lost from vibrations

Until now, we have been making the relatively unrealistic assumption that a vibration would never die out. For a realistic mass on a spring, there will be friction, and the kinetic and potential energy of the vibrations will therefore be gradually converted into heat. Similarly, a guitar string will slowly convert its kinetic and potential energy into sound. In all cases, the effect is to “pinch” the sinusoidal  $x - t$  graph more and more with passing time. Friction is not necessarily bad in this context — a musical instrument that never got rid of any of its energy would be completely silent! The dissipation of the energy in a vibration is known as damping.

### *self-check A*

Most people who try to draw graphs like those shown on the left will tend to shrink their wiggles horizontally as well as vertically. Why is this wrong?

▷ Answer, p. 534

In the graphs in figure b, I have not shown any point at which

the damped vibration finally stops completely. Is this realistic? Yes and no. If energy is being lost due to friction between two solid surfaces, then we expect the force of friction to be nearly independent of velocity. This constant friction force puts an upper limit on the total distance that the vibrating object can ever travel without replenishing its energy, since work equals force times distance, and the object must stop doing work when its energy is all converted into heat. (The friction force does reverse directions when the object turns around, but reversing the direction of the motion at the same time that we reverse the direction of the force makes it certain that the object is always doing positive work, not negative work.)

Damping due to a constant friction force is not the only possibility however, or even the most common one. A pendulum may be damped mainly by air friction, which is approximately proportional to  $v^2$ , while other systems may exhibit friction forces that are proportional to  $v$ . It turns out that friction proportional to  $v$  is the simplest case to analyze mathematically, and anyhow all the important physical insights can be gained by studying this case.

If the friction force is proportional to  $v$ , then as the vibrations die down, the frictional forces get weaker due to the lower speeds. The less energy is left in the system, the more miserly the system becomes with giving away any more energy. Under these conditions, the vibrations theoretically never die out completely, and mathematically, the loss of energy from the system is exponential: the system loses a fixed percentage of its energy per cycle. This is referred to as exponential decay.

A non-rigorous proof is as follows. The force of friction is proportional to  $v$ , and  $v$  is proportional to how far the objects travels in one cycle, so the frictional force is proportional to amplitude. The amount of work done by friction is proportional to the force and to the distance traveled, so the work done in one cycle is proportional to the square of the amplitude. Since both the work and the energy are proportional to  $A^2$ , the amount of energy taken away by friction in one cycle is a fixed percentage of the amount of energy the system has.

#### *self-check B*

Figure c shows an  $x$ - $t$  graph for a strongly damped vibration, which loses half of its amplitude with every cycle. What fraction of the energy is lost in each cycle?

▷ Answer, p. 534

It is customary to describe the amount of damping with a quantity called the quality factor,  $Q$ , defined as the number of cycles required for the energy to fall off by a factor of 535. (The origin of this obscure numerical factor is  $e^{2\pi}$ , where  $e = 2.71828\dots$  is the base of natural logarithms. Choosing this particular number causes some of our later equations to come out nice and simple.) The terminology arises from the fact that friction is often considered a bad



c / The amplitude is halved with each cycle.

thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

#### *Exponential decay in a trumpet*

*example 3*

▷ The vibrations of the air column inside a trumpet have a  $Q$  of about 10. This means that even after the trumpet player stops blowing, the note will keep sounding for a short time. If the player suddenly stops blowing, how will the sound intensity 20 cycles later compare with the sound intensity while she was still blowing?

▷ The trumpet's  $Q$  is 10, so after 10 cycles the energy will have fallen off by a factor of 535. After another 10 cycles we lose another factor of 535, so the sound intensity is reduced by a factor of  $535 \times 535 = 2.9 \times 10^5$ .

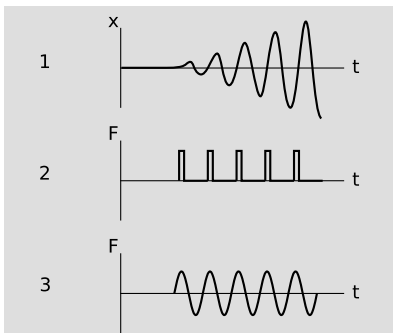
The decay of a musical sound is part of what gives it its character, and a good musical instrument should have the right  $Q$ , but the  $Q$  that is considered desirable is different for different instruments. A guitar is meant to keep on sounding for a long time after a string has been plucked, and might have a  $Q$  of 1000 or 10000. One of the reasons why a cheap synthesizer sounds so bad is that the sound suddenly cuts off after a key is released.

#### *Q of a stereo speaker*

*example 4*

Stereo speakers are not supposed to reverberate or “ring” after an electrical signal that stops suddenly. After all, the recorded music was made by musicians who knew how to shape the decays of their notes correctly. Adding a longer “tail” on every note would make it sound wrong. We therefore expect that stereo speaker will have a very low  $Q$ , and indeed, most speakers are designed with a  $Q$  of about 1. (Low-quality speakers with larger  $Q$  values are referred to as “boomy.”)

We will see later in the chapter that there are other reasons why a speaker should not have a high  $Q$ .



d / 1. Pushing a child on a swing gradually puts more and more energy into her vibrations.  
 2. A fairly realistic graph of the driving force acting on the child.  
 3. A less realistic, but more mathematically simple, driving force.

## 18.3 Putting energy into vibrations

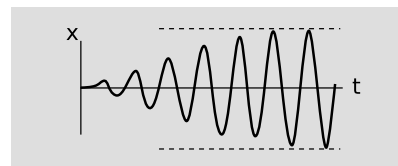
When pushing a child on a swing, you cannot just apply a constant force. A constant force will move the swing out to a certain angle, but will not allow the swing to start swinging. Nor can you give short pushes at randomly chosen times. That type of random pushing would increase the child's kinetic energy whenever you happened to be pushing in the same direction as her motion, but it would reduce her energy when your pushing happened to be in the opposite direction compared to her motion. To make her build up her energy, you need to make your pushes rhythmic, pushing at the same point in each cycle. In other words, your force needs to form a repeating pattern with the same frequency as the normal frequency

of vibration of the swing. Graph d/1 shows what the child's  $x - t$  graph would look like as you gradually put more and more energy into her vibrations. A graph of your *force* versus time would probably look something like graph 2. It turns out, however, that it is much simpler mathematically to consider a vibration with energy being pumped into it by a driving force that is itself a sine-wave, 3. A good example of this is your eardrum being driven by the force of a sound wave.

Now we know realistically that the child on the swing will not keep increasing her energy forever, nor does your eardrum end up exploding because a continuing sound wave keeps pumping more and more energy into it. In any realistic system, there is energy going out as well as in. As the vibrations increase in amplitude, there is an increase in the amount of energy taken away by damping with each cycle. This occurs for two reasons. Work equals force times distance (or, more accurately, the area under the force-distance curve). As the amplitude of the vibrations increases, the damping force is being applied over a longer distance. Furthermore, the damping force usually increases with velocity (we usually assume for simplicity that it is proportional to velocity), and this also serves to increase the rate at which damping forces remove energy as the amplitude increases. Eventually (and small children and our eardrums are thankful for this!), the amplitude approaches a maximum value,  $e$ , at which energy is removed by the damping force just as quickly as it is being put in by the driving force.

This process of approaching a maximum amplitude happens extremely quickly in many cases, e.g., the ear or a radio receiver, and we don't even notice that it took a millisecond or a microsecond for the vibrations to "build up steam." We are therefore mainly interested in predicting the behavior of the system once it has had enough time to reach essentially its maximum amplitude. This is known as the steady-state behavior of a vibrating system.

Now comes the interesting part: what happens if the frequency of the driving force is mismatched to the frequency at which the system would naturally vibrate on its own? We all know that a radio station doesn't have to be tuned in exactly, although there is only a small range over which a given station can be received. The designers of the radio had to make the range fairly small to make it possible eliminate unwanted stations that happened to be nearby in frequency, but it couldn't be too small or you wouldn't be able to adjust the knob accurately enough. (Even a digital radio can be tuned to 88.0 MHz and still bring in a station at 88.1 MHz.) The ear also has some natural frequency of vibration, but in this case the range of frequencies to which it can respond is quite broad. Evolution has made the ear's frequency response as broad as possible because it was to our ancestors' advantage to be able to hear everything from a low roars to a high-pitched shriek.



e / The amplitude approaches a maximum.



The remainder of this section develops four important facts about the response of a system to a driving force whose frequency is not necessarily the same as the system's natural frequency of vibration. The style is approximate and intuitive, but proofs are given in section 18.4.

First, although we know the ear has a frequency — about 4000 Hz — at which it would vibrate naturally, it does not vibrate at 4000 Hz in response to a low-pitched 200 Hz tone. It always responds at the frequency at which it is driven. Otherwise all pitches would sound like 4000 Hz to us. This is a general fact about driven vibrations:

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

Now let's think about the amplitude of the steady-state response. Imagine that a child on a swing has a natural frequency of vibration of 1 Hz, but we are going to try to make her swing back and forth at 3 Hz. We intuitively realize that quite a large force would be needed to achieve an amplitude of even 30 cm, i.e., the amplitude is less in proportion to the force. When we push at the natural frequency of 1 Hz, we are essentially just pumping energy back into the system to compensate for the loss of energy due to the damping (friction) force. At 3 Hz, however, we are not just counteracting friction. We are also providing an extra force to make the child's momentum reverse itself more rapidly than it would if gravity and the tension in the chain were the only forces acting. It is as if we are artificially increasing the  $k$  of the swing, but this is wasted effort because we spend just as much time decelerating the child (taking energy out of the system) as accelerating her (putting energy in).

Now imagine the case in which we drive the child at a very low frequency, say 0.02 Hz or about one vibration per minute. We are essentially just holding the child in position while very slowly walking back and forth. Again we intuitively recognize that the amplitude will be very small in proportion to our driving force. Imagine how hard it would be to hold the child at our own head-level when she is at the end of her swing! As in the too-fast 3 Hz case, we are spending most of our effort in artificially changing the  $k$  of the swing, but now rather than reinforcing the gravity and tension forces we are working against them, effectively reducing  $k$ . Only a very small part of our force goes into counteracting friction, and the rest is used in repetitively putting potential energy in on the upswing and taking it back out on the downswing, without any long-term gain.

We can now generalize to make the following statement, which

is true for all driven vibrations:

(2) A vibrating system resonates at its own natural frequency.<sup>1</sup> That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

*An opera singer breaking a wine glass* *example 5*

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back.

*Collapse of the Nimitz Freeway in an earthquake* *example 6*

I led off the chapter with the dramatic collapse of the Tacoma Narrows Bridge, mainly because it was well documented by a local physics professor, and an unknown person made a movie of the collapse. The collapse of a section of the Nimitz Freeway in Oakland, CA, during a 1989 earthquake is however a simpler example to analyze.

An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

Now all the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer's resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

*Collapse of the Tacoma Narrows Bridge* *example 7*

Let's now examine the more conceptually difficult case of the



f / The collapsed section of the Nimitz Freeway.

Tacoma Narrows Bridge. The surprise here is that the wind was steady. If the wind was blowing at constant velocity, why did it shake the bridge back and forth? The answer is a little complicated. Based on film footage and after-the-fact wind tunnel experiments, it appears that two different mechanisms were involved.

The first mechanism was the one responsible for the initial, relatively weak vibrations, and it involved resonance. As the wind moved over the bridge, it began acting like a kite or an airplane wing. As shown in the figure, it established swirling patterns of air flow around itself, of the kind that you can see in a moving cloud of smoke. As one of these swirls moved off of the bridge, there was an abrupt change in air pressure, which resulted in an up or down force on the bridge. We see something similar when a flag flaps in the wind, except that the flag's surface is usually vertical. This back-and-forth sequence of forces is exactly the kind of periodic driving force that would excite a resonance. The faster the wind, the more quickly the swirls would get across the bridge, and the higher the frequency of the driving force would be. At just the right velocity, the frequency would be the right one to excite the resonance. The wind-tunnel models, however, show that the pattern of vibration of the bridge excited by this mechanism would have been a different one than the one that finally destroyed the bridge.

The bridge was probably destroyed by a different mechanism, in which its vibrations at its own natural frequency of 0.2 Hz set up an alternating pattern of wind gusts in the air immediately around it, which then increased the amplitude of the bridge's vibrations. This vicious cycle fed upon itself, increasing the amplitude of the vibrations until the bridge finally collapsed.

As long as we're on the subject of collapsing bridges, it is worth bringing up the reports of bridges falling down when soldiers marching over them happened to step in rhythm with the bridge's natural frequency of oscillation. This is supposed to have happened in 1831 in Manchester, England, and again in 1849 in Anjou, France. Many modern engineers and scientists, however, are suspicious of the analysis of these reports. It is possible that the collapses had more to do with poor construction and overloading than with resonance. The Nimitz Freeway and Tacoma Narrows Bridge are far better documented, and occurred in an era when engineers' abilities to analyze the vibrations of a complex structure were much more advanced.

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*Emission and absorption of light waves by atoms*      example 8

In a very thin gas, the atoms are sufficiently far apart that they can act as individual vibrating systems. Although the vibrations are of a very strange and abstract type described by the theory of quantum mechanics, they nevertheless obey the same basic rules as ordinary mechanical vibrations. When a thin gas made of a cer-

tain element is heated, it emits light waves with certain specific frequencies, which are like a fingerprint of that element. As with all other vibrations, these atomic vibrations respond most strongly to a driving force that matches their own natural frequency. Thus if we have a relatively cold gas with light waves of various frequencies passing through it, the gas will absorb light at precisely those frequencies at which it would emit light if heated.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to  $Q$ .

This is fairly intuitive. The steady-state behavior is an equilibrium between energy input from the driving force and energy loss due to damping. A low- $Q$  oscillator, i.e., one with strong damping, dumps its energy faster, resulting in lower-amplitude steady-state motion.

*self-check C*

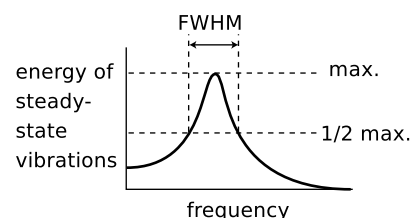
If an opera singer is shopping for a wine glass that she can impress her friends by breaking, what should she look for? ▷ Answer, p. 534

*Piano strings ringing in sympathy with a sung note* example 9

▷ A sufficiently loud musical note sung near a piano with the lid raised can cause the corresponding strings in the piano to vibrate. (A piano has a set of three strings for each note, all struck by the same hammer.) Why would this trick be unlikely to work with a violin?

▷ If you have heard the sound of a violin being plucked (the pizzicato effect), you know that the note dies away very quickly. In other words, a violin's  $Q$  is much lower than a piano's. This means that its resonances are much weaker in amplitude.

Our fourth and final fact about resonance is perhaps the most surprising. It gives us a way to determine numerically how wide a range of driving frequencies will produce a strong response. As shown in the graph, resonances do not suddenly fall off to zero outside a certain frequency range. It is usual to describe the width of a resonance by its full width at half-maximum (FWHM) as illustrated in figure g.



g / The definition of the full width at half maximum.

(4) The FWHM of a resonance is related to its  $Q$  and its resonant frequency  $f_{res}$  by the equation

$$\text{FWHM} = \frac{f_{res}}{Q} .$$

(This equation is only a good approximation when  $Q$  is large.)

Why? It is not immediately obvious that there should be any logical relationship between  $Q$  and the FWHM. Here's the idea. As we have seen already, the reason why the response of an oscillator is smaller away from resonance is that much of the driving force is being used to make the system act as if it had a different  $k$ . Roughly speaking, the half-maximum points on the graph correspond to the places where the amount of the driving force being wasted in this way is the same as the amount of driving force being used productively to replace the energy being dumped out by the damping force. If the damping force is strong, then a large amount of force is needed to counteract it, and we can waste quite a bit of driving force on changing  $k$  before it becomes comparable to the damping force. If, on the other hand, the damping force is weak, then even a small amount of force being wasted on changing  $k$  will become significant in proportion, and we cannot get very far from the resonant frequency before the two are comparable.

---

*Changing the pitch of a wind instrument* *example 10*

▷ A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the  $Q$  of a saxophone?

▷ Five percent is the width on one side of the resonance, so the full width is about 10%,  $\text{FWHM} / f_{\text{res}} = 0.1$ . This implies a  $Q$  of about 10, i.e., once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that has a low  $Q$ , so that they can produce the bluesy pitch-slides typical of their style. "Legit," i.e., classically oriented players, use a higher- $Q$  setup because their style only calls for enough pitch variation to produce a vibrato.)

---

*Decay of a saxophone tone* *example 11*

▷ If a typical saxophone setup has a  $Q$  of about 10, how long will it take for a 100-Hz tone played on a baritone saxophone to die down by a factor of 535 in energy, after the player suddenly stops blowing?

▷ A  $Q$  of 10 means that it takes 10 cycles for the vibrations to die down in energy by a factor of 535. Ten cycles at a frequency of 100 Hz would correspond to a time of 0.1 seconds, which is not very long. This is why a saxophone note doesn't "ring" like a note played on a piano or an electric guitar.

---

*$Q$  of a radio receiver* *example 12*

▷ A radio receiver used in the FM band needs to be tuned in to

within about 0.1 MHz for signals at about 100 MHz. What is its  $Q$ ?

▷  $Q = f_{\text{res}}/\text{FWHM} = 1000$ . This is an extremely high  $Q$  compared to most mechanical systems.

#### *Q of a stereo speaker*

#### *example 13*

We have already given one reason why a stereo speaker should have a low  $Q$ : otherwise it would continue ringing after the end of the musical note on the recording. The second reason is that we want it to be able to respond to a large range of frequencies.

#### *Nuclear magnetic resonance*

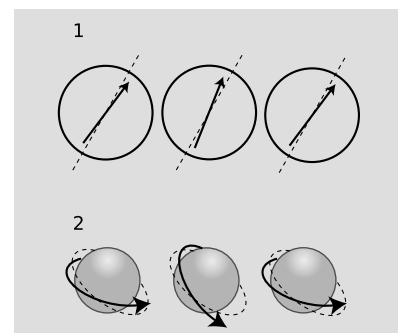
#### *example 14*

If you have ever played with a magnetic compass, you have undoubtedly noticed that if you shake it, it takes some time to settle down, h/1. As it settles down, it acts like a damped oscillator of the type we have been discussing. The compass needle is simply a small magnet, and the planet earth is a big magnet. The magnetic forces between them tend to bring the needle to an equilibrium position in which it lines up with the planet-earth-magnet.

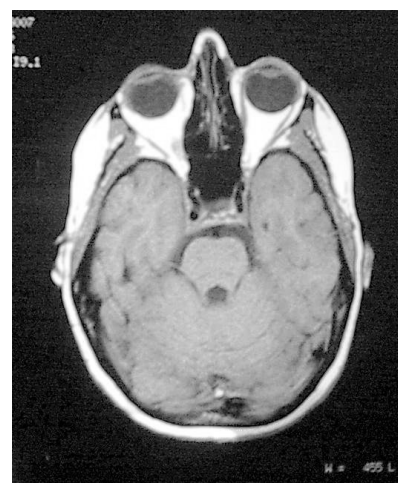
Essentially the same physics lies behind the technique called Nuclear Magnetic Resonance (NMR). NMR is a technique used to deduce the molecular structure of unknown chemical substances, and it is also used for making medical images of the inside of people's bodies. If you ever have an NMR scan, they will actually tell you you are undergoing "magnetic resonance imaging" or "MRI," because people are scared of the word "nuclear." In fact, the nuclei being referred to are simply the non-radioactive nuclei of atoms found naturally in your body.

Here's how NMR works. Your body contains large numbers of hydrogen atoms, each consisting of a small, lightweight electron orbiting around a large, heavy proton. That is, the nucleus of a hydrogen atom is just one proton. A proton is always spinning on its own axis, and the combination of its spin and its electrical charge cause it to behave like a tiny magnet. The principle identical to that of an electromagnet, which consists of a coil of wire through which electrical charges pass; the circling motion of the charges in the coil of wire makes it magnetic, and in the same way, the circling motion of the proton's charge makes it magnetic.

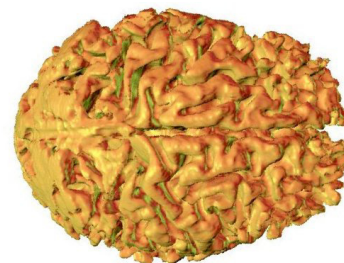
Now a proton in one of your body's hydrogen atoms finds itself surrounded by many other whirling, electrically charged particles: its own electron, plus the electrons and nuclei of the other nearby atoms. These neighbors act like magnets, and exert magnetic forces on the proton, h/2. The  $k$  of the vibrating proton is simply a measure of the total strength of these magnetic forces. Depending on the structure of the molecule in which the hydrogen atom finds itself, there will be a particular set of magnetic forces acting on the proton and a particular value of  $k$ . The NMR apparatus



h / Example 14. 1. A compass needle vibrates about the equilibrium position under the influence of the earth's magnetic forces. 2. The orientation of a proton's spin vibrates around its equilibrium direction under the influence of the magnetic forces coming from the surrounding electrons and nuclei.



i / A member of the author's family, who turned out to be healthy.



j / A three-dimensional computer reconstruction of the shape of a human brain, based on magnetic resonance data.

bombards the sample with radio waves, and if the frequency of the radio waves matches the resonant frequency of the proton, the proton will absorb radio-wave energy strongly and oscillate wildly. Its vibrations are damped not by friction, because there is no friction inside an atom, but by the reemission of radio waves.

By working backward through this chain of reasoning, one can determine the geometric arrangement of the hydrogen atom's neighboring atoms. It is also possible to locate atoms in space, allowing medical images to be made.

Finally, it should be noted that the behavior of the proton cannot be described entirely correctly by Newtonian physics. Its vibrations are of the strange and spooky kind described by the laws of quantum mechanics. It is impressive, however, that the few simple ideas we have learned about resonance can still be applied successfully to describe many aspects of this exotic system.

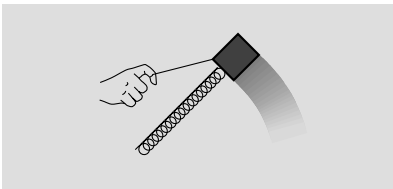
### Discussion question

**A** Nikola Tesla, one of the inventors of radio and an archetypical mad scientist, told a credulous reporter in 1912 the following story about an application of resonance. He built an electric vibrator that fit in his pocket, and attached it to one of the steel beams of a building that was under construction in New York. Although the article in which he was quoted didn't say so, he presumably claimed to have tuned it to the resonant frequency of the building. "In a few minutes, I could feel the beam trembling. Gradually the trembling increased in intensity and extended throughout the whole great mass of steel. Finally, the structure began to creak and weave, and the steelworkers came to the ground panic-stricken, believing that there had been an earthquake. ... [If] I had kept on ten minutes more, I could have laid that building flat in the street." Is this physically plausible?

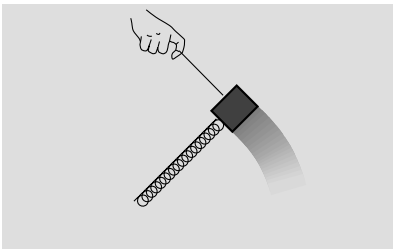
## 18.4 ★ Proofs

Our first goal is to predict the amplitude of the steady-state vibrations as a function of the frequency of the driving force and the amplitude of the driving force. With that equation in hand, we will then prove statements 2, 3, and 4 from section 18.3. We assume without proof statement 1, that the steady-state motion occurs at the same frequency as the driving force.

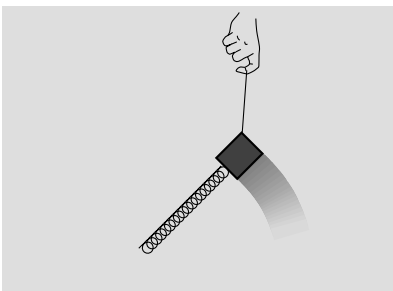
As with the proof in chapter 17, we make use of the fact that a sinusoidal vibration is the same as the projection of circular motion onto a line. We visualize the system shown in figures k-m, in which the mass swings in a circle on the end of a spring. The spring does not actually change its length at all, but it appears to from the flattened perspective of a person viewing the system edge-on. The radius of the circle is the amplitude,  $A$ , of the vibrations as seen edge-on. The damping force can be imagined as a backward drag force supplied by some fluid through which the mass is



k / Driving at a frequency above resonance.



l / Driving at resonance.



m / Driving at a frequency below resonance.

moving. As usual, we assume that the damping is proportional to velocity, and we use the symbol  $b$  for the proportionality constant,  $|F_d| = bv$ . The driving force, represented by a hand towing the mass with a string, has a tangential component  $|F_t|$  which counteracts the damping force,  $|F_t| = |F_d|$ , and a radial component  $F_r$  which works either with or against the spring's force, depending on whether we are driving the system above or below its resonant frequency.

The speed of the rotating mass is the circumference of the circle divided by the period,  $v = 2\pi A/T$ , its acceleration (which is directly inward) is  $a = v^2/r$ , and Newton's second law gives  $a = F/m = (kA + F_r)/m$ . We write  $f_o$  for  $\frac{1}{2\pi}\sqrt{k/m}$ . Straightforward algebra yields

$$[1] \quad \frac{F_r}{F_t} = \frac{2\pi m}{bf} (f^2 - f_o^2) \quad .$$

This is the ratio of the wasted force to the useful force, and we see that it becomes zero when the system is driven at resonance.

The amplitude of the vibrations can be found by attacking the equation  $|F_t| = bv = 2\pi bAf$ , which gives

$$[2] \quad A = \frac{|F_t|}{2\pi bf} \quad .(2)$$

However, we wish to know the amplitude in terms of  $|\mathbf{F}|$ , not  $|F_t|$ . From now on, let's drop the cumbersome magnitude symbols. With the Pythagorean theorem, it is easily proved that

$$[3] \quad F_t = \frac{F}{\sqrt{1 + \left(\frac{F_r}{F_t}\right)^2}} \quad , (3)$$

and equations 1-3 can then be combined to give the final result

$$[4] \quad A = \frac{F}{2\pi\sqrt{4\pi^2 m^2 (f^2 - f_o^2)^2 + b^2 f^2}} \quad .$$

### Statement 2: maximum amplitude at resonance

Equation [4] makes it plausible that the amplitude is maximized when the system is driven at close to its resonant frequency. At  $f = f_o$ , the first term inside the square root vanishes, and this makes the denominator as small as possible, causing the amplitude to be as big as possible. (Actually this is only approximately true, because it is possible to make  $A$  a little bigger by decreasing  $f$  a little below  $f_o$ , which makes the second term smaller. This technical issue is addressed in homework problem 3 on page 457.)

### Statement 3: amplitude at resonance proportional to $Q$

Equation [4] shows that the amplitude at resonance is proportional to  $1/b$ , and the  $Q$  of the system is inversely proportional to  $b$ , so the amplitude at resonance is proportional to  $Q$ .



#### Statement 4: FWHM related to $Q$

We will satisfy ourselves by proving only the proportionality  $FWHM \propto f_o/Q$ , not the actual equation  $FWHM = f_o/Q$ . The energy is proportional to  $A^2$ , i.e., to the inverse of the quantity inside the square root in equation [4]. At resonance, the first term inside the square root vanishes, and the half-maximum points occur at frequencies for which the whole quantity inside the square root is double its value at resonance, i.e., when the two terms are equal. At the half-maximum points, we have

$$\begin{aligned} f^2 - f_o^2 &= \left( f_o \pm \frac{FWHM}{2} \right)^2 - f_o^2 \\ &= \pm f_o \cdot FWHM + \frac{1}{4} FWHM^2 \end{aligned}$$

If we assume that the width of the resonance is small compared to the resonant frequency, then the  $FWHM^2$  term is negligible compared to the  $f_o \cdot FWHM$  term, and setting the terms in equation 4 equal to each other gives

$$4\pi^2 m^2 (f_o FWHM)^2 = b^2 f^2 \quad .$$

We are assuming that the width of the resonance is small compared to the resonant frequency, so  $f$  and  $f_o$  can be taken as synonyms. Thus,

$$FWHM = \frac{b}{2\pi m} \quad .$$

We wish to connect this to  $Q$ , which can be interpreted as the energy of the free (undriven) vibrations divided by the work done by damping in one cycle. The former equals  $kA^2/2$ , and the latter is proportional to the force,  $bv \propto bAf_o$ , multiplied by the distance traveled,  $A$ . (This is only a proportionality, not an equation, since the force is not constant.) We therefore find that  $Q$  is proportional to  $k/bf_o$ . The equation for the FWHM can then be restated as a proportionality  $FWHM \propto k/Qf_o m \propto f_o/Q$ .

## Summary

### Selected vocabulary

damping . . . . .	the dissipation of a vibration's energy into heat energy, or the frictional force that causes the loss of energy
quality factor . .	the number of oscillations required for a system's energy to fall off by a factor of 535 due to damping
driving force . . .	an external force that pumps energy into a vibrating system
resonance . . . .	the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration
steady state . . .	the behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force

### Notation

$Q$ . . . . .	the quality factor
$f_o$ . . . . .	the natural (resonant) frequency of a vibrating system, i.e., the frequency at which it would vibrate if it was simply kicked and left alone
$f$ . . . . .	the frequency at which the system actually vibrates, which in the case of a driven system is equal to the frequency of the driving force, not the natural frequency

## Summary

The energy of a vibration is always proportional to the square of the amplitude, assuming the amplitude is small. Energy is lost from a vibrating system for various reasons such as the conversion to heat via friction or the emission of sound. This effect, called damping, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss. A driving force that pumps energy into the system may drive the system at its own natural frequency or at some other frequency. When a vibrating system is driven by an external force, we are usually interested in its steady-state behavior, i.e., its behavior after it has had time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

The following are four important facts about a vibrating system being driven by an external force:

- (1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

(2) A vibrating system resonates at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to  $Q$ .

(4) The FWHM of a resonance is related to its  $Q$  and its resonant frequency  $f_o$  by the equation

$$\text{FWHM} = \frac{f_o}{Q}.$$

(This equation is only a good approximation when  $Q$  is large.)

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

**2** Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish's buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder's  $Q$  is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

**3** As noted in section 18.4, it is only approximately true that the amplitude has its maximum at  $f = (1/2\pi)\sqrt{k/m}$ . Being more careful, we should actually define two different symbols,  $f_0 = (1/2\pi)\sqrt{k/m}$  and  $f_o$  for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2 (f^2 - f_0^2)^2 + b^2 f^2}}.$$

Show that the maximum occurs not at  $f_o$  but rather at the frequency

$$f_o = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2}\text{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.

∫

4 (a) Let  $W$  be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy  $E$  that remains in the oscillations after  $n$  cycles of motion.

(b) From this, prove the equation

$$\left(1 - \frac{W}{E}\right)^Q = e^{-2\pi}$$

(recalling that the number 535 in the definition of  $Q$  is  $e^{2\pi}$ ).

(c) Use this to prove the approximation  $1/Q \approx (1/2\pi)W/E$ . (Hint: Use the approximation  $\ln(1+x) \approx x$ , which is valid for small values of  $x$ .)

5 The goal of this problem is to refine the proportionality  $\text{FWHM} \propto f_{\text{res}}/Q$  into the equation  $\text{FWHM} = f_{\text{res}}/Q$ , i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force  $F = -bv$  over one cycle of steady-state motion equals  $W_{\text{damp}} = -2\pi^2 b f A^2$ . Hint: It is less confusing to calculate the work done over half a cycle, from  $x = -A$  to  $x = +A$ , and then double it.

(b) Show that the fraction of the undriven oscillator's energy lost to damping over one cycle is  $|W_{\text{damp}}|/E = 4\pi^2 b f / k$ .

(c) Use the previous result, combined with the result of problem 4, to prove that  $Q$  equals  $k/2\pi b f$ .

(d) Combine the preceding result for  $Q$  with the equation  $\text{FWHM} = b/2\pi m$  from section 18.4 to prove the equation  $\text{FWHM} = f_{\text{res}}/Q$ .

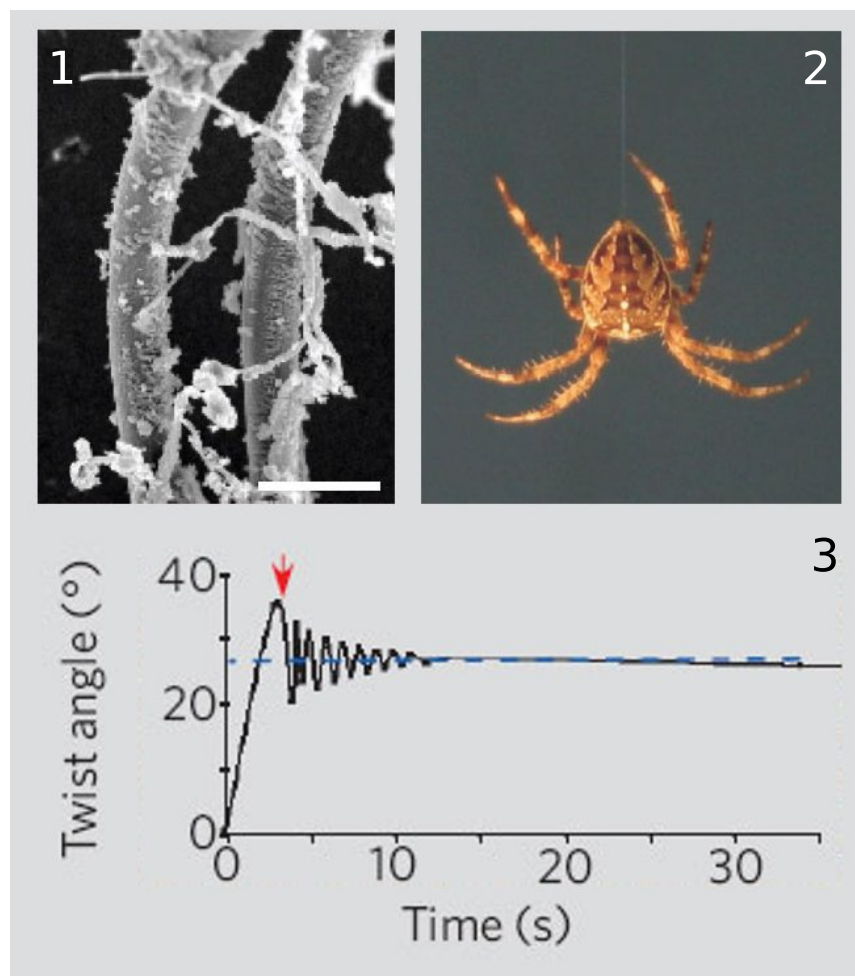
∫ ★

6 (a) We observe that the amplitude of a certain free oscillation decreases from  $A_0$  to  $A_0/Z$  after  $n$  oscillations. Find its  $Q$ . ✓

(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope's image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it's probably a bad thing for the spider if it twists back and forth while hanging like this. (We're referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn't be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low  $Q$ , so the vibrations damp out quickly.
2. It doesn't become brittle with repeated twisting as a copper wire would.
3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than  $\pm 35$  degrees, settling down to a new equilibrium at 27 degrees.
4. Over much longer time scales (hours), the thread eventually resets itself to its original equilibrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials have this "memory" property as well, but they typically need to be heated in order to make them go back to their original shapes.

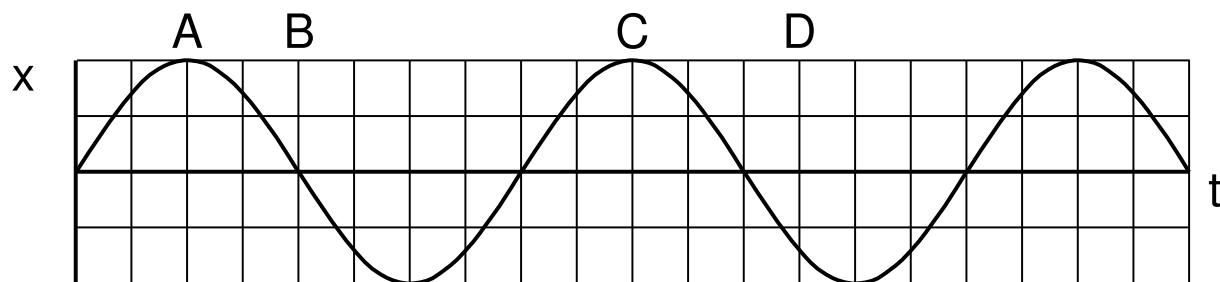
Focusing on property number 1, estimate the  $Q$  of spider silk from the graph. ✓



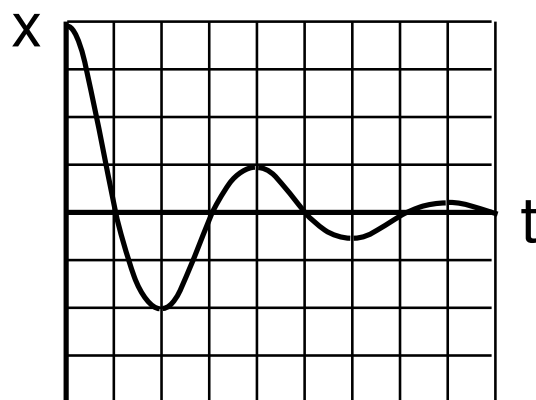
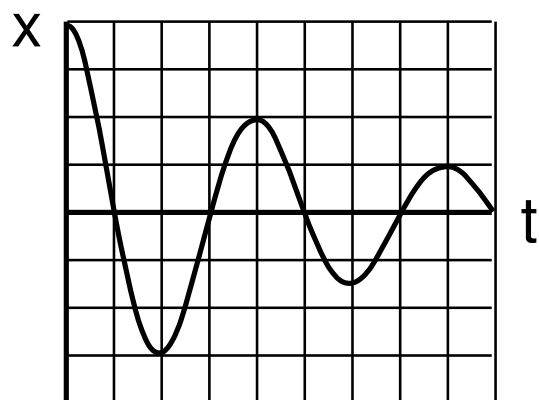
Problem 6.

## Exercise 18: Resonance

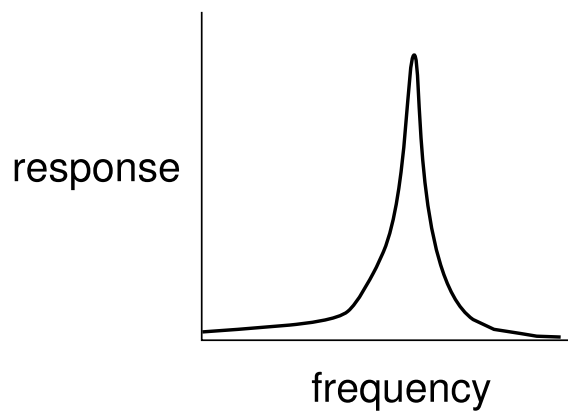
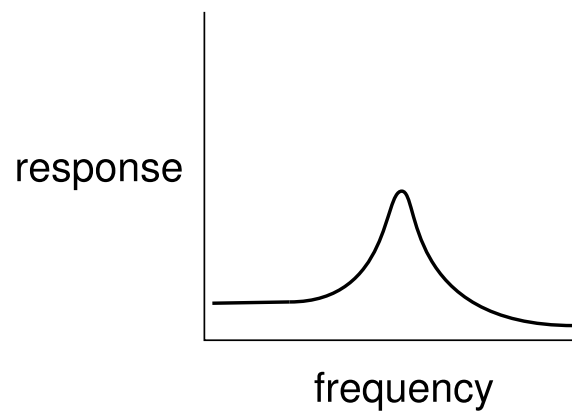
1. Compare the oscillator's energies at A, B, C, and D.



2. Compare the Q values of the two oscillators.



3. Match the x-t graphs in #2 with the amplitude-frequency graphs below.









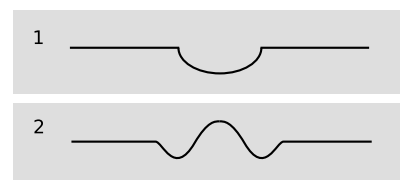
"The Great Wave Off Kanagawa," by Katsushika Hokusai (1760-1849).

## Chapter 19

# Free waves

Your vocal cords or a saxophone reed can vibrate, but being able to vibrate wouldn't be of much use unless the vibrations could be transmitted to the listener's ear by sound waves. What are waves and why do they exist? Put your fingertip in the middle of a cup of water and then remove it suddenly. You will have noticed two results that are surprising to most people. First, the flat surface of the water does not simply sink uniformly to fill in the volume vacated by your finger. Instead, ripples spread out, and the process of flattening out occurs over a long period of time, during which the water at the center vibrates above and below the normal water level. This type of wave motion is the topic of the present chapter. Second, you have found that the ripples bounce off of the walls of the cup, in much the same way that a ball would bounce off of a wall. In the next chapter we discuss what happens to waves that have a boundary around them. Until then, we confine ourselves to wave phenomena that can be analyzed as if the medium (e.g., the water) was infinite and the same everywhere.

It isn't hard to understand why removing your fingertip creates ripples rather than simply allowing the water to sink back down uniformly. The initial crater, (a), left behind by your finger has sloping sides, and the water next to the crater flows downhill to fill in the hole. The water far away, on the other hand, initially has



a / Dipping a finger in some water, 1, causes a disturbance that spreads outward, 2.

no way of knowing what has happened, because there is no slope for it to flow down. As the hole fills up, the rising water at the center gains upward momentum, and overshoots, creating a little hill where there had been a hole originally. The area just outside of this region has been robbed of some of its water in order to build the hill, so a depressed “moat” is formed, (b). This effect cascades outward, producing ripples.



b / The two circular patterns of ripples pass through each other. Unlike material objects, wave patterns can overlap in space, and when this happens they combine by addition.

## 19.1 Wave motion

There are three main ways in which wave motion differs from the motion of objects made of matter.

### 1. Superposition

The most profound difference is that waves do not display anything analogous to the normal forces between objects that come in contact. Two wave patterns can therefore overlap in the same region of space, as shown in figure b. Where the two waves coincide, they add together. For instance, suppose that at a certain location in at a certain moment in time, each wave would have had a crest 3 cm above the normal water level. The waves combine at this point to make a 6-cm crest. We use negative numbers to represent depressions in the water. If both waves would have had a troughs measuring -3 cm, then they combine to make an extra-deep -6 cm trough. A +3 cm crest and a -3 cm trough result in a height of zero, i.e., the waves momentarily cancel each other out at that point. This additive rule is referred to as the principle of superposition, “superposition” being merely a fancy word for “adding.”

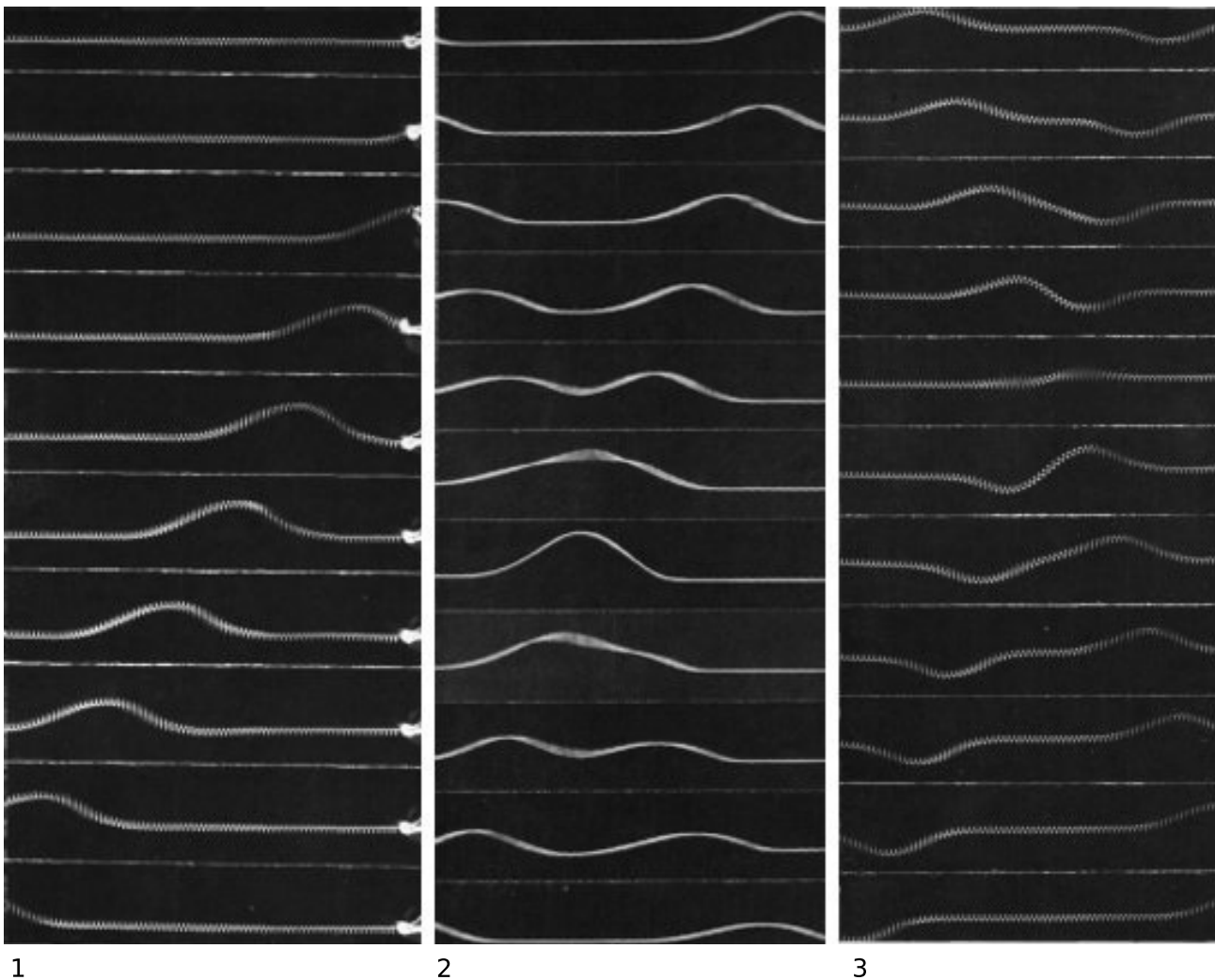
Superposition can occur not just with sinusoidal waves like the ones in the figure above but with waves of any shape. The figures on the following page show superposition of wave *pulses*. A pulse is simply a wave of very short duration. These pulses consist only of a single hump or trough. If you hit a clothesline sharply, you will observe pulses heading off in both directions. This is analogous to

the way ripples spread out in all directions when you make a disturbance at one point on water. The same occurs when the hammer on a piano comes up and hits a string.

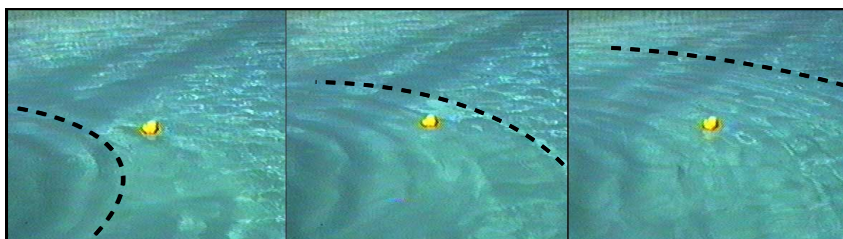
Experiments to date have not shown any deviation from the principle of superposition in the case of light waves. For other types of waves, it is typically a very good approximation for low-energy waves.

### Discussion question

**A** In figure c/3, the fifth frame shows the spring just about perfectly flat. If the two pulses have essentially canceled each other out perfectly, then why does the motion pick up again? Why doesn't the spring just stay flat?



c / These pictures show the motion of wave pulses along a spring. To make a pulse, one end of the spring was shaken by hand. Movies were filmed, and a series of frame chosen to show the motion. 1. A pulse travels to the left. 2. Superposition of two colliding positive pulses. 3. Superposition of two colliding pulses, one positive and one negative.



d / As the wave pattern passes the rubber duck, the duck stays put. The water isn't moving forward with the wave.

## 2. The medium is not transported with the wave.

Figure d shows a series of water waves before it has reached a rubber duck (left), having just passed the duck (middle) and having progressed about a meter beyond the duck (right). The duck bobs around its initial position, but is not carried along with the wave. This shows that the water itself does not flow outward with the wave. If it did, we could empty one end of a swimming pool simply by kicking up waves! We must distinguish between the motion of the medium (water in this case) and the motion of the wave pattern through the medium. The medium vibrates; the wave progresses through space.

### self-check A

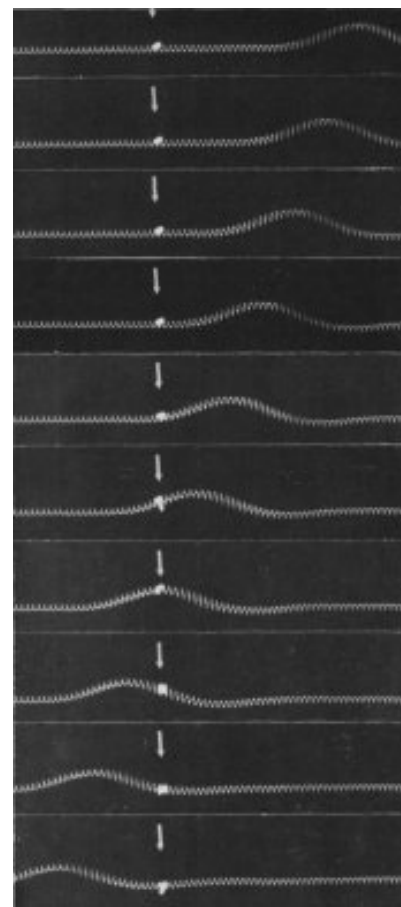
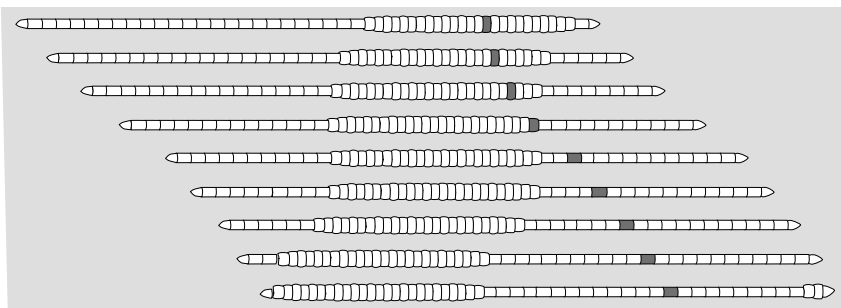
In figure e, you can detect the side-to-side motion of the spring because the spring appears blurry. At a certain instant, represented by a single photo, how would you describe the motion of the different parts of the spring? Other than the flat parts, do any parts of the spring have zero velocity?

▷ Answer, p. 534

### A worm

### example 1

The worm in the figure is moving to the right. The wave pattern, a pulse consisting of a compressed area of its body, moves to the left. In other words, the motion of the wave pattern is in the opposite direction compared to the motion of the medium.

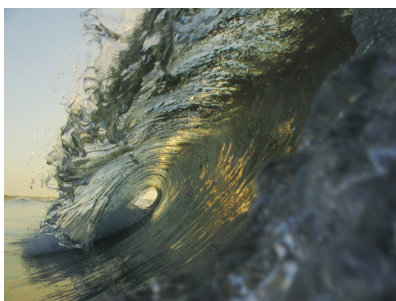


e / As the wave pulse goes by, the ribbon tied to the spring is not carried along. The motion of the wave pattern is to the right, but the medium (spring) is moving up and down, not to the right.

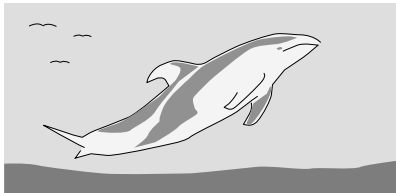
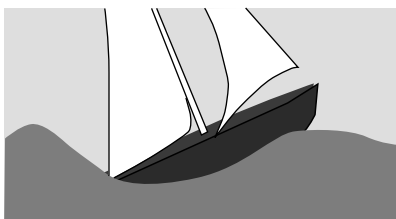




f / Example 2. The surfer is dragging his hand in the water.



g / Example 3: a breaking wave.



h / Example 4. The boat has run up against a limit on its speed because it can't climb over its own wave. Dolphins get around the problem by leaping out of the water.

### Surfing

### example 2

The incorrect belief that the medium moves with the wave is often reinforced by garbled secondhand knowledge of surfing. Anyone who has actually surfed knows that the front of the board pushes the water to the sides, creating a wake — the surfer can even drag his hand through the water, as in in figure f. If the water was moving along with the wave and the surfer, this wouldn't happen. The surfer is carried forward because forward is downhill, not because of any forward flow of the water. If the water was flowing forward, then a person floating in the water up to her neck would be carried along just as quickly as someone on a surfboard. In fact, it is even possible to surf down the back side of a wave, although the ride wouldn't last very long because the surfer and the wave would quickly part company.

### 3. A wave's velocity depends on the medium.

A material object can move with any velocity, and can be sped up or slowed down by a force that increases or decreases its kinetic energy. Not so with waves. The magnitude of a wave's velocity depends on the properties of the medium (and perhaps also on the shape of the wave, for certain types of waves). Sound waves travel at about 340 m/s in air, 1000 m/s in helium. If you kick up water waves in a pool, you will find that kicking harder makes waves that are taller (and therefore carry more energy), not faster. The sound waves from an exploding stick of dynamite carry a lot of energy, but are no faster than any other waves. Thus although both waves and physical objects carry energy as they move through space, the energy of the wave relates to its amplitude, not to its speed.

In the following section we will give an example of the physical relationship between the wave speed and the properties of the medium.

### Breaking waves

### example 3

The velocity of water waves increases with depth. The crest of a wave travels faster than the trough, and this can cause the wave to break.

Once a wave is created, the only reason its speed will change is if it enters a different medium or if the properties of the medium change. It is not so surprising that a change in medium can slow down a wave, but the reverse can also happen. A sound wave traveling through a helium balloon will slow down when it emerges into the air, but if it enters another balloon it will speed back up again! Similarly, water waves travel more quickly over deeper water, so a wave will slow down as it passes over an underwater ridge, but speed

up again as it emerges into deeper water.

### Hull speed

example 4

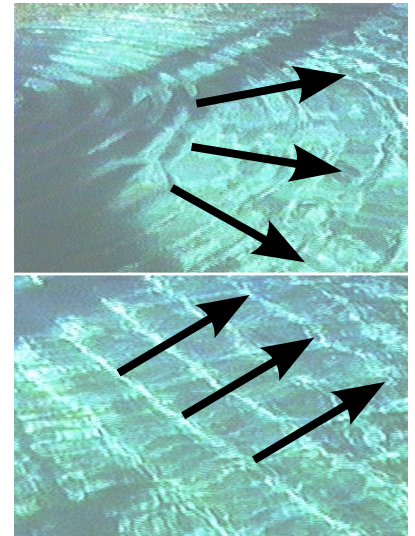
The speeds of most boats, and of some surface-swimming animals, are limited by the fact that they make a wave due to their motion through the water. The boat in figure h is going at the same speed as its own waves, and can't go any faster. No matter how hard the boat pushes against the water, it can't make the wave move ahead faster and get out of the way. The wave's speed depends only on the medium. Adding energy to the wave doesn't speed it up, it just increases its amplitude.

A water wave, unlike many other types of wave, has a speed that depends on its shape: a broader wave moves faster. The shape of the wave made by a boat tends to mold itself to the shape of the boat's hull, so a boat with a longer hull makes a broader wave that moves faster. The maximum speed of a boat whose speed is limited by this effect is therefore closely related to the length of its hull, and the maximum speed is called the hull speed. Sailboats designed for racing are not just long and skinny to make them more streamlined — they are also long so that their hull speeds will be high.

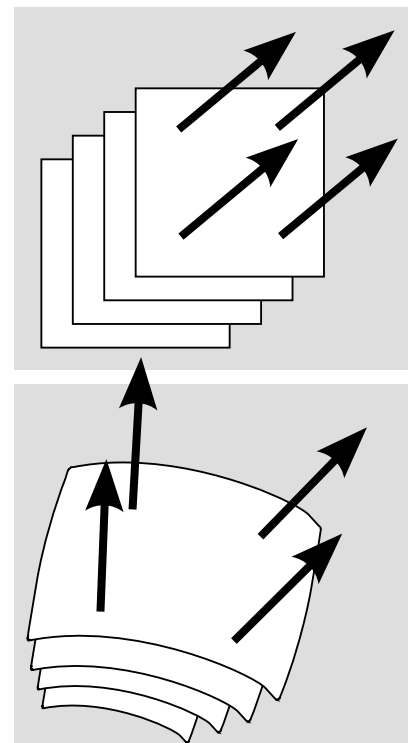
### Wave patterns

If the magnitude of a wave's velocity vector is preordained, what about its direction? Waves spread out in all directions from every point on the disturbance that created them. If the disturbance is small, we may consider it as a single point, and in the case of water waves the resulting wave pattern is the familiar circular ripple, i/1. If, on the other hand, we lay a pole on the surface of the water and wiggle it up and down, we create a linear wave pattern, i/2. For a three-dimensional wave such as a sound wave, the analogous patterns would be spherical waves and plane waves, j.

Infinitely many patterns are possible, but linear or plane waves are often the simplest to analyze, because the velocity vector is in the same direction no matter what part of the wave we look at. Since all the velocity vectors are parallel to one another, the problem is effectively one-dimensional. Throughout this chapter and the next, we will restrict ourselves mainly to wave motion in one dimension, while not hesitating to broaden our horizons when it can be done without too much complication.

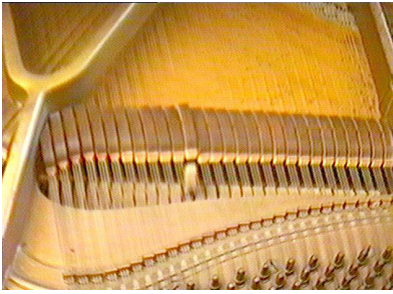


i / Circular and linear wave patterns.

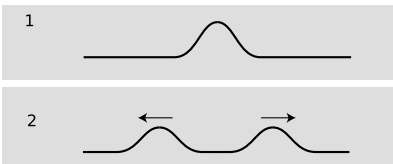


j / Plane and spherical wave patterns.

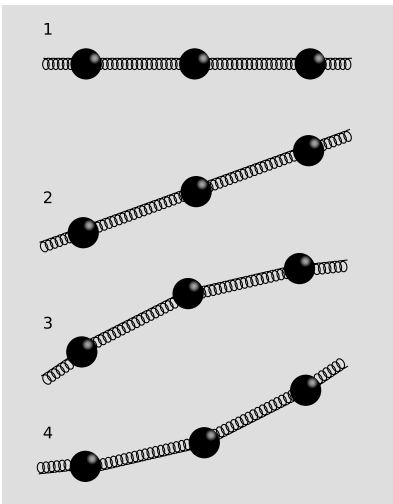




k / Hitting a key on a piano causes a hammer to come up from underneath and hit a string (actually a set of three strings). The result is a pair of pulses moving away from the point of impact.



l / A string is struck with a hammer, 1, and two pulses fly off, 2.



m / A continuous string can be modeled as a series of discrete masses connected by springs.

## Discussion questions

A [see above]

B Sketch two positive wave pulses on a string that are overlapping but not right on top of each other, and draw their superposition. Do the same for a positive pulse running into a negative pulse.

C A traveling wave pulse is moving to the right on a string. Sketch the velocity vectors of the various parts of the string. Now do the same for a pulse moving to the left.

D In a spherical sound wave spreading out from a point, how would the energy of the wave fall off with distance?

## 19.2 Waves on a string

So far you have learned some counterintuitive things about the behavior of waves, but intuition can be trained. The first half of this section aims to build your intuition by investigating a simple, one-dimensional type of wave: a wave on a string. If you have ever stretched a string between the bottoms of two open-mouthed cans to talk to a friend, you were putting this type of wave to work. Stringed instruments are another good example. Although we usually think of a piano wire simply as vibrating, the hammer actually strikes it quickly and makes a dent in it, which then ripples out in both directions. Since this chapter is about free waves, not bounded ones, we pretend that our string is infinitely long.

After the qualitative discussion, we will use simple approximations to investigate the speed of a wave pulse on a string. This quick and dirty treatment is then followed by a rigorous attack using the methods of calculus, which may be skipped by the student who has not studied calculus. How far you penetrate in this section is up to you, and depends on your mathematical self-confidence. If you skip some of the math, you should nevertheless absorb the significance of the result, discussed on p. 474.

### Intuitive ideas

Consider a string that has been struck, 1/1, resulting in the creation of two wave pulses, 2, one traveling to the left and one to the right. This is analogous to the way ripples spread out in all directions from a splash in water, but on a one-dimensional string, “all directions” becomes “both directions.”

We can gain insight by modeling the string as a series of masses connected by springs. (In the actual string the mass and the springiness are both contributed by the molecules themselves.) If we look at various microscopic portions of the string, there will be some areas that are flat, m/1, some that are sloping but not curved, 2, and some that are curved, 3 and 4. In example 1 it is clear that both the forces on the central mass cancel out, so it will not accelerate. The

same is true of 2, however. Only in curved regions such as 3 and 4 is an acceleration produced. In these examples, the vector sum of the two forces acting on the central mass is not zero. The important concept is that curvature makes force: the curved areas of a wave tend to experience forces resulting in an acceleration toward the mouth of the curve. Note, however, that an uncurved portion of the string need not remain motionless. It may move at constant velocity to either side.

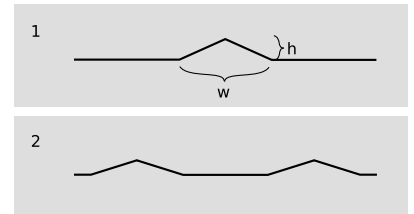
### Approximate treatment

We now carry out an approximate treatment of the speed at which two pulses will spread out from an initial indentation on a string. For simplicity, we imagine a hammer blow that creates a triangular dent,  $n/1$ . We will estimate the amount of time,  $t$ , required until each of the pulses has traveled a distance equal to the width of the pulse itself. The velocity of the pulses is then  $\pm w/t$ .

As always, the velocity of a wave depends on the properties of the medium, in this case the string. The properties of the string can be summarized by two variables: the tension,  $T$ , and the mass per unit length,  $\mu$  (Greek letter mu).

If we consider the part of the string encompassed by the initial dent as a single object, then this object has a mass of approximately  $\mu w$  (mass/length  $\times$  length = mass). (Here, and throughout the derivation, we assume that  $h$  is much less than  $w$ , so that we can ignore the fact that this segment of the string has a length slightly greater than  $w$ .) Although the downward acceleration of this segment of the string will be neither constant over time nor uniform across the string, we will pretend that it is constant for the sake of our simple estimate. Roughly speaking, the time interval between  $n/1$  and 2 is the amount of time required for the initial dent to accelerate from rest and reach its normal, flattened position. Of course the tip of the triangle has a longer distance to travel than the edges, but again we ignore the complications and simply assume that the segment as a whole must travel a distance  $h$ . Indeed, it might seem surprising that the triangle would so neatly spring back to a perfectly flat shape. It is an experimental fact that it does, but our analysis is too crude to address such details.

The string is kinked, i.e., tightly curved, at the edges of the triangle, so it is here that there will be large forces that do not cancel out to zero. There are two forces acting on the triangular hump, one of magnitude  $T$  acting down and to the right, and one of the same magnitude acting down and to the left. If the angle of the sloping sides is  $\theta$ , then the total force on the segment equals  $2T \sin \theta$ . Dividing the triangle into two right triangles, we see that  $\sin \theta$  equals  $h$  divided by the length of one of the sloping sides. Since  $h$  is much less than  $w$ , the length of the sloping side is essentially



$n/1$  A triangular pulse spreads out.

the same as  $w/2$ , so we have  $\sin \theta = 2h/w$ , and  $F = 4Th/w$ . The acceleration of the segment (actually the acceleration of its center of mass) is

$$\begin{aligned} a &= F/m \\ &= 4Th/\mu w^2 \end{aligned} \quad .$$

The time required to move a distance  $h$  under constant acceleration  $a$  is found by solving  $h = \frac{1}{2}at^2$  to yield

$$\begin{aligned} t &= \sqrt{\frac{2h}{a}} \\ &= w\sqrt{\frac{\mu}{2T}} \end{aligned} \quad .$$

Our final result for the velocity of the pulses is

$$\begin{aligned} |v| &= \frac{w}{t} \\ &= \sqrt{\frac{2T}{\mu}} \end{aligned} \quad .$$

The remarkable feature of this result is that the velocity of the pulses does not depend at all on  $w$  or  $h$ , i.e., any triangular pulse has the same speed. It is an experimental fact (and we will also prove rigorously in the following subsection) that any pulse of any kind, triangular or otherwise, travels along the string at the same speed. Of course, after so many approximations we cannot expect to have gotten all the numerical factors right. The correct result for the velocity of the pulses is

$$v = \sqrt{\frac{T}{\mu}} \quad .$$

The importance of the above derivation lies in the insight it brings—that all pulses move with the same speed—rather than in the details of the numerical result. The reason for our too-high value for the velocity is not hard to guess. It comes from the assumption that the acceleration was constant, when actually the total force on the segment would diminish as it flattened out.

### **Rigorous derivation using calculus (optional)**

After expending considerable effort for an approximate solution, we now display the power of calculus with a rigorous and completely general treatment that is nevertheless much shorter and easier. Let the flat position of the string define the  $x$  axis, so that  $y$  measures how far a point on the string is from equilibrium. The motion of the string is characterized by  $y(x, t)$ , a function of two variables. Knowing that the force on any small segment of string depends

on the curvature of the string in that area, and that the second derivative is a measure of curvature, it is not surprising to find that the infinitesimal force  $dF$  acting on an infinitesimal segment  $dx$  is given by

$$dF = T \frac{d^2 y}{dx^2} dx \quad .$$

(This can be proved by vector addition of the two infinitesimal forces acting on either side.) The acceleration is then  $a = dF/dm$ , or, substituting  $dm = \mu dx$ ,

$$\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2} \quad .$$

The second derivative with respect to time is related to the second derivative with respect to position. This is no more than a fancy mathematical statement of the intuitive fact developed above, that the string accelerates so as to flatten out its curves.

Before even bothering to look for solutions to this equation, we note that it already proves the principle of superposition, because the derivative of a sum is the sum of the derivatives. Therefore the sum of any two solutions will also be a solution.

Based on experiment, we expect that this equation will be satisfied by any function  $y(x, t)$  that describes a pulse or wave pattern moving to the left or right at the correct speed  $v$ . In general, such a function will be of the form  $y = f(x - vt)$  or  $y = f(x + vt)$ , where  $f$  is any function of one variable. Because of the chain rule, each derivative with respect to time brings out a factor of  $\pm v$ . Evaluating the second derivatives on both sides of the equation gives

$$(\pm v)^2 f'' = \frac{T}{\mu} f'' \quad .$$

Squaring gets rid of the sign, and we find that we have a valid solution for any function  $f$ , provided that  $v$  is given by

$$v = \sqrt{\frac{T}{\mu}} \quad .$$

## Significance of the result

This specific result for the speed of waves on a string,  $v = \sqrt{T/\mu}$ , is utterly unimportant. Don't memorize it. Don't take notes on it. Try to erase it from your memory.

What *is* important about this result is that it is an example of two things that are usually true, at least approximately, for mechanical waves in general:

1. The speed at which a wave moves does not depend on the size or shape of the wave.
2. The speed of a mechanical wave depends on a combination of two properties of the medium: some measure of its *inertia* and some measure of its *tightness*, i.e., the strength of the force trying to bring the medium back toward equilibrium.

### self-check B

- (a) What is it about the equation  $v = \sqrt{T/\mu}$  that relates to fact 1 above?  
(b) In the equation  $v = \sqrt{T/\mu}$ , which variable is a measure of inertia, and which is a measure of tightness? (c) Now suppose that we produce compressional wave pulses in a metal rod by tapping the end of the rod with a hammer. What physical properties of the rod would play the roles of inertia and tightness? How would you expect the speed of compressional waves in lead to compare with their speed in aluminum?

▷ Answer, p. 534

## 19.3 Sound and light waves

### Sound waves

The phenomenon of sound is easily found to have all the characteristics we expect from a wave phenomenon:

- Sound waves obey superposition. Sounds do not knock other sounds out of the way when they collide, and we can hear more than one sound at once if they both reach our ear simultaneously.
- The medium does not move with the sound. Even standing in front of a titanic speaker playing earsplitting music, we do not feel the slightest breeze.
- The velocity of sound depends on the medium. Sound travels faster in helium than in air, and faster in water than in helium. Putting more energy into the wave makes it more intense, not faster. For example, you can easily detect an echo when you clap your hands a short distance from a large, flat wall, and the delay of the echo is no shorter for a louder clap.

Although not all waves have a speed that is independent of the shape of the wave, and this property therefore is irrelevant to our collection of evidence that sound is a wave phenomenon, sound does nevertheless have this property. For instance, the music in a large concert hall or stadium may take on the order of a second to reach someone seated in the nosebleed section, but we do not notice or care, because the delay is the same for every sound. Bass, drums, and vocals all head outward from the stage at 340 m/s, regardless of their differing wave shapes.

If sound has all the properties we expect from a wave, then what type of wave is it? It must be a vibration of a physical medium such as air, since the speed of sound is different in different media, such as helium or water. Further evidence is that we don't receive sound signals that have come to our planet through outer space. The roars and whooshes of Hollywood's space ships are fun, but scientifically wrong.<sup>1</sup>

We can also tell that sound waves consist of compressions and expansions, rather than sideways vibrations like the shimmying of a snake. Only compressional vibrations would be able to cause your eardrums to vibrate in and out. Even for a very loud sound, the compression is extremely weak; the increase or decrease compared to normal atmospheric pressure is no more than a part per million. Our ears are apparently very sensitive receivers! Unlike a wave on a string, which vibrates in the direction perpendicular to the direction in which the wave pattern moves, a sound wave is a longitudinal wave, i.e., one in which the vibration is forward and backward along the direction of motion.

## Light waves

Entirely similar observations lead us to believe that light is a wave, although the concept of light as a wave had a long and tortuous history. It is interesting to note that Isaac Newton very influentially advocated a contrary idea about light. The belief that matter was made of atoms was stylish at the time among radical thinkers (although there was no experimental evidence for their existence), and it seemed logical to Newton that light as well should be made of tiny particles, which he called corpuscles (Latin for "small objects"). Newton's triumphs in the science of mechanics, i.e., the study of matter, brought him such great prestige that nobody bothered to

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<sup>1</sup>Outer space is not a perfect vacuum, so it is possible for sound waves to travel through it. However, if we want to create a sound wave, we typically do it by creating vibrations of a physical object, such as the sounding board of a guitar, the reed of a saxophone, or a speaker cone. The lower the density of the surrounding medium, the less efficiently the energy can be converted into sound and carried away. An isolated tuning fork, left to vibrate in interstellar space, would dissipate the energy of its vibration into internal heat at a rate many orders of magnitude greater than the rate of sound emission into the nearly perfect vacuum around it.

question his incorrect theory of light for 150 years. One persuasive proof that light is a wave is that according to Newton's theory, two intersecting beams of light should experience at least some disruption because of collisions between their corpuscles. Even if the corpuscles were extremely small, and collisions therefore very infrequent, at least some dimming should have been measurable. In fact, very delicate experiments have shown that there is no dimming.

The wave theory of light was entirely successful up until the 20th century, when it was discovered that not all the phenomena of light could be explained with a pure wave theory. It is now believed that both light and matter are made out of tiny chunks which have *both* wave and particle properties. For now, we will content ourselves with the wave theory of light, which is capable of explaining a great many things, from cameras to rainbows.

If light is a wave, what is waving? What is the medium that wiggles when a light wave goes by? It isn't air. A vacuum is impenetrable to sound, but light from the stars travels happily through zillions of miles of empty space. Light bulbs have no air inside them, but that doesn't prevent the light waves from leaving the filament. For a long time, physicists assumed that there must be a mysterious medium for light waves, and they called it the aether (not to be confused with the chemical). Supposedly the aether existed everywhere in space, and was immune to vacuum pumps. The details of the story are more fittingly reserved for later in this course, but the end result was that a long series of experiments failed to detect any evidence for the aether, and it is no longer believed to exist. Instead, light can be explained as a wave pattern made up of electrical and magnetic fields.

## 19.4 Periodic waves

### Period and frequency of a periodic wave

You choose a radio station by selecting a certain frequency. We have already defined period and frequency for vibrations, but what do they signify in the case of a wave? We can recycle our previous definition simply by stating it in terms of the vibrations that the wave causes as it passes a receiving instrument at a certain point in space. For a sound wave, this receiver could be an eardrum or a microphone. If the vibrations of the eardrum repeat themselves over and over, i.e., are periodic, then we describe the sound wave that caused them as periodic. Likewise we can define the period and frequency of a wave in terms of the period and frequency of the vibrations it causes. As another example, a periodic water wave would be one that caused a rubber duck to bob in a periodic manner as they passed by it.

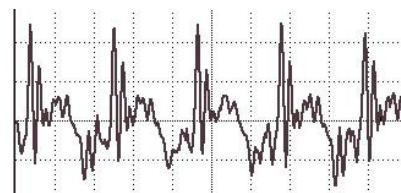
The period of a sound wave correlates with our sensory impression of musical pitch. A high frequency (short period) is a high note. The sounds that really define the musical notes of a song are only the ones that are periodic. It is not possible to sing a non-periodic sound like “sh” with a definite pitch.

The frequency of a light wave corresponds to color. Violet is the high-frequency end of the rainbow, red the low-frequency end. A color like brown that does not occur in a rainbow is not a periodic light wave. Many phenomena that we do not normally think of as light are actually just forms of light that are invisible because they fall outside the range of frequencies our eyes can detect. Beyond the red end of the visible rainbow, there are infrared and radio waves. Past the violet end, we have ultraviolet, x-rays, and gamma rays.

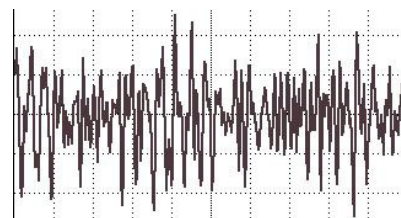
### Graphs of waves as a function of position

Some waves, light sound waves, are easy to study by placing a detector at a certain location in space and studying the motion as a function of time. The result is a graph whose horizontal axis is time. With a water wave, on the other hand, it is simpler just to look at the wave directly. This visual snapshot amounts to a graph of the height of the water wave as a function of *position*. Any wave can be represented in either way.

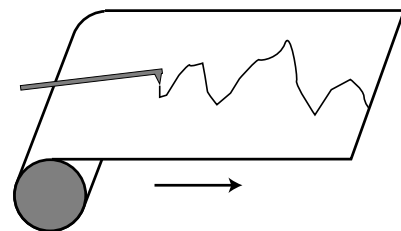
An easy way to visualize this is in terms of a strip chart recorder, an obsolescing device consisting of a pen that wiggles back and forth as a roll of paper is fed under it. It can be used to record a person’s electrocardiogram, or seismic waves too small to be felt as a noticeable earthquake but detectable by a seismometer. Taking the seismometer as an example, the chart is essentially a record of the ground’s wave motion as a function of time, but if the paper was set to feed at the same velocity as the motion of an earthquake wave, it



o / A graph of pressure versus time for a periodic sound wave, the vowel “ah.”



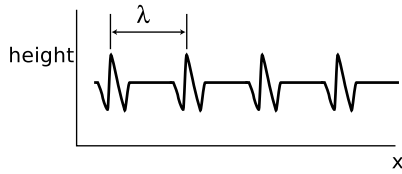
p / A similar graph for a non-periodic wave, “sh.”



q / A strip chart recorder.



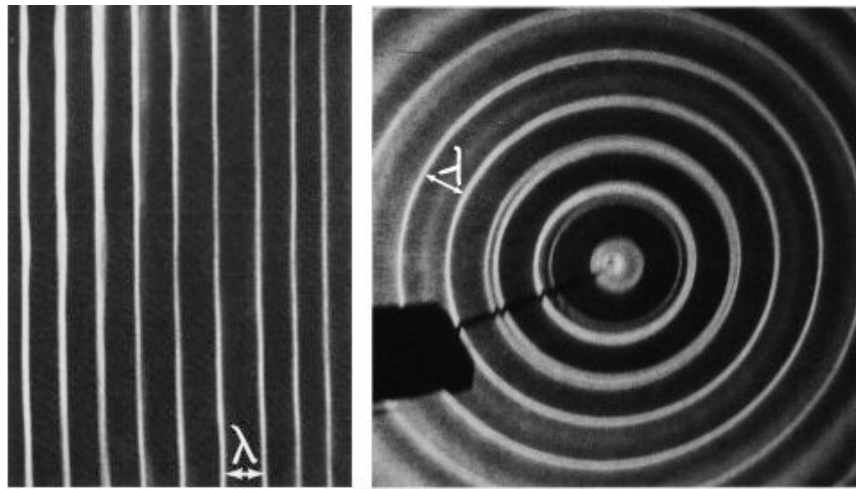
would also be a full-scale representation of the profile of the actual wave pattern itself. Assuming, as is usually the case, that the wave velocity is a constant number regardless of the wave's shape, knowing the wave motion as a function of time is equivalent to knowing it as a function of position.



r / A water wave profile created by a series of repeating pulses.

## Wavelength

Any wave that is periodic will also display a repeating pattern when graphed as a function of position. The distance spanned by one repetition is referred to as one *wavelength*. The usual notation for wavelength is  $\lambda$ , the Greek letter lambda. Wavelength is to space as period is to time.



s / Wavelengths of linear and circular water waves.

## Wave velocity related to frequency and wavelength

Suppose that we create a repetitive disturbance by kicking the surface of a swimming pool. We are essentially making a series of wave pulses. The wavelength is simply the distance a pulse is able to travel before we make the next pulse. The distance between pulses is  $\lambda$ , and the time between pulses is the period,  $T$ , so the speed of the wave is the distance divided by the time,

$$v = \lambda/T.$$

This important and useful relationship is more commonly written in terms of the frequency,

$$v = f\lambda \quad .$$

### Wavelength of radio waves

example 5

▷ The speed of light is  $3.0 \times 10^8$  m/s. What is the wavelength of the radio waves emitted by KKJZ, a station whose frequency is 88.1 MHz?

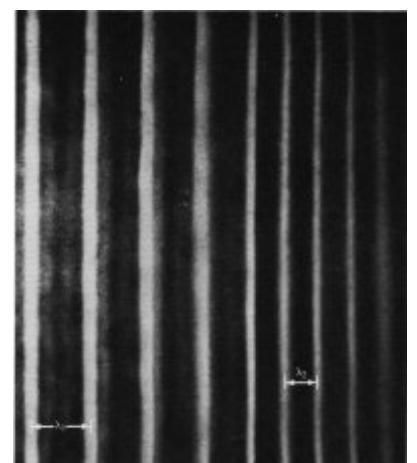
▷ Solving for wavelength, we have

$$\begin{aligned}\lambda &= v/f \\ &= (3.0 \times 10^8 \text{ m/s}) / (88.1 \times 10^6 \text{ s}^{-1}) \\ &= 3.4 \text{ m}\end{aligned}$$

The size of a radio antenna is closely related to the wavelength of the waves it is intended to receive. The match need not be exact (since after all one antenna can receive more than one wavelength!), but the ordinary “whip” antenna such as a car’s is 1/4 of a wavelength. An antenna optimized to receive KKJZ’s signal would have a length of  $3.4 \text{ m}/4 = 0.85 \text{ m}$ .



t / Ultrasound, i.e., sound with frequencies higher than the range of human hearing, was used to make this image of a fetus. The resolution of the image is related to the wavelength, since details smaller than about one wavelength cannot be resolved. High resolution therefore requires a short wavelength, corresponding to a high frequency.



u / A water wave traveling into a region with a different depth changes its wavelength.

The equation  $v = f\lambda$  defines a fixed relationship between any two of the variables if the other is held fixed. The speed of radio waves in air is almost exactly the same for all wavelengths and frequencies (it is exactly the same if they are in a vacuum), so there is a fixed relationship between their frequency and wavelength. Thus we can say either “Are we on the same wavelength?” or “Are we on the same frequency?”

A different example is the behavior of a wave that travels from a region where the medium has one set of properties to an area where the medium behaves differently. The frequency is now fixed,

because otherwise the two portions of the wave would otherwise get out of step, causing a kink or discontinuity at the boundary, which would be unphysical. (A more careful argument is that a kink or discontinuity would have infinite curvature, and waves tend to flatten out their curvature. An infinite curvature would flatten out infinitely fast, i.e., it could never occur in the first place.) Since the frequency must stay the same, any change in the velocity that results from the new medium must cause a change in wavelength.

The velocity of water waves depends on the depth of the water, so based on  $\lambda = v/f$ , we see that water waves that move into a region of different depth must change their wavelength, as shown in the figure on the left. This effect can be observed when ocean waves come up to the shore. If the deceleration of the wave pattern is sudden enough, the tip of the wave can curl over, resulting in a breaking wave.

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**A note on dispersive waves**

The discussion of wave velocity given here is actually an oversimplification for a wave whose velocity depends on its frequency and wavelength. Such a wave is called a dispersive wave. Nearly all the waves we deal with in this course are non-dispersive, but the issue becomes important in quantum physics, as discussed in more detail in optional section 35.2.

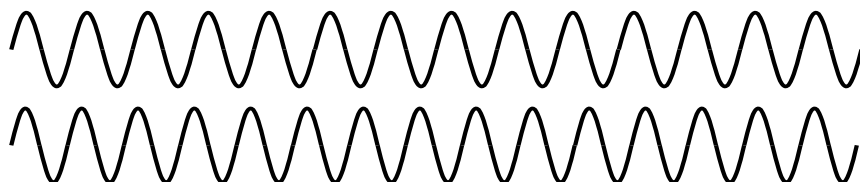
## Sinusoidal waves

Sinusoidal waves are the most important special case of periodic waves. In fact, many scientists and engineers would be uncomfortable with defining a waveform like the “ah” vowel sound as having a definite frequency and wavelength, because they consider only sine waves to be pure examples of a certain frequency and wavelengths. Their bias is not unreasonable, since the French mathematician Fourier showed that any periodic wave with frequency  $f$  can be constructed as a superposition of sine waves with frequencies  $f, 2f, 3f, \dots$ . In this sense, sine waves are the basic, pure building blocks of all waves. (Fourier’s result so surprised the mathematical community of France that he was ridiculed the first time he publicly presented his theorem.)

However, what definition to use is a matter of utility. Our sense of hearing perceives any two sounds having the same period as possessing the same pitch, regardless of whether they are sine waves or not. This is undoubtedly because our ear-brain system evolved to be able to interpret human speech and animal noises, which are periodic but not sinusoidal. Our eyes, on the other hand, judge a color as pure (belonging to the rainbow set of colors) only if it is a sine wave.

### Discussion question

**A** Suppose we superimpose two sine waves with equal amplitudes but slightly different frequencies, as shown in the figure. What will the superposition look like? What would this sound like if they were sound waves?



## 19.5 The Doppler effect

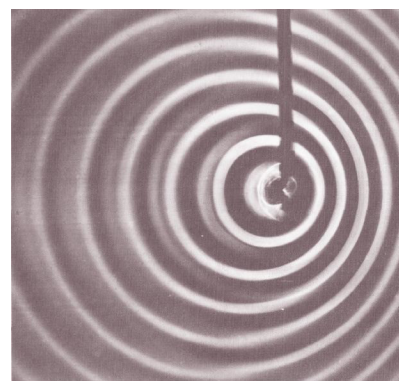
Figure v shows the wave pattern made by the tip of a vibrating rod which is moving across the water. If the rod had been vibrating in one place, we would have seen the familiar pattern of concentric circles, all centered on the same point. But since the source of the waves is moving, the wavelength is shortened on one side and lengthened on the other. This is known as the Doppler effect.

Note that the velocity of the waves is a fixed property of the medium, so for example the forward-going waves do not get an extra boost in speed as would a material object like a bullet being shot forward from an airplane.

We can also infer a change in frequency. Since the velocity is constant, the equation  $v = f\lambda$  tells us that the change in wavelength must be matched by an opposite change in frequency: higher frequency for the waves emitted forward, and lower for the ones emitted backward. The frequency Doppler effect is the reason for the familiar dropping-pitch sound of a race car going by. As the car approaches us, we hear a higher pitch, but after it passes us we hear a frequency that is lower than normal.

The Doppler effect will also occur if the observer is moving but the source is stationary. For instance, an observer moving toward a stationary source will perceive one crest of the wave, and will then be surrounded by the next crest sooner than she otherwise would have, because she has moved toward it and hastened her encounter with it. Roughly speaking, the Doppler effect depends only the relative motion of the source and the observer, not on their absolute state of motion (which is not a well-defined notion in physics) or on their velocity relative to the medium.

Restricting ourselves to the case of a moving source, and to waves emitted either directly along or directly against the direction of motion, we can easily calculate the wavelength, or equivalently the frequency, of the Doppler-shifted waves. Let  $v$  be the velocity of the waves, and  $v_s$  the velocity of the source. The wavelength of the



v / The pattern of waves made by a point source moving to the right across the water. Note the shorter wavelength of the forward-emitted waves and the longer wavelength of the backward-going ones.

forward-emitted waves is shortened by an amount  $v_s T$  equal to the distance traveled by the source over the course of one period. Using the definition  $f = 1/T$  and the equation  $v = f\lambda$ , we find for the wavelength of the Doppler-shifted wave the equation

$$\lambda' = \left(1 - \frac{v_s}{v}\right) \lambda \quad .$$

A similar equation can be used for the backward-emitted waves, but with a plus sign rather than a minus sign.

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*Doppler-shifted sound from a race car* *example 6*

▷ If a race car moves at a velocity of 50 m/s, and the velocity of sound is 340 m/s, by what percentage are the wavelength and frequency of its sound waves shifted for an observer lying along its line of motion?

▷ For an observer whom the car is approaching, we find

$$1 - \frac{v_s}{v} = 0.85 \quad ,$$

so the shift in wavelength is 15%. Since the frequency is inversely proportional to the wavelength for a fixed value of the speed of sound, the frequency is shifted upward by

$$1/0.85 = 1.18,$$

i.e., a change of 18%. (For velocities that are small compared to the wave velocities, the Doppler shifts of the wavelength and frequency are about the same.)

---

*Doppler shift of the light emitted by a race car* *example 7*

▷ What is the percent shift in the wavelength of the light waves emitted by a race car's headlights?

▷ Looking up the speed of light in the front of the book,  $v = 3.0 \times 10^8$  m/s, we find

$$1 - \frac{v_s}{v} = 0.99999983 \quad ,$$

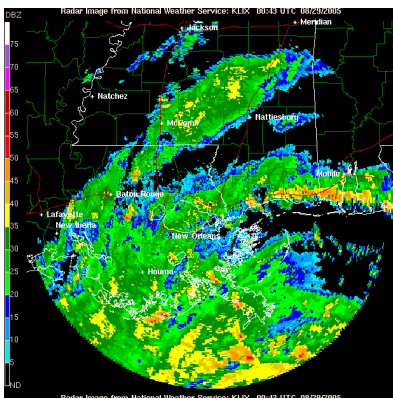
i.e., the percentage shift is only 0.000017%.

The second example shows that under ordinary earthbound circumstances, Doppler shifts of light are negligible because ordinary things go so much slower than the speed of light. It's a different story, however, when it comes to stars and galaxies, and this leads us to a story that has profound implications for our understanding of the origin of the universe.

---

*Doppler radar* *example 8*

The first use of radar was by Britain during World War II: antennas on the ground sent radio waves up into the sky, and detected the echoes when the waves were reflected from German planes.



w / Example 8. A Doppler radar image of Hurricane Katrina, in 2005.



Later, air forces wanted to mount radar antennas on airplanes, but then there was a problem, because if an airplane wanted to detect another airplane at a lower altitude, it would have to aim its radio waves downward, and then it would get echoes from the ground. The solution was the invention of Doppler radar, in which echoes from the ground were differentiated from echoes from other aircraft according to their Doppler shifts. A similar technology is used by meteorologists to map out rainclouds without being swamped by reflections from the ground, trees, and buildings.

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**Optional topic: Doppler shifts of light**

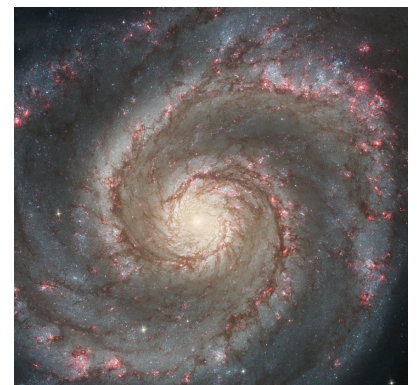
If Doppler shifts depend only on the relative motion of the source and receiver, then there is no way for a person moving with the source and another person moving with the receiver to determine who is moving and who isn't. Either can blame the Doppler shift entirely on the other's motion and claim to be at rest herself. This is entirely in agreement with the principle stated originally by Galileo that all motion is relative.

On the other hand, a careful analysis of the Doppler shifts of water or sound waves shows that it is only approximately true, at low speeds, that the shifts just depend on the relative motion of the source and observer. For instance, it is possible for a jet plane to keep up with its own sound waves, so that the sound waves appear to stand still to the pilot of the plane. The pilot then knows she is moving at exactly the speed of sound. The reason this doesn't disprove the relativity of motion is that the pilot is not really determining her absolute motion but rather her motion relative to the air, which is the medium of the sound waves.

Einstein realized that this solved the problem for sound or water waves, but would not salvage the principle of relative motion in the case of light waves, since light is not a vibration of any physical medium such as water or air. Beginning by imagining what a beam of light would look like to a person riding a motorcycle alongside it, Einstein eventually came up with a radical new way of describing the universe, in which space and time are distorted as measured by observers in different states of motion. As a consequence of this theory of relativity, he showed that light waves would have Doppler shifts that would exactly, not just approximately, depend only on the relative motion of the source and receiver. The resolution of the motorcycle paradox is given in example refeg:einstein-motorcycle on p. 666, and a quantitative discussion of Doppler shifts of light is given on p. 669.

## The Big Bang

As soon as astronomers began looking at the sky through telescopes, they began noticing certain objects that looked like clouds in deep space. The fact that they looked the same night after night meant that they were beyond the earth's atmosphere. Not knowing what they really were, but wanting to sound official, they called them "nebulae," a Latin word meaning "clouds" but sounding more impressive. In the early 20th century, astronomers realized that although some really were clouds of gas (e.g., the middle "star" of Orion's sword, which is visibly fuzzy even to the naked eye when



x / The galaxy M51. Under high magnification, the milky clouds reveal themselves to be composed of trillions of stars.



y / How do astronomers know what mixture of wavelengths a star emitted originally, so that they can tell how much the Doppler shift was? This image (obtained by the author with equipment costing about \$5, and no telescope) shows the mixture of colors emitted by the star Sirius. (If you have the book in black and white, blue is on the left and red on the right.) The star appears white or bluish-white to the eye, but any light looks white if it contains roughly an equal mixture of the rainbow colors, i.e., of all the pure sinusoidal waves with wavelengths lying in the visible range. Note the black “gap teeth.” These are the fingerprint of hydrogen in the outer atmosphere of Sirius. These wavelengths are selectively absorbed by hydrogen. Sirius is in our own galaxy, but similar stars in other galaxies would have the whole pattern shifted toward the red end, indicating they are moving away from us.



z / The telescope at Mount Wilson used by Hubble.

conditions are good), others were what we now call galaxies: virtual island universes consisting of trillions of stars (for example the Andromeda Galaxy, which is visible as a fuzzy patch through binoculars). Three hundred years after Galileo had resolved the Milky Way into individual stars through his telescope, astronomers realized that the universe is made of galaxies of stars, and the Milky Way is simply the visible part of the flat disk of our own galaxy, seen from inside.

This opened up the scientific study of cosmology, the structure and history of the universe as a whole, a field that had not been seriously attacked since the days of Newton. Newton had realized that if gravity was always attractive, never repulsive, the universe would have a tendency to collapse. His solution to the problem was to posit a universe that was infinite and uniformly populated with matter, so that it would have no geometrical center. The gravitational forces in such a universe would always tend to cancel out by symmetry, so there would be no collapse. By the 20th century, the belief in an unchanging and infinite universe had become conventional wisdom in science, partly as a reaction against the time that had been wasted trying to find explanations of ancient geological phenomena based on catastrophes suggested by biblical events like Noah’s flood.

In the 1920’s astronomer Edwin Hubble began studying the Doppler shifts of the light emitted by galaxies. A former college football player with a serious nicotine addiction, Hubble did not set out to change our image of the beginning of the universe. His autobiography seldom even mentions the cosmological discovery for which he is now remembered. When astronomers began to study the Doppler shifts of galaxies, they expected that each galaxy’s direction and velocity of motion would be essentially random. Some would be approaching us, and their light would therefore be Doppler-shifted to the blue end of the spectrum, while an equal number would be expected to have red shifts. What Hubble discovered instead was that except for a few very nearby ones, all the galaxies had red shifts, indicating that they were receding from us at a hefty fraction of the speed of light. Not only that, but the ones farther away were receding more quickly. The speeds were directly proportional to their distance from us.

Did this mean that the earth (or at least our galaxy) was the center of the universe? No, because Doppler shifts of light only depend on the relative motion of the source and the observer. If we see a distant galaxy moving away from us at 10% of the speed of light, we can be assured that the astronomers who live in that galaxy will see ours receding from them at the same speed in the opposite direction. The whole universe can be envisioned as a rising loaf of raisin bread. As the bread expands, there is more and more space between the raisins. The farther apart two raisins are, the

greater the speed with which they move apart.

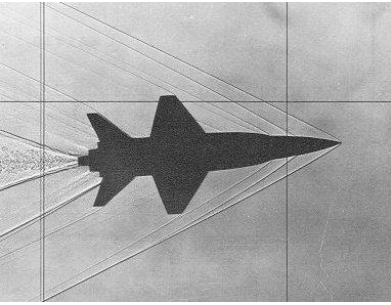
Extrapolating backward in time using the known laws of physics, the universe must have been denser and denser at earlier and earlier times. At some point, it must have been extremely dense and hot, and we can even detect the radiation from this early fireball, in the form of microwave radiation that permeates space. The phrase Big Bang was originally coined by the doubters of the theory to make it sound ridiculous, but it stuck, and today essentially all astronomers accept the Big Bang theory based on the very direct evidence of the red shifts and the cosmic microwave background radiation.

### **What the Big Bang is not**

Finally it should be noted what the Big Bang theory is not. It is not an explanation of *why* the universe exists. Such questions belong to the realm of religion, not science. Science can find ever simpler and ever more fundamental explanations for a variety of phenomena, but ultimately science takes the universe as it is according to observations.

Furthermore, there is an unfortunate tendency, even among many scientists, to speak of the Big Bang theory as a description of the very first event in the universe, which caused everything after it. Although it is true that time may have had a beginning (Einstein's theory of general relativity admits such a possibility), the methods of science can only work within a certain range of conditions such as temperature and density. Beyond a temperature of about  $10^9$  degrees C, the random thermal motion of subatomic particles becomes so rapid that its velocity is comparable to the speed of light. Early enough in the history of the universe, when these temperatures existed, Newtonian physics becomes less accurate, and we must describe nature using the more general description given by Einstein's theory of relativity, which encompasses Newtonian physics as a special case. At even higher temperatures, beyond about  $10^{33}$  degrees, physicists know that Einstein's theory as well begins to fall apart, but we don't know how to construct the even more general theory of nature that would work at those temperatures. No matter how far physics progresses, we will never be able to describe nature at infinitely high temperatures, since there is a limit to the temperatures we can explore by experiment and observation in order to guide us to the right theory. We are confident that we understand the basic physics involved in the evolution of the universe starting a few minutes after the Big Bang, and we may be able to push back to milliseconds or microseconds after it, but we cannot use the methods of science to deal with the beginning of time itself.





aa / Shock waves from by the X-15 rocket plane, flying at 3.5 times the speed of sound.



ab / As in figure aa, this plane shows a shock wave. The sudden decompression of the air causes water droplets to condense, forming a cloud.

### Discussion questions

**A** If an airplane travels at exactly the speed of sound, what would be the wavelength of the forward-emitted part of the sound waves it emitted? How should this be interpreted, and what would actually happen? What happens if it's going faster than the speed of sound? Can you use this to explain what you see in figure aa?

**B** If bullets go slower than the speed of sound, why can a supersonic fighter plane catch up to its own sound, but not to its own bullets?

**C** If someone inside a plane is talking to you, should their speech be Doppler shifted?

**D** The plane in figure ab was photographed when it was traveling at a speed close to the speed of sound. Comparing figures aa and ab, how can we tell from the angles of the cones that the speed is much lower in figure ab?

## Summary

### Selected vocabulary

superposition . .	the adding together of waves that overlap with each other
medium . . . . .	a physical substance whose vibrations constitute a wave
wavelength . . . .	the distance in space between repetitions of a periodic wave
Doppler effect . .	the change in a wave's frequency and wavelength due to the motion of the source or the observer or both

### Notation

$\lambda$ . . . . .	wavelength (Greek letter lambda)
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### Summary

Wave motion differs in three important ways from the motion of material objects:

(1) Waves obey the principle of superposition. When two waves collide, they simply add together.

(2) The medium is not transported along with the wave. The motion of any given point in the medium is a vibration about its equilibrium location, not a steady forward motion.

(3) The velocity of a wave depends on the medium, not on the amount of energy in the wave. (For some types of waves, notably water waves, the velocity may also depend on the shape of the wave.)

Sound waves consist of increases and decreases (typically very small ones) in the density of the air. Light is a wave, but it is a vibration of electric and magnetic fields, not of any physical medium. Light can travel through a vacuum.

A periodic wave is one that creates a periodic motion in a receiver as it passes it. Such a wave has a well-defined period and frequency, and it will also have a wavelength, which is the distance in space between repetitions of the wave pattern. The velocity, frequency, and wavelength of a periodic wave are related by the equation

$$v = f\lambda.$$

A wave emitted by a moving source will be shifted in wavelength and frequency. The shifted wavelength is given by the equation

$$\lambda' = \left(1 - \frac{v_s}{v}\right) \lambda \quad ,$$

where  $v$  is the velocity of the waves and  $v_s$  is the velocity of the source, taken to be positive or negative so as to produce a Doppler-lengthened wavelength if the source is receding and a Doppler-shortened one if it approaches. A similar shift occurs if the observer

is moving, and in general the Doppler shift depends approximately only on the relative motion of the source and observer if their velocities are both small compared to the waves' velocity. (This is not just approximately but exactly true for light waves, and as required by Einstein's Theory of Relativity.)

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

1 The following is a graph of the height of a water wave as a function of *position*, at a certain moment in time.



Trace this graph onto another piece of paper, and then sketch below it the corresponding graphs that would be obtained if

- (a) the amplitude and frequency were doubled while the velocity remained the same;
- (b) the frequency and velocity were both doubled while the amplitude remained unchanged;
- (c) the wavelength and amplitude were reduced by a factor of three while the velocity was doubled.

*Explain all your answers.* [Problem by Arnold Arons.]

2 (a) The graph shows the height of a water wave pulse as a function of position. Draw a graph of height as a function of time for a specific point on the water. Assume the pulse is traveling to the right.



Problem 2.

- (b) Repeat part a, but assume the pulse is traveling to the left.
- (c) Now assume the original graph was of height as a function of time, and draw a graph of height as a function of position, assuming the pulse is traveling to the right.
- (d) Repeat part c, but assume the pulse is traveling to the left.

*Explain all your answers.* [Problem by Arnold Arons.]

3 The figure shows one wavelength of a steady sinusoidal wave traveling to the right along a string. Define a coordinate system in which the positive  $x$  axis points to the right and the positive  $y$  axis up, such that the flattened string would have  $y = 0$ . Copy the figure, and label with  $y = 0$  all the appropriate parts of the string. Similarly, label with  $v = 0$  all parts of the string whose velocities are zero, and with  $a = 0$  all parts whose accelerations are zero. There is more than one point whose velocity is of the greatest magnitude. Pick one of these, and indicate the direction of its velocity vector. Do the same for a point having the maximum magnitude of acceleration. Explain all your answers.



Problem 3.

[Problem by Arnold Arons.]

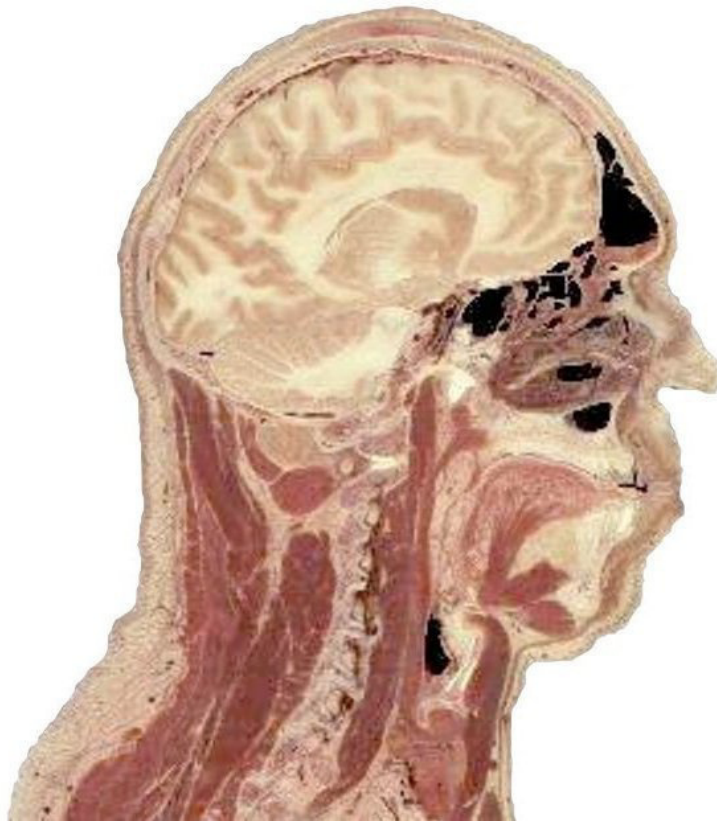
**4** Find an equation for the relationship between the Doppler-shifted frequency of a wave and the frequency of the original wave, for the case of a stationary observer and a source moving directly toward or away from the observer.

**5** Suggest a quantitative experiment to look for any deviation from the principle of superposition for surface waves in water. Make it simple and practical.

**6** The musical note middle C has a frequency of 262 Hz. What are its period and wavelength? ✓

**7** Singing that is off-pitch by more than about 1% sounds bad. How fast would a singer have to be moving relative to the rest of a band to make this much of a change in pitch due to the Doppler effect?

**8** In section 19.2, we saw that the speed of waves on a string depends on the ratio of  $T/\mu$ , i.e., the speed of the wave is greater if the string is under more tension, and less if it has more inertia. This is true in general: the speed of a mechanical wave always depends on the medium's inertia in relation to the restoring force (tension, stiffness, resistance to compression,...) Based on these ideas, explain why the speed of sound in air is significantly greater on a hot day, while the speed of sound in liquids and solids shows almost no variation with temperature.



A cross-sectional view of a human body, showing the vocal tract.

## Chapter 20

# Bounded waves

Speech is what separates humans most decisively from animals. No other species can master syntax, and even though chimpanzees can learn a vocabulary of hand signs, there is an unmistakable difference between a human infant and a baby chimp: starting from birth, the human experiments with the production of complex speech sounds.

Since speech sounds are instinctive for us, we seldom think about them consciously. How do we do control sound waves so skillfully? Mostly we do it by changing the shape of a connected set of hollow cavities in our chest, throat, and head. Somehow by moving the boundaries of this space in and out, we can produce all the vowel sounds. Up until now, we have been studying only those properties of waves that can be understood as if they existed in an infinite, open space. In this chapter we address what happens when a wave is confined within a certain space, or when a wave pattern encounters the boundary between two different media, as when a light wave moving through air encounters a glass windowpane.

a / A diver photographed this fish, and its reflection, from underwater. The reflection is the one on top, and is formed by light waves that went up to the surface of the water, but were then reflected back down into the water.



## 20.1 Reflection, transmission, and absorption

### Reflection and transmission

Sound waves can echo back from a cliff, and light waves are reflected from the surface of a pond. We use the word reflection, normally applied only to light waves in ordinary speech, to describe any such case of a wave rebounding from a barrier. Figure b shows a circular water wave being reflected from a straight wall. In this chapter, we will concentrate mainly on reflection of waves that move in one dimension, as in figure c.

Wave reflection does not surprise us. After all, a material object such as a rubber ball would bounce back in the same way. But waves are not objects, and there are some surprises in store.

First, only part of the wave is usually reflected. Looking out through a window, we see light waves that passed through it, but a person standing outside would also be able to see her reflection in the glass. A light wave that strikes the glass is partly *reflected* and partly *transmitted* (passed) by the glass. The energy of the original wave is split between the two. This is different from the behavior of the rubber ball, which must go one way or the other, not both.

Second, consider what you see if you are swimming underwater and you look up at the surface. You see your own reflection. This is utterly counterintuitive, since we would expect the light waves to burst forth to freedom in the wide-open air. A material projectile shot up toward the surface would never rebound from the water-air

boundary! Figure a shows a similar example.

What is it about the difference between two media that causes waves to be partly reflected at the boundary between them? Is it their density? Their chemical composition? Ultimately all that matters is the speed of the wave in the two media. *A wave is partially reflected and partially transmitted at the boundary between media in which it has different speeds.* For example, the speed of light waves in window glass is about 30% less than in air, which explains why windows always make reflections. Figures d/1 and 2 show examples of wave pulses being reflected at the boundary between two coil springs of different weights, in which the wave speed is different.

Reflections such as b and c, where a wave encounters a massive fixed object, can usually be understood on the same basis as cases like d/1 and 2 later in this section, where two media meet. Example c, for instance, is like a more extreme version of example d/1. If the heavy coil spring in d/1 was made heavier and heavier, it would end up acting like the fixed wall to which the light spring in c has been attached.

#### self-check A

In figure c, the reflected pulse is upside-down, but its depth is just as big as the original pulse's height. How does the energy of the reflected pulse compare with that of the original? ▷ Answer, p. 535

#### Fish have internal ears.

#### example 1

Why don't fish have ear-holes? The speed of sound waves in a fish's body is not much different from their speed in water, so sound waves are not strongly reflected from a fish's skin. They pass right through its body, so fish can have internal ears.

#### Whale songs traveling long distances

#### example 2

Sound waves travel at drastically different speeds through rock, water, and air. Whale songs are thus strongly reflected at both the bottom and the surface. The sound waves can travel hundreds of miles, bouncing repeatedly between the bottom and the surface, and still be detectable. Sadly, noise pollution from ships has nearly shut down this cetacean version of the internet.

#### Long-distance radio communication.

#### example 3

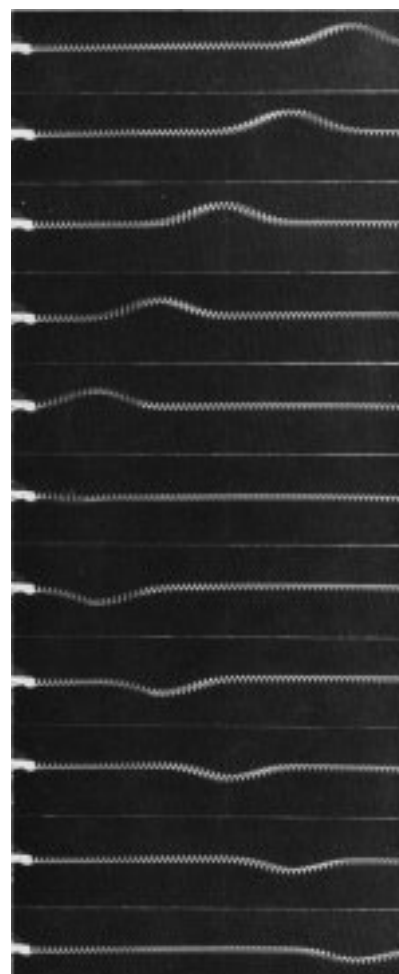
Radio communication can occur between stations on opposite sides of the planet. The mechanism is similar to the one explained in example 2, but the three media involved are the earth, the atmosphere, and the ionosphere.

#### self-check B

Sonar is a method for ships and submarines to detect each other by producing sound waves and listening for echoes. What properties would an underwater object have to have in order to be invisible to sonar? ▷ Answer, p. 535



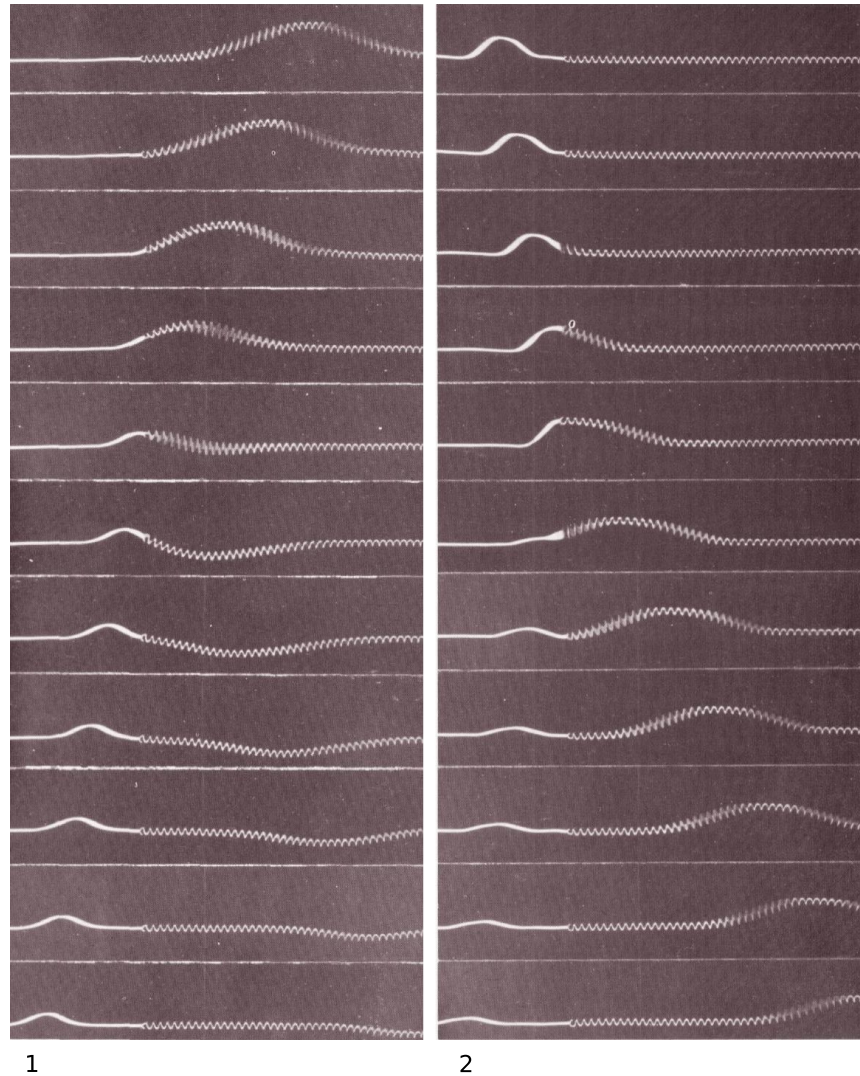
b / Circular water waves are reflected from a boundary on the left.



c / A wave on a spring, initially traveling to the left, is reflected from the fixed end.



The use of the word “reflection” naturally brings to mind the creation of an image by a mirror, but this might be confusing, because we do not normally refer to “reflection” when we look at surfaces that are not shiny. Nevertheless, reflection is how we see the surfaces of all objects, not just polished ones. When we look at a sidewalk, for example, we are actually seeing the reflecting of the sun from the concrete. The reason we don’t see an image of the sun at our feet is simply that the rough surface blurs the image so drastically.



d / 1. A wave in the lighter spring, where the wave speed is greater, travels to the left and is then partly reflected and partly transmitted at the boundary with the heavier coil spring, which has a lower wave speed. The reflection is inverted. 2. A wave moving to the right in the heavier spring is partly reflected at the boundary with the lighter spring. The reflection is uninverted.

## Inverted and uninverted reflections

Notice how the pulse reflected back to the right in example d/1 comes back upside-down, whereas the one reflected back to the left in 2 returns in its original upright form. This is true for other waves as well. In general, there are two possible types of reflections, a reflection back into a faster medium and a reflection back into a slower medium. One type will always be an inverting reflection and one noninverting.

It's important to realize that when we discuss inverted and uninverted reflections on a string, we are talking about whether the wave is flipped across the direction of motion (i.e., upside-down in these drawings). The reflected pulse will always be reversed front to back, as shown in figure e. This is because it is traveling in the other direction. The leading edge of the pulse is what gets reflected first, so it is still ahead when it starts back to the left — it's just that “ahead” is now in the opposite direction.

## Absorption

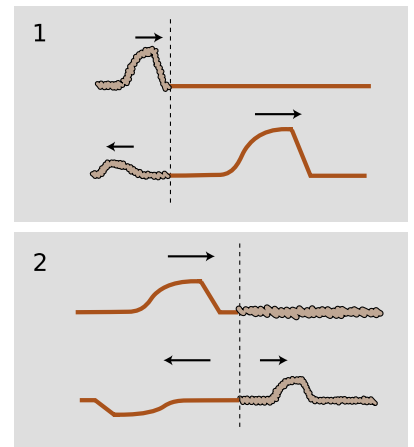
So far we have tacitly assumed that wave energy remains as wave energy, and is not converted to any other form. If this was true, then the world would become more and more full of sound waves, which could never escape into the vacuum of outer space. In reality, any mechanical wave consists of a traveling pattern of vibrations of some physical medium, and vibrations of matter always produce heat, as when you bend a coat-hanger back and forth and it becomes hot. We can thus expect that in mechanical waves such as water waves, sound waves, or waves on a string, the wave energy will gradually be converted into heat. This is referred to as *absorption*.

The wave suffers a decrease in amplitude, as shown in figure f. The decrease in amplitude amounts to the same fractional change for each unit of distance covered. For example, if a wave decreases from amplitude 2 to amplitude 1 over a distance of 1 meter, then after traveling another meter it will have an amplitude of  $1/2$ . That is, the reduction in amplitude is exponential. This can be proven as follows. By the principle of superposition, we know that a wave of amplitude 2 must behave like the superposition of two identical waves of amplitude 1. If a single amplitude-1 wave would die down to amplitude  $1/2$  over a certain distance, then two amplitude-1 waves superposed on top of one another to make amplitude  $1 + 1 = 2$  must die down to amplitude  $1/2 + 1/2 = 1$  over the same distance.

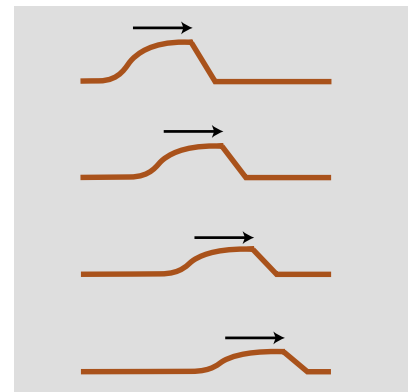
### self-check C

As a wave undergoes absorption, it loses energy. Does this mean that it slows down? ▷ Answer, p. 535

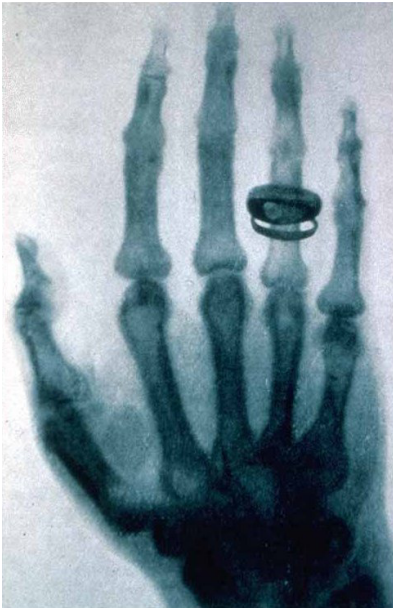
In many cases, this frictional heating effect is quite weak. Sound waves in air, for instance, dissipate into heat extremely slowly, and the sound of church music in a cathedral may reverberate for as much



e / 1. An uninverted reflection. The reflected pulse is reversed front to back, but is not upside-down. 2. An inverted reflection. The reflected pulse is reversed both front to back and top to bottom.



f / A pulse traveling through a highly absorptive medium.



g / X-rays are light waves with a very high frequency. They are absorbed strongly by bones, but weakly by flesh.

as 3 or 4 seconds before it becomes inaudible. During this time it has traveled over a kilometer! Even this very gradual dissipation of energy occurs mostly as heating of the church's walls and by the leaking of sound to the outside (where it will eventually end up as heat). Under the right conditions (humid air and low frequency), a sound wave in a straight pipe could theoretically travel hundreds of kilometers before being noticeably attenuated.

In general, the absorption of mechanical waves depends a great deal on the chemical composition and microscopic structure of the medium. Ripples on the surface of antifreeze, for instance, die out extremely rapidly compared to ripples on water. For sound waves and surface waves in liquids and gases, what matters is the viscosity of the substance, i.e., whether it flows easily like water or mercury or more sluggishly like molasses or antifreeze. This explains why our intuitive expectation of strong absorption of sound in water is incorrect. Water is a very weak absorber of sound (viz. whale songs and sonar), and our incorrect intuition arises from focusing on the wrong property of the substance: water's high density, which is irrelevant, rather than its low viscosity, which is what matters.

Light is an interesting case, since although it can travel through matter, it is not itself a vibration of any material substance. Thus we can look at the star Sirius,  $10^{14}$  km away from us, and be assured that none of its light was absorbed in the vacuum of outer space during its 9-year journey to us. The Hubble Space Telescope routinely observes light that has been on its way to us since the early history of the universe, billions of years ago. Of course the energy of light can be dissipated if it does pass through matter (and the light from distant galaxies is often absorbed if there happen to be clouds of gas or dust in between).

#### *Soundproofing*

#### *example 4*

Typical amateur musicians setting out to soundproof their garages tend to think that they should simply cover the walls with the densest possible substance. In fact, sound is not absorbed very strongly even by passing through several inches of wood. A better strategy for soundproofing is to create a sandwich of alternating layers of materials in which the speed of sound is very different, to encourage reflection.

The classic design is alternating layers of fiberglass and plywood. The speed of sound in plywood is very high, due to its stiffness, while its speed in fiberglass is essentially the same as its speed in air. Both materials are fairly good sound absorbers, but sound waves passing through a few inches of them are still not going to be absorbed sufficiently. The point of combining them is that a sound wave that tries to get out will be strongly reflected at each of the fiberglass-plywood boundaries, and will bounce back and forth many times like a ping pong ball. Due to all the back-

and-forth motion, the sound may end up traveling a total distance equal to ten times the actual thickness of the soundproofing before it escapes. This is the equivalent of having ten times the thickness of sound-absorbing material.

#### *The swim bladder*

#### *example 5*

The swim bladder of a fish, which was first discussed in homework problem 2 in chapter 18, is often located right next to the fish's ear. As discussed in example 1 on page 493, the fish's body is nearly transparent to sound, so it's actually difficult to get any of the sound wave energy to deposit itself in the fish so that the fish can hear it! The physics here is almost exactly the same as the physics of example 4 above, with the gas-filled swim bladder playing the role of the low-density material.

#### *Radio transmission*

#### *example 6*

A radio transmitting station, such as a commercial station or an amateur "ham" radio station, must have a length of wire or cable connecting the amplifier to the antenna. The cable and the antenna act as two different media for radio waves, and there will therefore be partial reflection of the waves as they come from the cable to the antenna. If the waves bounce back and forth many times between the amplifier and the antenna, a great deal of their energy will be absorbed. There are two ways to attack the problem. One possibility is to design the antenna so that the speed of the waves in it is as close as possible to the speed of the waves in the cable; this minimizes the amount of reflection. The other method is to connect the amplifier to the antenna using a type of wire or cable that does not strongly absorb the waves. Partial reflection then becomes irrelevant, since all the wave energy will eventually exit through the antenna.

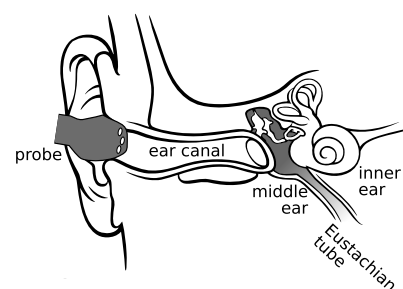
#### *The tympanogram*

#### *example 7*

The tympanogram is a medical procedure used to diagnose problems with the middle ear.

The middle ear is a chamber, normally filled with air, lying between the eardrum (tympanic membrane) and the inner ear. It contains a tiny set of bones that act as a system of levers to amplify the motion of the eardrum and transmit it to the inner ear. The air pressure in the inner ear is normally equalized via the Eustachian tube, which connects to the throat; when you feel uncomfortable pressure in your ear while flying, it's because the pressure has not yet equalized. Ear infections or allergies can cause the middle ear to become filled with fluid, and the Eustachian tube can also become blocked, so that the pressure in the inner ear cannot become equalized.

The tympanometer has a probe that is inserted into the ear, with several holes. One hole is used to send a 226 Hz sound wave into



h / A tympanometer, example 7.

the ear canal. The ear has evolved so as to transmit a maximum amount of wave motion to the inner ear. Any change in its physical properties will change its behavior from its normal optimum, so that more sound energy than normal is reflected back. A second hole in the probe senses the reflected wave. If the reflection is stronger than normal, there is probably something wrong with the inner ear.

The full physical analysis is fairly complex. The middle ear has some of the characteristics of a mass oscillating on a spring, but it also has some of the characteristics of a medium that carries waves. Crudely, we could imagine that an infected, fluid-filled middle ear would act as a medium that differed greatly from the air in the outer ear, causing a large amount of reflection.

Equally crudely, we could forget about the wave ideas and think of the middle ear purely as a mass on a spring. We expect resonant behavior, and there is in fact such a resonance, which is typically at a frequency of about 600 Hz in adults, so the 226 Hz frequency emitted by the probe is actually quite far from resonance. If the mechanisms of the middle ear are jammed and cannot vibrate, then it is not possible for energy of the incoming sound wave to be turned into energy of vibration in the middle ear, and therefore by conservation of energy we would expect all of the sound to be reflected.

Sometimes the middle ear's mechanisms can get jammed because of abnormally high or low pressure, because the Eustachian tube is blocked and cannot equalize the pressure with the outside environment. Diagnosing such a condition is the purpose of the third hole in the probe, which is used to vary the pressure in the ear canal. The amount of reflection is measured as a function of this pressure. If the reflection is minimized for some value of the pressure that is different than atmospheric pressure, it indicates that that is the value of the pressure in the middle ear; when the pressures are equalized, the forces on the eardrum cancel out, and it can relax to its normal position, unjamming the middle ear's mechanisms.

### Discussion question

**A** A sound wave that underwent a pressure-inverting reflection would have its compressions converted to expansions and vice versa. How would its energy and frequency compare with those of the original sound? Would it sound any different? What happens if you swap the two wires where they connect to a stereo speaker, resulting in waves that vibrate in the opposite way?

## 20.2 ★ Quantitative treatment of reflection

In this optional section we analyze the reasons why reflections occur at a speed-changing boundary, predict quantitatively the intensities of reflection and transmission, and discuss how to predict for any type of wave which reflections are inverting and which are noninverting. The gory details are likely to be of interest mainly to students with concentrations in the physical sciences, but all readers are encouraged at least to skim the first two subsections for physical insight.

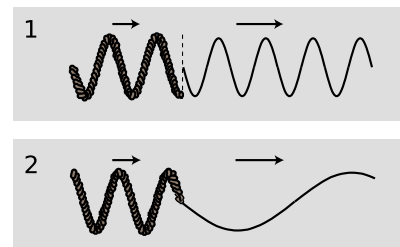
### Why reflection occurs

To understand the fundamental reasons for what does occur at the boundary between media, let's first discuss what doesn't happen. For the sake of concreteness, consider a sinusoidal wave on a string. If the wave progresses from a heavier portion of the string, in which its velocity is low, to a lighter-weight part, in which it is high, then the equation  $v = f\lambda$  tells us that it must change its frequency, or its wavelength, or both. If only the frequency changed, then the parts of the wave in the two different portions of the string would quickly get out of step with each other, producing a discontinuity in the wave,  $i/1$ . This is unphysical, so we know that the wavelength must change while the frequency remains constant, 2.

But there is still something unphysical about figure 2. The sudden change in the shape of the wave has resulted in a sharp kink at the boundary. This can't really happen, because the medium tends to accelerate in such a way as to eliminate curvature. A sharp kink corresponds to an infinite curvature at one point, which would produce an infinite acceleration, which would not be consistent with the smooth pattern of wave motion envisioned in figure 2. Waves can have kinks, but not stationary kinks.

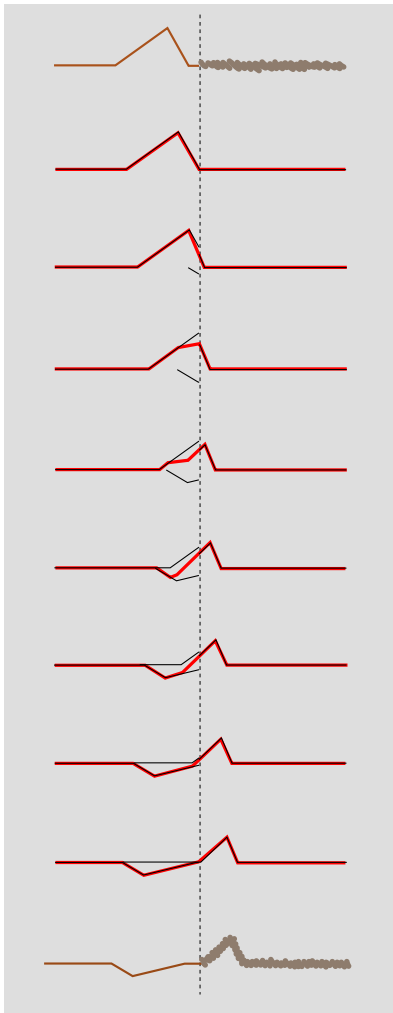
We conclude that without positing partial reflection of the wave, we cannot simultaneously satisfy the requirements of (1) continuity of the wave, and (2) no sudden changes in the slope of the wave. (The student who has studied calculus will recognize this as amounting to an assumption that both the wave and its derivative are continuous functions.)

Does this amount to a proof that reflection occurs? Not quite. We have only proven that certain types of wave motion are not valid solutions. In the following subsection, we prove that a valid solution can always be found in which a reflection occurs. Now in physics, we normally assume (but seldom prove formally) that the equations of motion have a unique solution, since otherwise a given set of initial conditions could lead to different behavior later on, but the Newtonian universe is supposed to be deterministic. Since the solution must be unique, and we derive below a valid solution involving a reflected pulse, we will have ended up with what amounts



i/1. A change in frequency without a change in wavelength would produce a discontinuity in the wave. 2. A simple change in wavelength without a reflection would result in a sharp kink in the wave.





j / A pulse being partially reflected and partially transmitted at the boundary between two strings in which the speed of waves is different. The top drawing shows the pulse heading to the right, toward the heavier string. For clarity, all but the first and last drawings are schematic. Once the reflected pulse begins to emerge from the boundary, it adds together with the trailing parts of the incident pulse. Their sum, shown as a wider line, is what is actually observed.

to a proof of reflection.

### Intensity of reflection

We will now show, in the case of waves on a string, that it is possible to satisfy the physical requirements given above by constructing a reflected wave, and as a bonus this will produce an equation for the proportions of reflection and transmission and a prediction as to which conditions will lead to inverted and which to uninverted reflection. We assume only that the principle of superposition holds, which is a good approximation for waves on a string of sufficiently small amplitude.

Let the unknown amplitudes of the reflected and transmitted waves be  $R$  and  $T$ , respectively. An inverted reflection would be represented by a negative value of  $R$ . We can without loss of generality take the incident (original) wave to have unit amplitude. Superposition tells us that if, for instance, the incident wave had double this amplitude, we could immediately find a corresponding solution simply by doubling  $R$  and  $T$ .

Just to the left of the boundary, the height of the wave is given by the height 1 of the incident wave, plus the height  $R$  of the part of the reflected wave that has just been created and begun heading back, for a total height of  $1 + R$ . On the right side immediately next to the boundary, the transmitted wave has a height  $T$ . To avoid a discontinuity, we must have

$$1 + R = T \quad .$$

Next we turn to the requirement of equal slopes on both sides of the boundary. Let the slope of the incoming wave be  $s$  immediately to the left of the junction. If the wave was 100% reflected, and without inversion, then the slope of the reflected wave would be  $-s$ , since the wave has been reversed in direction. In general, the slope of the reflected wave equals  $-sR$ , and the slopes of the superposed waves on the left side add up to  $s - sR$ . On the right, the slope depends on the amplitude,  $T$ , but is also changed by the stretching or compression of the wave due to the change in speed. If, for example, the wave speed is twice as great on the right side, then the slope is cut in half by this effect. The slope on the right is therefore  $s(v_1/v_2)T$ , where  $v_1$  is the velocity in the original medium and  $v_2$  the velocity in the new medium. Equality of slopes gives  $s - sR = s(v_1/v_2)T$ , or

$$1 - R = \frac{v_1}{v_2}T \quad .$$

Solving the two equations for the unknowns  $R$  and  $T$  gives

$$R = \frac{v_2 - v_1}{v_2 + v_1} \quad \text{and} \quad T = \frac{2v_2}{v_2 + v_1} \quad .$$

The first equation shows that there is no reflection unless the two wave speeds are different, and that the reflection is inverted in reflection back into a fast medium.

The energies of the transmitted and reflected waves always add up to the same as the energy of the original wave. There is never any abrupt loss (or gain) in energy when a wave crosses a boundary. (Conversion of wave energy to heat occurs for many types of waves, but it occurs throughout the medium.) The equation for  $T$ , surprisingly, allows the amplitude of the transmitted wave to be greater than 1, i.e., greater than that of the incident wave. This does not violate conservation of energy, because this occurs when the second string is less massive, reducing its kinetic energy, and the transmitted pulse is broader and less strongly curved, which lessens its potential energy.

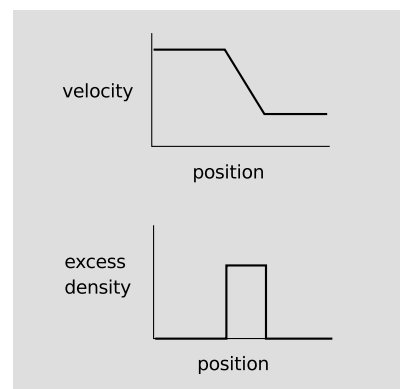
### Inverted and uninverted reflections in general

For waves on a string, reflections back into a faster medium are inverted, while those back into a slower medium are uninverted. Is this true for all types of waves? The rather subtle answer is that it depends on what property of the wave you are discussing.

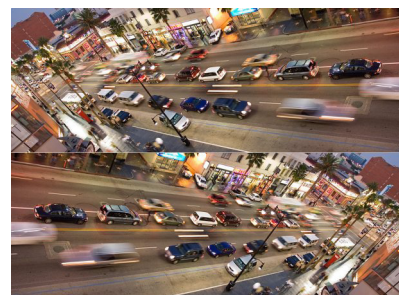
Let's start by considering wave disturbances of freeway traffic. Anyone who has driven frequently on crowded freeways has observed the phenomenon in which one driver taps the brakes, starting a chain reaction that travels backward down the freeway as each person in turn exercises caution in order to avoid rear-ending anyone. The reason why this type of wave is relevant is that it gives a simple, easily visualized example of our description of a wave depends on which aspect of the wave we have in mind. In steadily flowing freeway traffic, both the density of cars and their velocity are constant all along the road. Since there is no disturbance in this pattern of constant velocity and density, we say that there is no wave. Now if a wave is touched off by a person tapping the brakes, we can either describe it as a region of high density or as a region of decreasing velocity.

The freeway traffic wave is in fact a good model of a sound wave, and a sound wave can likewise be described either by the density (or pressure) of the air or by its speed. Likewise many other types of waves can be described by either of two functions, one of which is often the derivative of the other with respect to position.

Now let's consider reflections. If we observe the freeway wave in a mirror, the high-density area will still appear high in density, but velocity in the opposite direction will now be described by a negative number. A person observing the mirror image will draw the same density graph, but the velocity graph will be flipped across the  $x$  axis, and its original region of negative slope will now have positive slope. Although I don't know any physical situation that would



k / A disturbance in freeway traffic.



l / In the mirror image, the areas of positive excess traffic density are still positive, but the velocities of the cars have all been reversed, so areas of positive excess velocity have been turned into negative ones.



correspond to the reflection of a traffic wave, we can immediately apply the same reasoning to sound waves, which often do get reflected, and determine that a reflection can either be density-inverting and velocity-noninverting or density-noninverting and velocity-inverting.

This same type of situation will occur over and over as one encounters new types of waves, and to apply the analogy we need only determine which quantities, like velocity, become negated in a mirror image and which, like density, stay the same.

A light wave, for instance consists of a traveling pattern of electric and magnetic fields. All you need to know in order to analyze the reflection of light waves is how electric and magnetic fields behave under reflection; you don't need to know any of the detailed physics of electricity and magnetism. An electric field can be detected, for example, by the way one's hair stands on end. The direction of the hair indicates the direction of the electric field. In a mirror image, the hair points the other way, so the electric field is apparently reversed in a mirror image. The behavior of magnetic fields, however, is a little tricky. The magnetic properties of a bar magnet, for instance, are caused by the aligned rotation of the outermost orbiting electrons of the atoms. In a mirror image, the direction of rotation is reversed, say from clockwise to counterclockwise, and so the magnetic field is reversed twice: once simply because the whole picture is flipped and once because of the reversed rotation of the electrons. In other words, magnetic fields do not reverse themselves in a mirror image. We can thus predict that there will be two possible types of reflection of light waves. In one, the electric field is inverted and the magnetic field uninverted. In the other, the electric field is uninverted and the magnetic field inverted.

## 20.3 Interference effects



m / Seen from this angle, the optical coating on the lenses of these binoculars appears purple and green. (The color varies depending on the angle from which the coating is viewed, and the angle varies across the faces of the lenses because of their curvature.)

If you look at the front of a pair of high-quality binoculars, you will notice a greenish-blue coating on the lenses. This is advertised as a coating to prevent reflection. Now reflection is clearly undesirable — we want the light to go *in* the binoculars — but so far I've described reflection as an unalterable fact of nature, depending only on the properties of the two wave media. The coating can't change the speed of light in air or in glass, so how can it work? The key is that the coating itself is a wave medium. In other words, we have a three-layer sandwich of materials: air, coating, and glass. We will analyze the way the coating works, not because optical coatings are an important part of your education but because it provides a good example of the general phenomenon of wave interference effects.

There are two different interfaces between media: an air-coating boundary and a coating-glass boundary. Partial reflection and partial transmission will occur at each boundary. For ease of visual-

ization let's start by considering an equivalent system consisting of three dissimilar pieces of string tied together, and a wave pattern consisting initially of a single pulse. Figure n/1 shows the incident pulse moving through the heavy rope, in which its velocity is low. When it encounters the lighter-weight rope in the middle, a faster medium, it is partially reflected and partially transmitted. (The transmitted pulse is bigger, but nevertheless has only part of the original energy.) The pulse transmitted by the first interface is then partially reflected and partially transmitted by the second boundary, 3. In figure 4, two pulses are on the way back out to the left, and a single pulse is heading off to the right. (There is still a weak pulse caught between the two boundaries, and this will rattle back and forth, rapidly getting too weak to detect as it leaks energy to the outside with each partial reflection.)

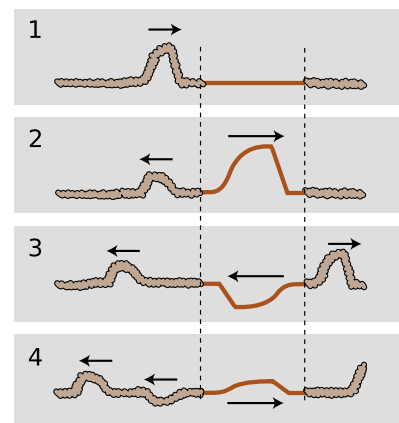
Note how, of the two reflected pulses in 4, one is inverted and one uninverted. One underwent reflection at the first boundary (a reflection back into a slower medium is uninverted), but the other was reflected at the second boundary (reflection back into a faster medium is inverted).

Now let's imagine what would have happened if the incoming wave pattern had been a long sinusoidal wave train instead of a single pulse. The first two waves to reemerge on the left could be in phase, o/1, or out of phase, 2, or anywhere in between. The amount of lag between them depends entirely on the width of the middle segment of string. If we choose the width of the middle string segment correctly, then we can arrange for destructive interference to occur, 2, with cancellation resulting in a very weak reflected wave.

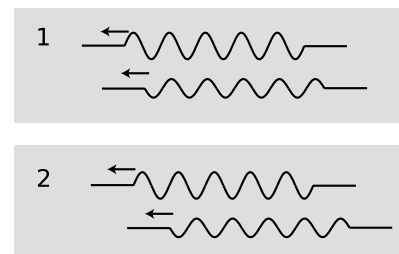
This whole analysis applies directly to our original case of optical coatings. Visible light from most sources does consist of a stream of short sinusoidal wave-trains such as the ones drawn above. The only real difference between the waves-on-a-rope example and the case of an optical coating is that the first and third media are air and glass, in which light does not have the same speed. However, the general result is the same as long as the air and the glass have light-wave speeds that either both greater than the coating's or both less than the coating's.

The business of optical coatings turns out to be a very arcane one, with a plethora of trade secrets and "black magic" techniques handed down from master to apprentice. Nevertheless, the ideas you have learned about waves in general are sufficient to allow you to come to some definite conclusions without any further technical knowledge. The self-check and discussion questions will direct you along these lines of thought.

The example of an optical coating was typical of a wide variety of wave interference effects. With a little guidance, you are now ready to figure out for yourself other examples such as the rainbow



n / A rope consisting of three sections, the middle one being lighter.



o / Two reflections, are superimposed. One reflection is inverted.



p / A soap bubble displays interference effects.

pattern made by a compact disc, a layer of oil on a puddle, or a soap bubble.

*self-check D*

1. Color corresponds to wavelength of light waves. Is it possible to choose a thickness for an optical coating that will produce destructive interference for all colors of light?

2. How can you explain the rainbow colors on the soap bubble in figure p?  
▷ Answer, p. 535

**Discussion questions**

**A** Is it possible to get *complete* destructive interference in an optical coating, at least for light of one specific wavelength?

**B** Sunlight consists of sinusoidal wave-trains containing on the order of a hundred cycles back-to-back, for a length of something like a tenth of a millimeter. What happens if you try to make an optical coating thicker than this?

**C** Suppose you take two microscope slides and lay one on top of the other so that one of its edges is resting on the corresponding edge of the bottom one. If you insert a sliver of paper or a hair at the opposite end, a wedge-shaped layer of air will exist in the middle, with a thickness that changes gradually from one end to the other. What would you expect to see if the slides were illuminated from above by light of a single color? How would this change if you gradually lifted the lower edge of the top slide until the two slides were finally parallel?

**D** An observation like the one described in discussion question C was used by Newton as evidence *against* the wave theory of light! If Newton didn't know about inverting and noninverting reflections, what would have seemed inexplicable to him about the region where the air layer had zero or nearly zero thickness?

## 20.4 Waves bounded on both sides

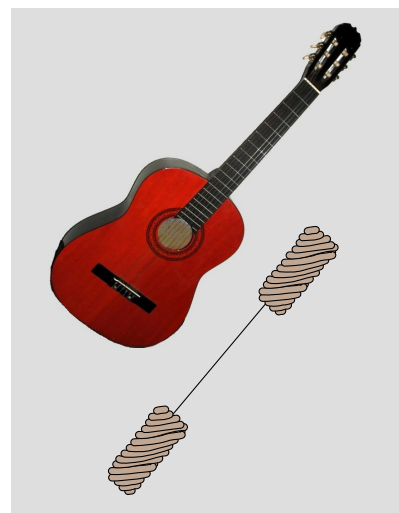
In the examples discussed in section 20.3, it was theoretically true that a pulse would be trapped permanently in the middle medium, but that pulse was not central to our discussion, and in any case it was weakening severely with each partial reflection. Now consider a guitar string. At its ends it is tied to the body of the instrument itself, and since the body is very massive, the behavior of the waves when they reach the end of the string can be understood in the same way as if the actual guitar string was attached on the ends to strings that were extremely massive,  $q$ . Reflections are most intense when the two media are very dissimilar. Because the wave speed in the body is so radically different from the speed in the string, we should expect nearly 100% reflection.

Although this may seem like a rather bizarre physical model of the actual guitar string, it already tells us something interesting about the behavior of a guitar that we would not otherwise have understood. The body, far from being a passive frame for attaching the strings to, is actually the exit path for the wave energy in the strings. With every reflection, the wave pattern on the string loses a tiny fraction of its energy, which is then conducted through the body and out into the air. (The string has too little cross-section to make sound waves efficiently by itself.) By changing the properties of the body, moreover, we should expect to have an effect on the manner in which sound escapes from the instrument. This is clearly demonstrated by the electric guitar, which has an extremely massive, solid wooden body. Here the dissimilarity between the two wave media is even more pronounced, with the result that wave energy leaks out of the string even more slowly. This is why an electric guitar with no electric pickup can hardly be heard at all, and it is also the reason why notes on an electric guitar can be sustained for longer than notes on an acoustic guitar.

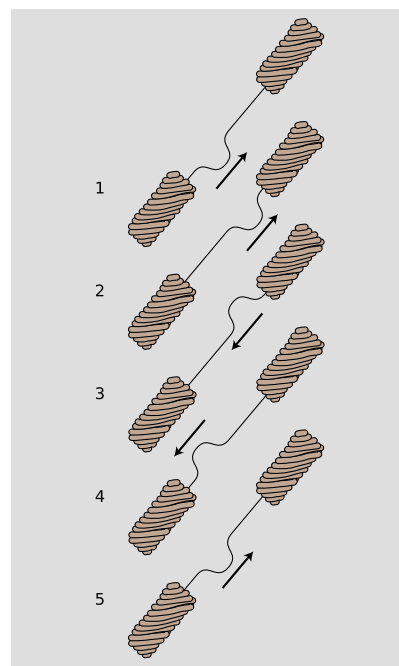
If we initially create a disturbance on a guitar string, how will the reflections behave? In reality, the finger or pick will give the string a triangular shape before letting it go, and we may think of this triangular shape as a very broad “dent” in the string which will spread out in both directions. For simplicity, however, let’s just imagine a wave pattern that initially consists of a single, narrow pulse traveling up the neck,  $r/1$ . After reflection from the top end, it is inverted,  $3$ . Now something interesting happens: figure 5 is identical to figure 1. After two reflections, the pulse has been inverted twice and has changed direction twice. It is now back where it started. The motion is periodic. This is why a guitar produces sounds that have a definite sensation of pitch.

### *self-check E*

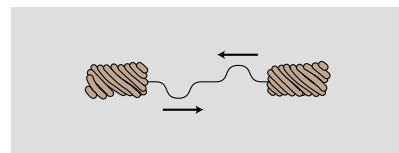
Notice that from  $r/1$  to  $r/5$ , the pulse has passed by every point on the string exactly twice. This means that the total distance it has traveled



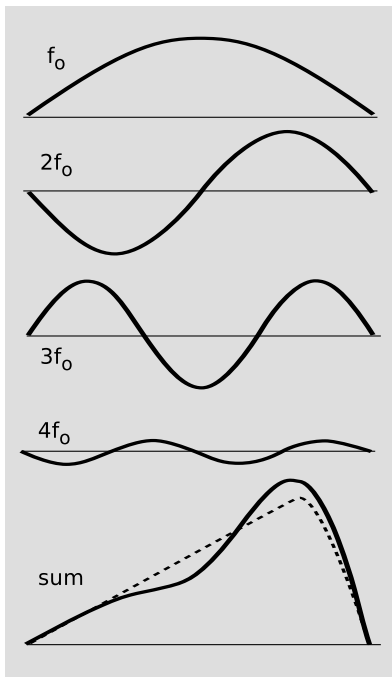
$q$  / A model of a guitar string.



$r$  / The motion of a pulse on the string.



$s$  / A tricky way to double the frequency.



t / Using the sum of four sine waves to approximate the triangular initial shape of a plucked guitar string.

equals  $2L$ , where  $L$  is the length of the string. Given this fact, what are the period and frequency of the sound it produces, expressed in terms of  $L$  and  $v$ , the velocity of the wave? ▷ Answer, p. 535

Note that if the waves on the string obey the principle of superposition, then the velocity must be independent of amplitude, and the guitar will produce the same pitch regardless of whether it is played loudly or softly. In reality, waves on a string obey the principle of superposition approximately, but not exactly. The guitar, like just about any acoustic instrument, is a little out of tune when played loudly. (The effect is more pronounced for wind instruments than for strings, but wind players are able to compensate for it.)

Now there is only one hole in our reasoning. Suppose we somehow arrange to have an initial setup consisting of two identical pulses heading toward each other, as in figure s. They will pass through each other, undergo a single inverting reflection, and come back to a configuration in which their positions have been exactly interchanged. This means that the period of vibration is half as long. The frequency is twice as high.

This might seem like a purely academic possibility, since nobody actually plays the guitar with two picks at once! But in fact it is an example of a very general fact about waves that are bounded on both sides. A mathematical theorem called Fourier's theorem states that any wave can be created by superposing sine waves. Figure t shows how even by using only four sine waves with appropriately chosen amplitudes, we can arrive at a sum which is a decent approximation to the realistic triangular shape of a guitar string being plucked. The one-hump wave, in which half a wavelength fits on the string, will behave like the single pulse we originally discussed. We call its frequency  $f_0$ . The two-hump wave, with one whole wavelength, is very much like the two-pulse example. For the reasons discussed above, its frequency is  $2f_0$ . Similarly, the three-hump and four-hump waves have frequencies of  $3f_0$  and  $4f_0$ .

Theoretically we would need to add together infinitely many such wave patterns to describe the initial triangular shape of the string exactly, although the amplitudes required for the very high frequency parts would be very small, and an excellent approximation could be achieved with as few as ten waves.

We thus arrive at the following very general conclusion. Whenever a wave pattern exists in a medium bounded on both sides by media in which the wave speed is very different, the motion can be broken down into the motion of a (theoretically infinite) series of sine waves, with frequencies  $f_0, 2f_0, 3f_0, \dots$  Except for some technical details, to be discussed below, this analysis applies to a vast range of sound-producing systems, including the air column within the human vocal tract. Because sounds composed of this kind of pattern of frequencies are so common, our ear-brain system has evolved so

as to perceive them as a single, fused sensation of tone.

## Musical applications

Many musicians claim to be able to pick out by ear several of the frequencies  $2f_o$ ,  $3f_o$ , ..., called overtones or *harmonics* of the fundamental  $f_o$ , but they are kidding themselves. In reality, the overtone series has two important roles in music, neither of which depends on this fictitious ability to “hear out” the individual overtones.

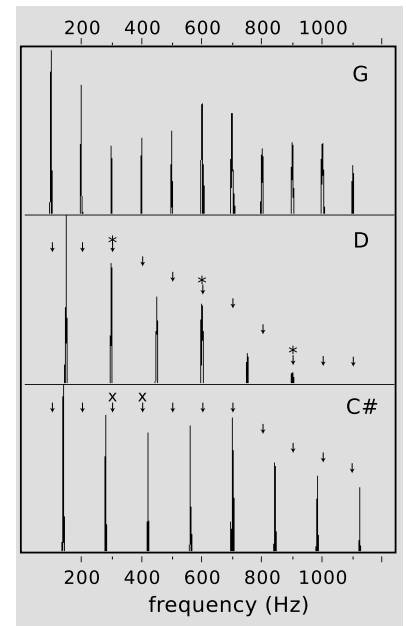
First, the relative strengths of the overtones is an important part of the personality of a sound, called its timbre (rhymes with “amber”). The characteristic tone of the brass instruments, for example, is a sound that starts out with a very strong harmonic series extending up to very high frequencies, but whose higher harmonics die down drastically as the attack changes to the sustained portion of the note.

Second, although the ear cannot separate the individual harmonics of a single musical tone, it is very sensitive to clashes between the overtones of notes played simultaneously, i.e., in harmony. We tend to perceive a combination of notes as being dissonant if they have overtones that are close but not the same. Roughly speaking, strong overtones whose frequencies differ by more than 1% and less than 10% cause the notes to sound dissonant. It is important to realize that the term “dissonance” is not a negative one in music. No matter how long you search the radio dial, you will never hear more than three seconds of music without at least one dissonant combination of notes. Dissonance is a necessary ingredient in the creation of a musical cycle of tension and release. Musically knowledgeable people don’t use the word “dissonant” as a criticism of music, although dissonance can be used in a clumsy way, or without providing any contrast between dissonance and consonance.

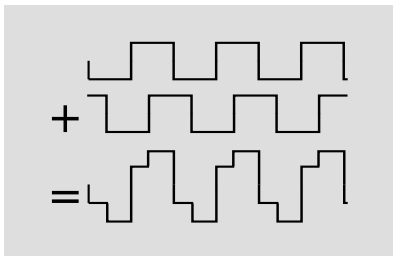
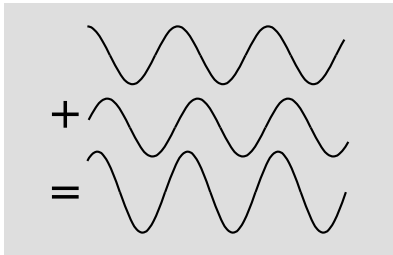
## Standing waves

Figure v shows sinusoidal wave patterns made by shaking a rope. I used to enjoy doing this at the bank with the pens on chains, back in the days when people actually went to the bank. You might think that I and the person in the photos had to practice for a long time in order to get such nice sine waves. In fact, a sine wave is the only shape that can create this kind of wave pattern, called a standing wave, which simply vibrates back and forth in one place without moving. The sine wave just creates itself automatically when you find the right frequency, because no other shape is possible.

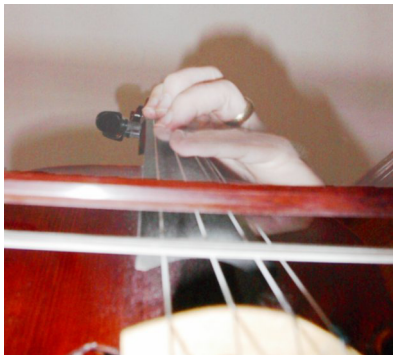
If you think about it, it’s not even obvious that sine waves should be able to do this trick. After all, waves are supposed to travel at a set speed, aren’t they? The speed isn’t supposed to be zero! Well, we can actually think of a standing wave as a superposition of a moving



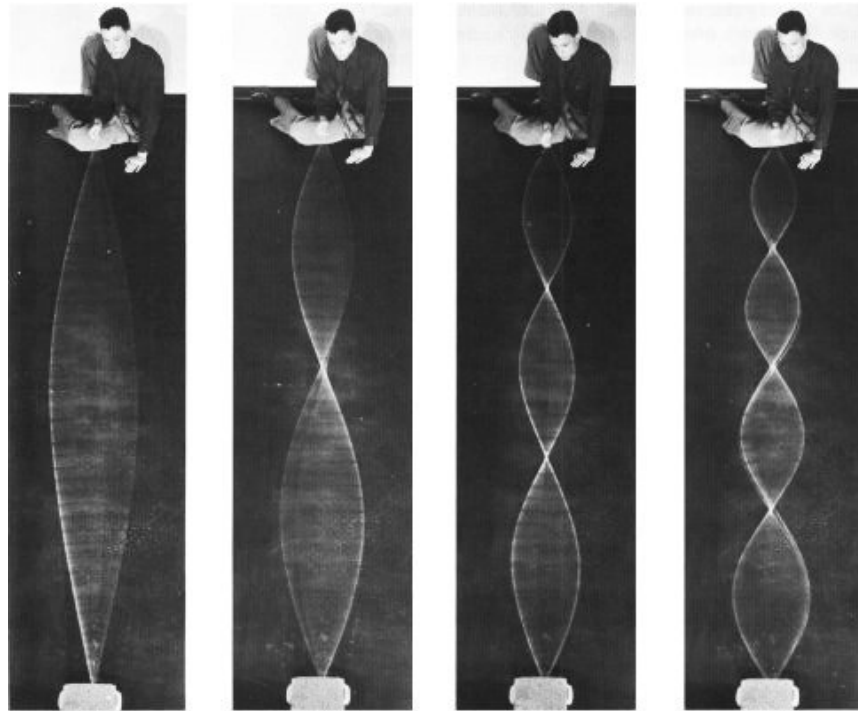
u / Graphs of loudness versus frequency for the vowel “ah,” sung as three different musical notes. G is consonant with D, since every overtone of G that is close to an overtone of D (\*) is at exactly the same frequency. G and C# are dissonant together, since some of the overtones of G (x) are close to, but not right on top of, those of C#.



w / Sine waves add to make sine waves. Other functions don't have this property.



x / Example 8.



v / Standing waves on a spring.

sine wave with its own reflection, which is moving the opposite way. Sine waves have the unique mathematical property, w, that the sum of sine waves of equal wavelength is simply a new sine wave with the same wavelength. As the two sine waves go back and forth, they always cancel perfectly at the ends, and their sum appears to stand still.

Standing wave patterns are rather important, since atoms are really standing-wave patterns of electron waves. You are a standing wave!

#### Harmonics on string instruments

#### example 8

Figure x shows a violinist playing what string players refer to as a natural harmonic. The term “harmonic” is used here in a somewhat different sense than in physics. The musician’s pinkie is pressing very lightly against the string — not hard enough to make it touch the fingerboard — at a point precisely at the center of the string’s length. As shown in the diagram, this allows the string to vibrate at frequencies  $2f_0, 4f_0, 6f_0, \dots$ , which have stationary points at the center of the string, but not at the odd multiples  $f_0, 3f_0, \dots$ . Since all the overtones are multiples of  $2f_0$ , the ear perceives  $2f_0$  as the basic frequency of the note. In musical terms, doubling the frequency corresponds to raising the pitch by an octave. The technique can be used in order to make it easier to play high notes in rapid passages, or for its own sake, because of the change in timbre.



## Standing-wave patterns of air columns

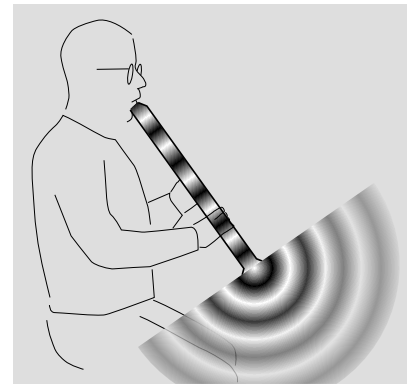
The air column inside a wind instrument behaves very much like the wave-on-a-string example we've been concentrating on so far, the main difference being that we may have either inverting or noninverting reflections at the ends.

Some organ pipes are closed at both ends. The speed of sound is different in metal than in air, so there is a strong reflection at the closed ends, and we can have standing waves. These reflections are both density-noninverting, so we get symmetric standing-wave patterns, such as the one shown in figure z/1.

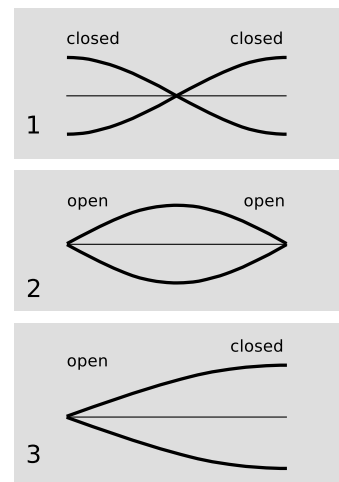
Figure y shows the sound waves in and around a bamboo Japanese flute called a shakuhachi, which is *open* at both ends of the air column. We can only have a standing wave pattern if there are reflections at the ends, but that is very counterintuitive — why is there any reflection at all, if the sound wave is free to emerge into open space, and there is no change in medium? Recall the reason why we got reflections at a change in medium: because the wavelength changes, so the wave has to readjust itself from one pattern to another, and the only way it can do that without developing a kink is if there is a reflection. Something similar is happening here. The only difference is that the wave is adjusting from being a plane wave to being a spherical wave. The reflections at the open ends are density-inverting, z/2, so the wave pattern is pinched off at the ends. Comparing panels 1 and 2 of the figure, we see that although the wave patterns are different, in both cases the wavelength is the same: in the lowest-frequency standing wave, half a wavelength fits inside the tube. Thus, it isn't necessary to memorize which type of reflection is inverting and which is inverting. It's only necessary to know that the tubes are symmetric.

Finally, we can have an asymmetric tube: closed at one end and open at the other. A common example is the pan pipes, aa, which are closed at the bottom and open at the top. The standing wave with the lowest frequency is therefore one in which  $1/4$  of a wavelength fits along the length of the tube, as shown in figure z/3.

Sometimes an instrument's physical appearance can be misleading. A concert flute, ab, is closed at the mouth end and open at the other, so we would expect it to behave like an asymmetric air column; in reality, it behaves like a symmetric air column open at both ends, because the embouchure hole (the hole the player blows over) acts like an open end. The clarinet and the saxophone look similar, having a mouthpiece and reed at one end and an open end at the other, but they act different. In fact the clarinet's air column has patterns of vibration that are asymmetric, the saxophone symmetric. The discrepancy comes from the difference between the conical tube of the sax and the cylindrical tube of the clarinet. The adjustment of the wave pattern from a plane wave to a spherical



y / Surprisingly, sound waves undergo partial reflection at the open ends of tubes as well as closed ones.



z / Graphs of excess density versus position for the lowest-frequency standing waves of three types of air columns. Points on the axis have normal air density.





aa / A pan pipe is an asymmetric air column, open at the top and closed at the bottom.



ab / A concert flute looks like an asymmetric air column, open at the mouth end and closed at the other. However, its patterns of vibration are symmetric, because the embouchure hole acts like an open end.

wave is more gradual at the flaring bell of the saxophone.

#### *self-check F*

Draw a graph of density versus position for the first overtone of the air column in a tube open at one end and closed at the other. This will be the next-to-longest possible wavelength that allows for a point of maximum vibration at one end and a point of no vibration at the other. How many times shorter will its wavelength be compared to the wavelength of the lowest-frequency standing wave, shown in the figure? Based on this, how many times greater will its frequency be? ▷ Answer, p. 535

## Summary

### Selected vocabulary

reflection . . . . .	the bouncing back of part of a wave from a boundary
transmission . . .	the continuation of part of a wave through a boundary
absorption . . . .	the gradual conversion of wave energy into heating of the medium
standing wave . .	a wave pattern that stays in one place

### Notation

$\lambda$ . . . . .	wavelength (Greek letter lambda)
---------------------	----------------------------------

### Summary

Whenever a wave encounters the boundary between two media in which its speeds are different, part of the wave is reflected and part is transmitted. The reflection is always reversed front-to-back, but may also be inverted in amplitude. Whether the reflection is inverted depends on how the wave speeds in the two media compare, e.g., a wave on a string is uninverted when it is reflected back into a segment of string where its speed is lower. The greater the difference in wave speed between the two media, the greater the fraction of the wave energy that is reflected. Surprisingly, a wave in a dense material like wood will be strongly reflected back into the wood at a wood-air boundary.

A one-dimensional wave confined by highly reflective boundaries on two sides will display motion which is periodic. For example, if both reflections are inverting, the wave will have a period equal to twice the time required to traverse the region, or to that time divided by an integer. An important special case is a sinusoidal wave; in this case, the wave forms a stationary pattern composed of a superposition of sine waves moving in opposite direction.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** Light travels faster in warmer air. Use this fact to explain the formation of a mirage appearing like the shiny surface of a pool of water when there is a layer of hot air above a road. (For simplicity, pretend that there is actually a sharp boundary between the hot layer and the cooler layer above it.)

**2** (a) Using the equations from optional section 20.2, compute the amplitude of light that is reflected back into air at an air-water interface, relative to the amplitude of the incident wave. The speeds of light in air and water are  $3.0 \times 10^8$  and  $2.2 \times 10^8$  m/s, respectively.

(b) Find the energy of the reflected wave as a fraction of the incident energy. [Hint: The answers to the two parts are not the same.]

✓

**3** A concert flute produces its lowest note, at about 262 Hz, when half of a wavelength fits inside its tube. Compute the length of the flute.

▷ Answer, p. 536

**4** (a) A good tenor saxophone player can play all of the following notes without changing her fingering, simply by altering the tightness of her lips: E♭ (150 Hz), E♭ (300 Hz), B♭ (450 Hz), and E♭ (600 Hz). How is this possible? (I'm not asking you to analyze the coupling between the lips, the reed, the mouthpiece, and the air column, which is very complicated.)

(b) Some saxophone players are known for their ability to use this technique to play “freak notes,” i.e., notes above the normal range of the instrument. Why isn't it possible to play notes below the normal range using this technique?

C	261.6 Hz
D	293.7
E	329.6
F	349.2
G	392.0
A	440.0
B♭	466.2

**5** The table gives the frequencies of the notes that make up the key of F major, starting from middle C and going up through all seven notes. (a) Calculate the first four or five harmonics of C and G, and determine whether these two notes will be consonant or dissonant. (b) Do the same for C and B♭. [Hint: Remember that harmonics that differ by about 1-10% cause dissonance.]

Problem 5.

**6** Brass and wind instruments go up in pitch as the musician warms up. Suppose a particular trumpet's frequency goes up by 1.2%. Let's consider possible physical reasons for the change in pitch. (a) Solids generally expand with increasing temperature, because the stronger random motion of the atoms tends to bump them apart. Brass expands by  $1.88 \times 10^{-5}$  per degree C. Would this tend to raise the pitch, or lower it? Estimate the size of the effect in comparison with the observed change in frequency. (b) The speed

of sound in a gas is proportional to the square root of the absolute temperature, where zero absolute temperature is  $-273$  degrees C. As in part a, analyze the size and direction of the effect.

**7** Your exhaled breath contains about 4.5% carbon dioxide, and is therefore more dense than fresh air by about 2.3%. By analogy with the treatment of waves on a string in section 19.2, we expect that the speed of sound will be inversely proportional to the square root of the density of the gas. Calculate the effect on the frequency produced by a wind instrument.

# Three essential mathematical skills

More often than not when a search-and-rescue team finds a hiker dead in the wilderness, it turns out that the person was not carrying some item from a short list of essentials, such as water and a map. There are three mathematical essentials in this course.

## 1. Converting units

basic technique: section 0.9, p. 30; conversion of area, volume, etc.: section 1.1, p. 41

Examples:

$$0.7 \cancel{\text{kg}} \times \frac{10^3 \text{ g}}{1 \cancel{\text{kg}}} = 700 \text{ g} \quad .$$

To check that we have the conversion factor the right way up ( $10^3$  rather than  $1/10^3$ ), we note that the smaller unit of grams has been *compensated* for by making the number larger.

For units like  $\text{m}^2$ ,  $\text{kg}/\text{m}^3$ , etc., we have to raise the conversion factor to the appropriate power:

$$4 \text{ m}^3 \times \left( \frac{10^3 \text{ mm}}{1 \text{ m}} \right)^3 = 4 \times 10^9 \cancel{\text{m}^3} \times \frac{\text{mm}^3}{\cancel{\text{m}^3}} = 4 \times 10^9 \text{ mm}^3$$

Examples with solutions — p. 37, #6; p. 59, #10

Problems you can check at [lightandmatter.com/area1checker.html](http://lightandmatter.com/area1checker.html) — p. 37, #5; p. 37, #4; p. 37, #7; p. 59, #1; p. 60, #19

## 2. Reasoning about ratios and proportionalities

The technique is introduced in section 1.2, p. 43, in the context of area and volume, but it applies more generally to any relationship in which one variable depends on another raised to some power.

Example: When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost per kilometer of travel  $C$  is given by

$$C = kw^4 \quad ,$$

where  $w$  is the weight per axle and  $k$  is a constant. The weight per axle is about 13 times higher for a semi-trailer than for my Honda Fit. How many times greater is the cost imposed on the federal government when the semi travels a given distance on an interstate freeway?

▷ First we convert the equation into a proportionality by throwing out  $k$ , which is the same for both vehicles:

$$C \propto w^4$$

Next we convert this proportionality to a statement about ratios:

$$\frac{C_1}{C_2} = \left( \frac{w_1}{w_2} \right)^4 \approx 29,000$$

Since the gas taxes I pay to drive my Fit are nowhere near 29,000 times more than those paid to drive the truck the same distance, the federal government is effectively awarding a massive subsidy to the trucking company. Plus my Fit is cuter.

Examples with solutions — p. 59, #11; p. 59, #12; p. 60, #17; p. 115, #16; p. 116, #22; p. 238, #6; p. 260, #10; p. 261, #15; p. 263, #19; p. 289, #8; p. 289, #9

Problems you can check at [lightandmatter.com/area1checker.html](http://lightandmatter.com/area1checker.html) — p. 60, #16; p. 60, #18; p. 61, #23; p. 62, #24; p. 62, #25; p. 189, #7; p. 238, #8; p. 259, #5; p. 260, #8; p. 263, #21; p. 288, #4; p. 395, #2

### 3. Vector addition

section 7.3, p. 196

Example: The  $\Delta \mathbf{r}$  vector from San Diego to Los Angeles has magnitude 190 km and direction  $129^\circ$  counterclockwise from east. The one from LA to Las Vegas is 370 km at  $38^\circ$  counterclockwise from east. Find the distance and direction from San Diego to Las Vegas.

▷ Graphical addition is discussed on p. 196. Here we concentrate on analytic addition, which involves adding the  $x$  components to find the total  $x$  component, and similarly for  $y$ . The trig needed in order to find the components of the second leg (LA to Vegas) is laid out in figure d on p. 194 and explained in detail in example 3 on p. 195:

$$\begin{aligned}\Delta x_2 &= (370 \text{ km}) \cos 38^\circ = 292 \text{ km} \\ \Delta y_2 &= (370 \text{ km}) \sin 38^\circ = 228 \text{ km}\end{aligned}$$

(Since these are intermediate results, we keep an extra sig fig to avoid accumulating too much rounding error.) Once we understand the trig for one example, we don't need to reinvent the wheel every time. The pattern is completely universal, provided that we first make sure to get the angle expressed according to the usual trig convention, counterclockwise from the  $x$  axis. Applying the pattern to the first leg, we have:

$$\begin{aligned}\Delta x_1 &= (190 \text{ km}) \cos 129^\circ = -120 \text{ km} \\ \Delta y_1 &= (190 \text{ km}) \sin 129^\circ = 148 \text{ km}\end{aligned}$$

For the vector directly from San Diego to Las Vegas, we have

$$\begin{aligned}\Delta x &= \Delta x_1 + \Delta x_2 = 172 \text{ km} \\ \Delta y &= \Delta y_1 + \Delta y_2 = 376 \text{ km} \quad .\end{aligned}$$

The distance from San Diego to Las Vegas is found using the Pythagorean theorem,

$$\sqrt{(172 \text{ km})^2 + (376 \text{ km})^2} = 410 \text{ km}$$

(rounded to two sig figs because it's one of our final results). The direction is one of the two possible values of the inverse tangent

$$\tan^{-1}(\Delta y/\Delta x) = \{65^\circ, 245^\circ\} \quad .$$

Consulting a sketch shows that the first of these values is the correct one.

Examples with solutions — p. 218, #8; p. 218, #9; p. 362, #8

Problems you can check at [lightandmatter.com/area1checker.html](http://lightandmatter.com/area1checker.html) — p. 202, #3; p. 202, #4; p. 217, #1; p. 217, #3; p. 220, #16; p. 259, #3; p. 264, #23; p. 361, #3

## Hints for volume 1

### Hints for chapter 10

#### Page 262, problem 17:

If you try to calculate the two forces and subtract, your calculator will probably give a result of zero due to rounding. Instead, reason about the fractional amount by which the quantity  $1/r^2$  will change. As a warm-up, you may wish to observe the percentage change in  $1/r^2$  that results from changing  $r$  from 1 to 1.01.

### Hints for chapter 15

#### Page 398, problem 18:

The first two parts can be done more easily by setting  $a = 1$ , since the value of  $a$  only changes the distance scale. One way to do part b is by graphing.

## Solutions to selected problems for volume 1

### Solutions for chapter 0

#### Page 37, problem 6:

$$134 \text{ mg} \times \frac{10^{-3} \text{ g}}{1 \text{ mg}} \times \frac{10^{-3} \text{ kg}}{1 \text{ g}} = 1.34 \times 10^{-4} \text{ kg}$$

#### Page 38, problem 8:

(a) Let's do 10.0 g and 1000 g. The arithmetic mean is 505 grams. It comes out to be 0.505 kg, which is consistent. (b) The geometric mean comes out to be 100 g or 0.1 kg, which is consistent. (c) If we multiply meters by meters, we get square meters. Multiplying grams by grams should give square grams! This sounds strange, but it makes sense. Taking the square root of square grams ( $\text{g}^2$ ) gives grams again. (d) No. The superduper mean of two quantities with units of grams wouldn't even be something with units of grams! Related to this shortcoming is the fact that the superduper mean would fail the kind of consistency test carried out in the first two parts of the problem.

#### Page 38, problem 12:

(a) They're all defined in terms of the ratio of side of a triangle to another. For instance, the tangent is the length of the opposite side over the length of the adjacent side. Dividing meters by meters gives a unitless result, so the tangent, as well as the other trig functions, is unitless. (b) The tangent function gives a unitless result, so the units on the right-hand side had better cancel out. They do, because the top of the fraction has units of meters squared, and so does the bottom.

### Solutions for chapter 1

#### Page 59, problem 10:

$$1 \text{ mm}^2 \times \left( \frac{1 \text{ cm}}{10 \text{ mm}} \right)^2 = 10^{-2} \text{ cm}^2$$

#### Page 59, problem 11:

The bigger scope has a diameter that's ten times greater. Area scales as the square of the linear

dimensions, so  $A \propto d^2$ , or in the language of ratios  $A_1/A_2 = (d_1/d_2)^2 = 100$ . Its light-gathering power is a hundred times greater.

**Page 59, problem 12:**

Since they differ by two steps on the Richter scale, the energy of the bigger quake is  $10^4$  times greater. The wave forms a hemisphere, and the surface area of the hemisphere over which the energy is spread is proportional to the square of its radius,  $A \propto r^2$ , or  $r \propto \sqrt{A}$ , which means  $r_1/r_2 = \sqrt{A_1/A_2}$ . If the amount of vibration was the same, then the surface areas must be in the ratio  $A_1/A_2 = 10^4$ , which means that the ratio of the radii is  $10^2$ .

**Page 60, problem 17:**

The cone of mixed gin and vermouth is the same shape as the cone of vermouth, but its linear dimensions are doubled. Translating the proportionality  $V \propto L^3$  into an equation about ratios, we have  $V_1/V_2 = (L_1/L_2)^3 = 8$ . Since the ratio of the whole thing to the vermouth is 8, the ratio of gin to vermouth is 7.

**Page 60, problem 19:**

The proportionality  $V \propto L^3$  can be restated in terms of ratios as  $V_1/V_2 = (L_1/L_2)^3 = (1/10)^3 = 1/1000$ , so scaling down the linear dimensions by a factor of  $1/10$  reduces the volume by  $1/1000$ , to a milliliter.

**Page 61, problem 21:**

Let's estimate the Great Wall's volume, and then figure out how many bricks that would represent. The wall is famous because it covers pretty much all of China's northern border, so let's say it's 1000 km long. From pictures, it looks like it's about 10 m high and 10 m wide, so the total volume would be  $10^6 \text{ m} \times 10 \text{ m} \times 10 \text{ m} = 10^8 \text{ m}^3$ . If a single brick has a volume of 1 liter, or  $10^{-3} \text{ m}^3$ , then this represents about  $10^{11}$  bricks. If one person can lay 10 bricks in an hour (taking into account all the preparation, etc.), then this would be  $10^{10}$  man-hours.

**Page 62, problem 26:**

Directly guessing the number of jelly beans would be like guessing volume directly. That would be a mistake. Instead, we start by estimating the linear dimensions, in units of beans. The contents of the jar look like they're about 10 beans deep. Although the jar is a cylinder, its exact geometrical shape doesn't really matter for the purposes of our order-of-magnitude estimate. Let's pretend it's a rectangular jar. The horizontal dimensions are also something like 10 beans, so it looks like the jar has about  $10 \times 10 \times 10$  or  $\sim 10^3$  beans inside.

**Solutions for chapter 2**

**Page 88, problem 4:**

$$\begin{aligned} 1 \text{ light-year} &= v\Delta t \\ &= (3 \times 10^8 \text{ m/s}) (1 \text{ year}) \\ &= (3 \times 10^8 \text{ m/s}) \left[ (1 \text{ year}) \times \left( \frac{365 \text{ days}}{1 \text{ year}} \right) \times \left( \frac{24 \text{ hours}}{1 \text{ day}} \right) \times \left( \frac{3600 \text{ s}}{1 \text{ hour}} \right) \right] \\ &= 9.5 \times 10^{15} \text{ m} \end{aligned}$$

**Page 88, problem 5:**

Velocity is relative, so having to lean tells you nothing about the train's velocity. Fullerton is moving at a huge speed relative to Beijing, but that doesn't produce any noticeable effect in



either city. The fact that you have to lean tells you that the train is *changing* its speed, but it doesn't tell you what the train's current speed is.

**Page 88, problem 7:**

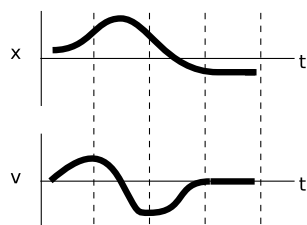
To the person riding the moving bike, bug A is simply going in circles. The only difference between the motions of the two wheels is that one is traveling through space, but motion is relative, so this doesn't have any effect on the bugs. It's equally hard for each of them.

**Page 89, problem 10:**

In one second, the ship moves  $v$  meters to the east, and the person moves  $v$  meters north relative to the deck. Relative to the water, he traces the diagonal of a triangle whose length is given by the Pythagorean theorem,  $(v^2 + v^2)^{1/2} = \sqrt{2}v$ . Relative to the water, he is moving at a 45-degree angle between north and east.

**Solutions for chapter 3**

**Page 115, problem 14:**



**Page 115, problem 15:**

Taking  $g$  to be  $10 \text{ m/s}^2$ , the bullet loses  $10 \text{ m/s}$  of speed every second, so it will take  $10 \text{ s}$  to come to a stop, and then another  $10 \text{ s}$  to come back down, for a total of  $20 \text{ s}$ .

**Page 115, problem 16:**

$\Delta x = \frac{1}{2}at^2$ , so for a fixed value of  $\Delta x$ , we have  $t \propto 1/\sqrt{a}$ . Translating this into the language of ratios gives  $t_M/t_E = \sqrt{a_E/a_M} = \sqrt{3} = 1.7$ .

**Page 115, problem 17:**

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= 10 - 3t^2 \\ a &= \frac{dv}{dt} \\ &= -6t \\ &= -18 \text{ m/s}^2 \end{aligned}$$

**Page 116, problem 18:**

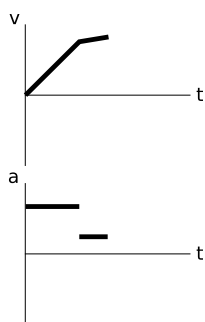
(a) Solving for  $\Delta x = \frac{1}{2}at^2$  for  $a$ , we find  $a = 2\Delta x/t^2 = 5.51 \text{ m/s}^2$ . (b)  $v = \sqrt{2a\Delta x} = 66.6 \text{ m/s}$ . (c) The actual car's final velocity is less than that of the idealized constant-acceleration car. If the real car and the idealized car covered the quarter mile in the same time but the real car was moving more slowly at the end than the idealized one, the real car must have been going faster than the idealized car at the beginning of the race. The real car apparently has a greater acceleration at the beginning, and less acceleration at the end. This makes sense, because every car has some maximum speed, which is the speed beyond which it cannot accelerate.

**Page 116, problem 19:**

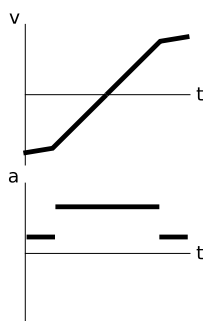
Since the lines are at intervals of one m/s and one second, each box represents one meter. From  $t = 0$  to  $t = 2$  s, the area under the curve represents a positive  $\Delta x$  of 6 m. (The triangle has half the area of the  $2 \times 6$  rectangle it fits inside.) After  $t = 2$  s, the area above the curve represents negative  $\Delta x$ . To get  $-6$  m worth of area, we need to go out to  $t = 6$  s, at which point the triangle under the axis has a width of 4 s and a height of 3 m/s, for an area of 6 m (half of  $3 \times 4$ ).

**Page 116, problem 20:**

(a) We choose a coordinate system with positive pointing to the right. Some people might expect that the ball would slow down once it was on the more gentle ramp. This may be true if there is significant friction, but Galileo's experiments with inclined planes showed that when friction is negligible, a ball rolling on a ramp has constant acceleration, not constant speed. The speed stops increasing as quickly once the ball is on the more gentle slope, but it still keeps on increasing. The  $a$ - $t$  graph can be drawn by inspecting the slope of the  $v$ - $t$  graph.

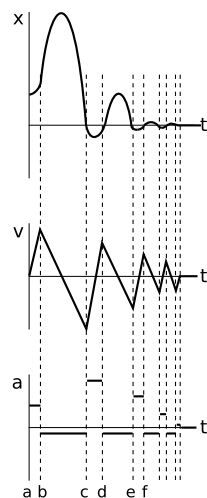


(b) The ball will roll back down, so the second half of the motion is the same as in part a. In the first (rising) half of the motion, the velocity is negative, since the motion is in the opposite direction compared to the positive  $x$  axis. The acceleration is again found by inspecting the slope of the  $v$ - $t$  graph.

**Page 116, problem 21:**

This is a case where it's probably easiest to draw the acceleration graph first. While the ball is in the air (bc, de, etc.), the only force acting on it is gravity, so it must have the same, constant acceleration during each hop. Choosing a coordinate system where the positive  $x$  axis points up, this becomes a negative acceleration (force in the opposite direction compared to the axis). During the short times between hops when the ball is in contact with the ground (cd, ef, etc.), it experiences a large acceleration, which turns around its velocity very rapidly. These short positive accelerations probably aren't constant, but it's hard to know how they'd really look. We just idealize them as constant accelerations. Similarly, the hand's force on the ball during the time ab is probably not constant, but we can draw it that way, since we don't know

how to draw it more realistically. Since our acceleration graph consists of constant-acceleration segments, the velocity graph must consist of line segments, and the position graph must consist of parabolas. On the  $x$  graph, I chose zero to be the height of the center of the ball above the floor when the ball is just lying on the floor. When the ball is touching the floor and compressed, as in interval  $cd$ , its center is below this level, so its  $x$  is negative.



**Page 116, problem 22:**

We have  $v_f^2 = 2a\Delta x$ , so the distance is proportional to the square of the velocity. To get up to half the speed, the ball needs  $1/4$  the distance, i.e.,  $L/4$ .

**Solutions for chapter 4**

**Page 143, problem 7:**

$a = \Delta v / \Delta t$ , and also  $a = F/m$ , so

$$\begin{aligned}\Delta t &= \frac{\Delta v}{a} \\ &= \frac{m\Delta v}{F} \\ &= \frac{(1000 \text{ kg})(50 \text{ m/s} - 20 \text{ m/s})}{3000 \text{ N}} \\ &= 10 \text{ s}\end{aligned}$$

**Page 144, problem 10:**

- (a) This is a measure of the box's resistance to a change in its state of motion, so it measures the box's mass. The experiment would come out the same in lunar gravity.
- (b) This is a measure of how much gravitational force it feels, so it's a measure of weight. In lunar gravity, the box would make a softer sound when it hit.
- (c) As in part a, this is a measure of its resistance to a change in its state of motion: its mass. Gravity isn't involved at all.

**Solutions for chapter 5**

**Page 172, problem 14:**

- (a)

top spring's rightward force on connector  
 ...connector's leftward force on top spring  
 bottom spring's rightward force on connector  
 ...connector's leftward force on bottom spring  
 hand's leftward force on connector  
 ...connector's rightward force on hand

Looking at the three forces on the connector, we see that the hand's force must be double the force of either spring. The value of  $x - x_o$  is the same for both springs and for the arrangement as a whole, so the spring constant must be  $2k$ . This corresponds to a stiffer spring (more force to produce the same extension).

(b) Forces in which the left spring participates:

hand's leftward force on left spring  
 ...left spring's rightward force on hand  
 right spring's rightward force on left spring  
 ...left spring's leftward force on right spring

Forces in which the right spring participates:

left spring's leftward force on right spring  
 ...right spring's rightward force on left spring  
 wall's rightward force on right spring  
 ...right spring's leftward force on wall

Since the left spring isn't accelerating, the total force on it must be zero, so the two forces acting on it must be equal in magnitude. The same applies to the two forces acting on the right spring. The forces between the two springs are connected by Newton's third law, so all eight of these forces must be equal in magnitude. Since the value of  $x - x_o$  for the whole setup is double what it is for either spring individually, the spring constant of the whole setup must be  $k/2$ , which corresponds to a less stiff spring.

**Page 172, problem 16:**

(a) Spring constants in parallel add, so the spring constant has to be proportional to the cross-sectional area. Two springs in series give half the spring constant, three springs in series give  $1/3$ , and so on, so the spring constant has to be inversely proportional to the length. Summarizing, we have  $k \propto A/L$ . (b) With the Young's modulus, we have  $k = (A/L)E$ . The spring constant has units of N/m, so the units of  $E$  would have to be N/m<sup>2</sup>.

**Page 173, problem 18:**

(a) The swimmer's acceleration is caused by the water's force on the swimmer, and the swimmer makes a backward force on the water, which accelerates the water backward. (b) The club's normal force on the ball accelerates the ball, and the ball makes a backward normal force on the club, which decelerates the club. (c) The bowstring's normal force accelerates the arrow, and the arrow also makes a backward normal force on the string. This force on the string causes the string to accelerate less rapidly than it would if the bow's force was the only one acting on it. (d) The tracks' backward frictional force slows the locomotive down. The locomotive's forward frictional force causes the whole planet earth to accelerate by a tiny amount, which is too small to measure because the earth's mass is so great.

**Page 173, problem 20:**

The person's normal force on the box is paired with the box's normal force on the person. The

dirt's frictional force on the box pairs with the box's frictional force on the dirt. The earth's gravitational force on the box matches the box's gravitational force on the earth.

**Page 174, problem 26:**

- (a) A liter of water has a mass of 1.0 kg. The mass is the same in all three locations. Mass indicates how much an object resists a change in its motion. It has nothing to do with gravity.  
 (b) The term “weight” refers to the force of gravity on an object. The bottle's weight on earth is  $F_W = mg = 9.8$  N. Its weight on the moon is about one sixth that value, and its weight in interstellar space is zero.

**Solutions for chapter 6**

**Page 188, problem 5:**

- (a) The easiest strategy is to find the time spent aloft, and then find the range. The vertical motion and the horizontal motion are independent. The vertical motion has acceleration  $-g$ , and the cannonball spends enough time in the air to reverse its vertical velocity component completely, so we have

$$\begin{aligned}\Delta v_y &= v_{yf} - v_{yo} \\ &= -2v \sin \theta \quad .\end{aligned}$$

The time spent aloft is therefore

$$\begin{aligned}\Delta t &= \Delta v_y / a_y \\ &= 2v \sin \theta / g \quad .\end{aligned}$$

During this time, the horizontal distance traveled is

$$\begin{aligned}R &= v_x \Delta t \\ &= 2v^2 \sin \theta \cos \theta / g \quad .\end{aligned}$$

- (b) The range becomes zero at both  $\theta = 0$  and at  $\theta = 90^\circ$ . The  $\theta = 0$  case gives zero range because the ball hits the ground as soon as it leaves the mouth of the cannon. A 90-degree angle gives zero range because the cannonball has no horizontal motion.

**Solutions for chapter 8**

**Page 218, problem 8:**

We want to find out about the velocity vector  $v_{BG}$  of the bullet relative to the ground, so we need to add Annie's velocity relative to the ground  $v_{AG}$  to the bullet's velocity vector  $v_{BA}$  relative to her. Letting the positive  $x$  axis be east and  $y$  north, we have

$$\begin{aligned}v_{BA,x} &= (140 \text{ mi/hr}) \cos 45^\circ \\ &= 100 \text{ mi/hr} \\ v_{BA,y} &= (140 \text{ mi/hr}) \sin 45^\circ \\ &= 100 \text{ mi/hr}\end{aligned}$$

and

$$\begin{aligned}v_{AG,x} &= 0 \\ v_{AG,y} &= 30 \text{ mi/hr} \quad .\end{aligned}$$

The bullet's velocity relative to the ground therefore has components

$$\begin{aligned}v_{BG,x} &= 100 \text{ mi/hr and} \\v_{BG,y} &= 130 \text{ mi/hr} \quad .\end{aligned}$$

Its speed on impact with the animal is the magnitude of this vector

$$\begin{aligned}|v_{BG}| &= \sqrt{(100 \text{ mi/hr})^2 + (130 \text{ mi/hr})^2} \\&= 160 \text{ mi/hr}\end{aligned}$$

(rounded off to 2 significant figures).

**Page 218, problem 9:**

Since its velocity vector is constant, it has zero acceleration, and the sum of the force vectors acting on it must be zero. There are three forces acting on the plane: thrust, lift, and gravity. We are given the first two, and if we can find the third we can infer its mass. The sum of the  $y$  components of the forces is zero, so

$$\begin{aligned}0 &= F_{thrust,y} + F_{lift,y} + F_{W,y} \\&= |\mathbf{F}_{thrust}| \sin \theta + |\mathbf{F}_{lift}| \cos \theta - mg \quad .\end{aligned}$$

The mass is

$$\begin{aligned}m &= (|\mathbf{F}_{thrust}| \sin \theta + |\mathbf{F}_{lift}| \cos \theta) / g \\&= 6.9 \times 10^4 \text{ kg}\end{aligned}$$

**Page 218, problem 10:**

(a) Since the wagon has no acceleration, the total forces in both the  $x$  and  $y$  directions must be zero. There are three forces acting on the wagon:  $\mathbf{F}_T$ ,  $\mathbf{F}_W$ , and the normal force from the ground,  $\mathbf{F}_N$ . If we pick a coordinate system with  $x$  being horizontal and  $y$  vertical, then the angles of these forces measured counterclockwise from the  $x$  axis are  $90^\circ - \phi$ ,  $270^\circ$ , and  $90^\circ + \theta$ , respectively. We have

$$\begin{aligned}F_{x,total} &= |\mathbf{F}_T| \cos(90^\circ - \phi) + |\mathbf{F}_W| \cos(270^\circ) + |\mathbf{F}_N| \cos(90^\circ + \theta) \\F_{y,total} &= |\mathbf{F}_T| \sin(90^\circ - \phi) + |\mathbf{F}_W| \sin(270^\circ) + |\mathbf{F}_N| \sin(90^\circ + \theta) \quad ,\end{aligned}$$

which simplifies to

$$\begin{aligned}0 &= |\mathbf{F}_T| \sin \phi - |\mathbf{F}_N| \sin \theta \\0 &= |\mathbf{F}_T| \cos \phi - |\mathbf{F}_W| + |\mathbf{F}_N| \cos \theta.\end{aligned}$$

The normal force is a quantity that we are not given and do not wish to find, so we should choose it to eliminate. Solving the first equation for  $|\mathbf{F}_N| = (\sin \phi / \sin \theta) |\mathbf{F}_T|$ , we eliminate  $|\mathbf{F}_N|$  from the second equation,

$$0 = |\mathbf{F}_T| \cos \phi - |\mathbf{F}_W| + |\mathbf{F}_T| \sin \phi \cos \theta / \sin \theta$$

and solve for  $|\mathbf{F}_T|$ , finding

$$|\mathbf{F}_T| = \frac{|\mathbf{F}_W|}{\cos \phi + \sin \phi \cos \theta / \sin \theta} \quad .$$

Multiplying both the top and the bottom of the fraction by  $\sin \theta$ , and using the trig identity for  $\sin(\theta + \phi)$  gives the desired result,

$$|\mathbf{F}_T| = \frac{\sin \theta}{\sin(\theta + \phi)} |\mathbf{F}_W| \quad .$$

(b) The case of  $\phi = 0$ , i.e., pulling straight up on the wagon, results in  $|\mathbf{F}_T| = |\mathbf{F}_W|$ : we simply support the wagon and it glides up the slope like a chair-lift on a ski slope. In the case of  $\phi = 180^\circ - \theta$ ,  $|\mathbf{F}_T|$  becomes infinite. Physically this is because we are pulling directly into the ground, so no amount of force will suffice.

**Page 219, problem 11:**

(a) If there was no friction, the angle of repose would be zero, so the coefficient of static friction,  $\mu_s$ , will definitely matter. We also make up symbols  $\theta$ ,  $m$  and  $g$  for the angle of the slope, the mass of the object, and the acceleration of gravity. The forces form a triangle just like the one in section 8.3, but instead of a force applied by an external object, we have static friction, which is less than  $\mu_s |\mathbf{F}_N|$ . As in that example,  $|\mathbf{F}_s| = mg \sin \theta$ , and  $|\mathbf{F}_s| < \mu_s |\mathbf{F}_N|$ , so

$$mg \sin \theta < \mu_s |\mathbf{F}_N| \quad .$$

From the same triangle, we have  $|\mathbf{F}_N| = mg \cos \theta$ , so

$$mg \sin \theta < \mu_s mg \cos \theta \quad .$$

Rearranging,

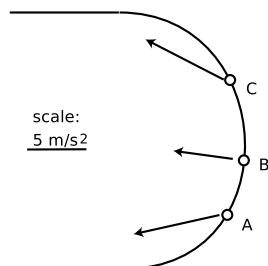
$$\theta < \tan^{-1} \mu_s \quad .$$

(b) Both  $m$  and  $g$  canceled out, so the angle of repose would be the same on an asteroid.

**Solutions for chapter 9**

**Page 237, problem 5:**

Each cyclist has a radial acceleration of  $v^2/r = 5 \text{ m/s}^2$ . The tangential accelerations of cyclists A and B are  $375 \text{ N}/75 \text{ kg} = 5 \text{ m/s}^2$ .



**Page 238, problem 6:**

(a) The inward normal force must be sufficient to produce circular motion, so

$$|\mathbf{F}_N| = mv^2/r \quad .$$

We are searching for the minimum speed, which is the speed at which the static friction force is just barely able to cancel out the downward gravitational force. The maximum force of static friction is

$$|\mathbf{F}_s| = \mu_s |\mathbf{F}_N| \quad ,$$

and this cancels the gravitational force, so

$$|\mathbf{F}_s| = mg \quad .$$

Solving these three equations for  $v$  gives

$$v = \sqrt{\frac{gr}{\mu_s}} \quad .$$

(b) Greater by a factor of  $\sqrt{3}$ .

**Page 238, problem 7:**

The inward force must be supplied by the inward component of the normal force,

$$|\mathbf{F}_N| \sin \theta = mv^2/r \quad .$$

The upward component of the normal force must cancel the downward force of gravity,

$$|\mathbf{F}_N| \cos \theta = mg.$$

Eliminating  $|\mathbf{F}_N|$  and solving for  $\theta$ , we find

$$\theta = \tan^{-1} \left( \frac{v^2}{gr} \right) \quad .$$

**Solutions for chapter 10**

**Page 260, problem 10:**

Newton's law of gravity tells us that her weight will be 6000 times smaller because of the asteroid's smaller mass, but  $13^2 = 169$  times greater because of its smaller radius. Putting these two factors together gives a reduction in weight by a factor of  $6000/169$ , so her weight will be  $(400 \text{ N})(169)/(6000) = 11 \text{ N}$ .

**Page 261, problem 11:**

Newton's law of gravity says  $F = Gm_1m_2/r^2$ , and Newton's second law says  $F = m_2a$ , so  $Gm_1m_2/r^2 = m_2a$ . Since  $m_2$  cancels,  $a$  is independent of  $m_2$ .

**Page 261, problem 12:**

Newton's second law gives

$$F = m_D a_D \quad ,$$

where  $F$  is Ida's force on Dactyl. Using Newton's universal law of gravity,  $F = Gm_I m_D / r^2$ , and the equation  $a = v^2 / r$  for circular motion, we find

$$Gm_I m_D / r^2 = m_D v^2 / r.$$

Dactyl's mass cancels out, giving

$$Gm_I / r^2 = v^2 / r.$$

Dactyl's velocity equals the circumference of its orbit divided by the time for one orbit:  $v = 2\pi r / T$ . Inserting this in the above equation and solving for  $m_I$ , we find

$$m_I = \frac{4\pi^2 r^3}{GT^2} \quad ,$$



so Ida's density is

$$\begin{aligned}\rho &= m_I/V \\ &= \frac{4\pi^2 r^3}{GVT^2} \quad .\end{aligned}$$

**Page 261, problem 15:**

Newton's law of gravity depends on the inverse square of the distance, so if the two planets' masses had been equal, then the factor of  $0.83/0.059 = 14$  in distance would have caused the force on planet c to be  $14^2 = 2.0 \times 10^2$  times weaker. However, planet c's mass is 3.0 times greater, so the force on it is only smaller by a factor of  $2.0 \times 10^2/3.0 = 65$ .

**Page 262, problem 16:**

The reasoning is reminiscent of section 10.2. From Newton's second law we have

$$F = ma = mv^2/r = m(2\pi r/T)^2/r = 4\pi^2 mr/T^2 \quad ,$$

and Newton's law of gravity gives  $F = GMm/r^2$ , where  $M$  is the mass of the earth. Setting these expressions equal to each other, we have

$$4\pi^2 mr/T^2 = GMm/r^2 \quad ,$$

which gives

$$\begin{aligned}r &= \left( \frac{GMT^2}{4\pi^2} \right)^{1/3} \\ &= 4.22 \times 10^4 \text{ km} \quad .\end{aligned}$$

This is the distance from the center of the earth, so to find the altitude, we need to subtract the radius of the earth. The altitude is  $3.58 \times 10^4$  km.

**Page 262, problem 17:**

Any fractional change in  $r$  results in double that amount of fractional change in  $1/r^2$ . For example, raising  $r$  by 1% causes  $1/r^2$  to go down by very nearly 2%. A 27-day orbit is  $1/13.5$  of a year, so the fractional change in  $1/r^2$  is

$$2 \times \frac{(1/13.5) \text{ cm}}{3.84 \times 10^5 \text{ km}} \times \frac{1 \text{ km}}{10^5 \text{ cm}} = 4 \times 10^{-12}$$

**Page 263, problem 19:**

(a) The asteroid's mass depends on the cube of its radius, and for a given mass the surface gravity depends on  $r^{-2}$ . The result is that surface gravity is directly proportional to radius. Half the gravity means half the radius, or one eighth the mass. (b) To agree with a, Earth's mass would have to be  $1/8$  Jupiter's. We assumed spherical shapes and equal density. Both planets are at least roughly spherical, so the only way out of the contradiction is if Jupiter's density is significantly less than Earth's.

**Solutions for chapter 11**

**Page 289, problem 7:**

A force is an interaction between two objects, so while the bullet is in the air, there is no force. There is only a force while the bullet is in contact with the book. There is energy the whole

time, and the total amount doesn't change. The bullet has some kinetic energy, and transfers some of it to the book as heat, sound, and the energy required to tear a hole through the book.

**Page 289, problem 8:**

(a) The energy stored in the gasoline is being changed into heat via frictional heating, and also probably into sound and into energy of water waves. Note that the kinetic energy of the propeller and the boat are not changing, so they are not involved in the energy transformation. (b) The cruising speed would be greater by a factor of the cube root of 2, or about a 26% increase.

**Page 289, problem 9:**

We don't have actual masses and velocities to plug in to the equation, but that's OK. We just have to reason in terms of ratios and proportionalities. Kinetic energy is proportional to mass and to the square of velocity, so B's kinetic energy equals

$$(13.4 \text{ J})(3.77)/(2.34)^2 = 9.23 \text{ J}$$

**Page 289, problem 11:**

Room temperature is about 20°C. The fraction of the energy that actually goes into heating the water is

$$\frac{(250 \text{ g})/(0.24 \text{ g} \cdot ^\circ\text{C}/\text{J}) \times (100^\circ\text{C} - 20^\circ\text{C})}{(1.25 \times 10^3 \text{ J/s})(126 \text{ s})} = 0.53$$

So roughly half of the energy is wasted. The wasted energy might be in several forms: heating of the cup, heating of the oven itself, or leakage of microwaves from the oven.

**Solutions for chapter 12**

**Page 303, problem 5:**

$$\begin{aligned} E_{total,i} &= E_{total,f} \\ PE_i + \text{heat}_i &= PE_f + KE_f + \text{heat}_f \\ \frac{1}{2}mv^2 &= PE_i - PE_f + \text{heat}_i - \text{heat}_f \\ &= -\Delta PE - \Delta \text{heat} \\ v &= \sqrt{2 \left( \frac{-\Delta PE - \Delta \text{heat}}{m} \right)} \\ &= 6.4 \text{ m/s} \end{aligned}$$

**Page 304, problem 7:**

Let  $\theta$  be the angle by which he has progressed around the pipe. Conservation of energy gives

$$\begin{aligned} E_{total,i} &= E_{total,f} \\ PE_i &= PE_f + KE_f \end{aligned}$$

Let's make  $PE = 0$  at the top, so

$$0 = mgr(\cos \theta - 1) + \frac{1}{2}mv^2 \quad .$$

While he is still in contact with the pipe, the radial component of his acceleration is

$$a_r = \frac{v^2}{r} \quad ,$$

and making use of the previous equation we find

$$a_r = 2g(1 - \cos \theta) \quad .$$

There are two forces on him, a normal force from the pipe and a downward gravitation force from the earth. At the moment when he loses contact with the pipe, the normal force is zero, so the radial component,  $mg \cos \theta$ , of the gravitational force must equal  $ma_r$ ,

$$mg \cos \theta = 2mg(1 - \cos \theta) \quad ,$$

which gives

$$\cos \theta = \frac{2}{3} \quad .$$

The amount by which he has dropped is  $r(1 - \cos \theta)$ , which equals  $r/3$  at this moment.

**Page 304, problem 9:**

(a) Example: As one child goes up on one side of a see-saw, another child on the other side comes down. (b) Example: A pool ball hits another pool ball, and transfers some KE.

**Page 304, problem 11:**

Suppose the river is 1 m deep, 100 m wide, and flows at a speed of 10 m/s, and that the falls are 100 m tall. In 1 second, the volume of water flowing over the falls is  $10^3 \text{ m}^3$ , with a mass of  $10^6 \text{ kg}$ . The potential energy released in one second is  $(10^6 \text{ kg})(g)(100 \text{ m}) = 10^9 \text{ J}$ , so the power is  $10^9 \text{ W}$ . A typical household might have 10 hundred-watt appliances turned on at any given time, so it consumes about  $10^3$  watts on the average. The plant could supply a about million households with electricity.

**Solutions for chapter 13**

**Page 332, problem 18:**

No. Work describes how energy was transferred by some process. It isn't a measurable property of a system.

**Solutions for chapter 14**

**Page 362, problem 8:**

Let  $m$  be the mass of the little puck and  $M = 2.3m$  be the mass of the big one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector  $p_x = Mv$  and  $p_y = mv$ , the direction of the vector is  $\tan^{-1}(p_y/p_x) = 23^\circ$  counterclockwise from the big puck's original direction of motion.

**Page 363, problem 11:**

Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancel out on the average. Cooling changes individual molecular momenta, but not the total.

**Page 363, problem 14:**

By conservation of momentum, the total momenta of the pieces after the explosion is the same as the momentum of the firework before the explosion. However, there is no law of conservation of kinetic energy, only a law of conservation of energy. The chemical energy in the gunpowder is converted into heat and kinetic energy when it explodes. All we can say about the kinetic energy of the pieces is that their total is greater than the kinetic energy before the explosion.

**Page 363, problem 15:**

(a) Particle  $i$  had velocity  $v_i$  in the center-of-mass frame, and has velocity  $v_i + u$  in the new frame. The total kinetic energy is

$$\frac{1}{2}m_1(\mathbf{v}_1 + \mathbf{u})^2 + \dots,$$

where “...” indicates that the sum continues for all the particles. Rewriting this in terms of the vector dot product, we have

$$\frac{1}{2}m_1(\mathbf{v}_1 + \mathbf{u}) \cdot (\mathbf{v}_1 + \mathbf{u}) + \dots = \frac{1}{2}m_1(\mathbf{v}_1 \cdot \mathbf{v}_1 + 2\mathbf{u} \cdot \mathbf{v}_1 + \mathbf{u} \cdot \mathbf{u}) + \dots.$$

When we add up all the terms like the first one, we get  $K_{cm}$ . Adding up all the terms like the third one, we get  $M|\mathbf{u}|^2/2$ . The terms like the second term cancel out:

$$m_1\mathbf{u} \cdot \mathbf{v}_1 + \dots = \mathbf{u} \cdot (m_1\mathbf{v}_1 + \dots),$$

where the sum in brackets equals the total momentum in the center-of-mass frame, which is zero by definition.

(b) Changing frames of reference doesn't change the distances between the particles, so the potential energies are all unaffected by the change of frames of reference. Suppose that in a given frame of reference, frame 1, energy is conserved in some process: the initial and final energies add up to be the same. First let's transform to the center-of-mass frame. The potential energies are unaffected by the transformation, and the total kinetic energy is simply reduced by the quantity  $M|\mathbf{u}_1|^2/2$ , where  $\mathbf{u}_1$  is the velocity of frame 1 relative to the center of mass. Subtracting the same constant from the initial and final energies still leaves them equal. Now we transform to frame 2. Again, the effect is simply to change the initial and final energies by adding the same constant.

**Page 364, problem 16:**

A conservation law is about addition: it says that when you add up a certain thing, the total always stays the same. Funkosity would violate the additive nature of conservation laws, because a two-kilogram mass would have twice as much funkosity as a pair of one-kilogram masses moving at the same speed.

**Solutions for chapter 15****Page 398, problem 20:**

The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to  $90^\circ$  that we can pretend they're 90 degrees, considering the rough nature of the other assumptions and measurements. The result is  $(300 \text{ N})(2.5 \text{ cm}) = (F)(8 \text{ cm})$ , or  $F = 90 \text{ N}$ .

**Page 399, problem 28:**

The foot of the rod is moving in a circle relative to the center of the rod, with speed  $v = \pi b/T$ , and acceleration  $v^2/(b/2) = (\pi^2/8)g$ . This acceleration is initially upward, and is greater in magnitude than  $g$ , so the foot of the rod will lift off without dragging. We could also worry about whether the foot of the rod would make contact with the floor again before the rod finishes up flat on its back. This is a question that can be settled by graphing, or simply by

inspection of figure m on page 377. The key here is that the two parts of the acceleration are both independent of  $m$  and  $b$ , so the result is universal, and it does suffice to check a graph in a single example. In practical terms, this tells us something about how difficult the trick is to do. Because  $\pi^2/8 = 1.23$  isn't much greater than unity, a hit that is just a little too weak (by a factor of  $1.23^{1/2} = 1.11$ ) will cause a fairly obvious qualitative change in the results. This is easily observed if you try it a few times with a pencil.

## Answers to self-checks for volume 1

### Answers to self-checks for chapter 0

#### Page 17, self-check A:

If only he has the special powers, then his results can never be reproduced.

#### Page 19, self-check B:

They would have had to weigh the rays, or check for a loss of weight in the object from which they were have emitted. (For technical reasons, this was not a measurement they could actually do, hence the opportunity for disagreement.)

#### Page 25, self-check C:

A dictionary might define “strong” as “possessing powerful muscles,” but that’s not an operational definition, because it doesn’t say how to measure strength numerically. One possible operational definition would be the number of pounds a person can bench press.

#### Page 29, self-check D:

A microsecond is 1000 times longer than a nanosecond, so it would seem like 1000 seconds, or about 20 minutes.

#### Page 30, self-check E:

Exponents have to do with multiplication, not addition. The first line should be 100 times longer than the second, not just twice as long.

#### Page 33, self-check F:

The various estimates differ by 5 to 10 million. The CIA’s estimate includes a ridiculous number of gratuitous significant figures. Does the CIA understand that every day, people in are born in, die in, immigrate to, and emigrate from Nigeria?

#### Page 33, self-check G:

(1) 4; (2) 2; (3) 2

### Answers to self-checks for chapter 1

#### Page 42, self-check A:

$$1 \text{ yd}^2 \times (3 \text{ ft}/1 \text{ yd})^2 = 9 \text{ ft}^2$$

$$1 \text{ yd}^3 \times (3 \text{ ft}/1 \text{ yd})^3 = 27 \text{ ft}^3$$

#### Page 48, self-check B:

$$C_1/C_2 = (w_1/w_2)^4$$

### Answers to self-checks for chapter 2

#### Page 71, self-check A:

Coasting on a bike and coasting on skates give one-dimensional center-of-mass motion, but running and pedaling require moving body parts up and down, which makes the center of mass move up and down. The only example of rigid-body motion is coasting on skates. (Coasting on

a bike is not rigid-body motion, because the wheels twist.)

**Page 71, self-check B:**

By shifting his weight around, he can cause the center of mass not to coincide with the geometric center of the wheel.

**Page 72, self-check C:**

(1) a point in time; (2) time in the abstract sense; (3) a time interval

**Page 73, self-check D:**

Zero, because the “after” and “before” values of  $x$  are the same.

**Page 80, self-check E:**

(1) The effect only occurs during blastoff, when their velocity is changing. Once the rocket engines stop firing, their velocity stops changing, and they no longer feel any effect. (2) It is only an observable effect of your motion relative to the air.

**Answers to self-checks for chapter 3**

**Page 93, self-check A:**

Its speed increases at a steady rate, so in the next second it will travel 19 cm.

**Answers to self-checks for chapter 4**

**Page 135, self-check A:**

(1) The case of  $\rho = 0$  represents an object falling in a vacuum, i.e., there is no density of air. The terminal velocity would be infinite. Physically, we know that an object falling in a vacuum would never stop speeding up, since there would be no force of air friction to cancel the force of gravity. (2) The 4-cm ball would have a mass that was greater by a factor of  $4 \times 4 \times 4$ , but its cross-sectional area would be greater by a factor of  $4 \times 4$ . Its terminal velocity would be greater by a factor of  $\sqrt{4^3/4^2} = 2$ . (3) It isn't of any general importance. It's just an example of one physical situation. You should not memorize it.

**Page 137, self-check B:**

(1) This is motion, not force. (2) This is a description of how the sub is able to get the water to produce a forward force on it. (3) The sub runs out of energy, not force.

**Answers to self-checks for chapter 5**

**Page 149, self-check A:**

The sprinter pushes backward against the ground, and by Newton's third law, the ground pushes forward on her. (Later in the race, she is no longer accelerating, but the ground's forward force is needed in order to cancel out the backward forces, such as air friction.)

**Page 156, self-check B:**

(1) It's kinetic friction, because her uniform is sliding over the dirt. (2) It's static friction, because even though the two surfaces are moving relative to the landscape, they're not slipping over each other. (3) Only kinetic friction creates heat, as when you rub your hands together. If you move your hands up and down together without sliding them across each other, no heat is produced by the static friction.

**Page 157, self-check C:**

By the POFOSTITO mnemonic, we know that each of the bird's forces on the trunk will be of the same type as the corresponding force of the tree on the bird, but in the opposite direction. The bird's feet make a normal force on the tree that is to the right and a static frictional force

that is downward.

**Page 157, self-check D:**

Frictionless ice can certainly make a normal force, since otherwise a hockey puck would sink into the ice. Friction is not possible without a normal force, however: we can see this from the equation, or from common sense, e.g., while sliding down a rope you do not get any friction unless you grip the rope.

**Page 158, self-check E:**

(1) Normal forces are always perpendicular to the surface of contact, which means right or left in this figure. Normal forces are repulsive, so the cliff's force on the feet is to the right, i.e., away from the cliff. (2) Frictional forces are always parallel to the surface of contact, which means right or left in this figure. Static frictional forces are in the direction that would tend to keep the surfaces from slipping over each other. If the wheel was going to slip, its surface would be moving to the left, so the static frictional force on the wheel must be in the direction that would prevent this, i.e., to the right. This makes sense, because it is the static frictional force that accelerates the dragster. (3) Normal forces are always perpendicular to the surface of contact. In this diagram, that means either up and to the left or down and to the right. Normal forces are repulsive, so the ball is pushing the bat away from itself. Therefore the ball's force is down and to the right on this diagram.

**Answers to self-checks for chapter 6**

**Page 181, self-check A:**

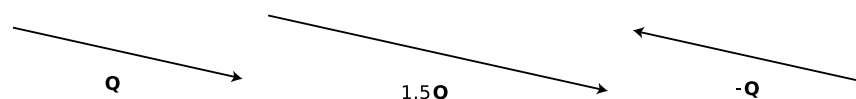
The wind increases the ball's overall speed. If you think about it in terms of overall speed, it's not so obvious that the increased speed is exactly sufficient to compensate for the greater distance. However, it becomes much simpler if you think about the forward motion and the sideways motion as two separate things. Suppose the ball is initially moving at one meter per second. Even if it picks up some sideways motion from the wind, it's still getting closer to the wall by one meter every second.

**Answers to self-checks for chapter 7**

**Page 193, self-check A:**

$$\mathbf{v} = \Delta \mathbf{r} / \Delta t$$

**Page 193, self-check B:**



**Page 198, self-check C:**

$\mathbf{A} - \mathbf{B}$  is equivalent to  $\mathbf{A} + (-\mathbf{B})$ , which can be calculated graphically by reversing  $\mathbf{B}$  to form  $-\mathbf{B}$ , and then adding it to  $\mathbf{A}$ .

**Answers to self-checks for chapter 8**

**Page 208, self-check A:**

(1) It is speeding up, because the final velocity vector has the greater magnitude. (2) The result would be zero, which would make sense. (3) Speeding up produced a  $\Delta \mathbf{v}$  vector in the same direction as the motion. Slowing down would have given a  $\Delta \mathbf{v}$  that pointed backward.

**Page 209, self-check B:**

As we have already seen, the projectile has  $a_x = 0$  and  $a_y = -g$ , so the acceleration vector is pointing straight down.

### Answers to self-checks for chapter 9

#### Page 227, self-check A:

(1) Uniform. They have the same motion as the drum itself, which is rotating as one solid piece. No part of the drum can be rotating at a different speed from any other part. (2) Nonuniform. Gravity speeds it up on the way down and slows it down on the way up.

### Answers to self-checks for chapter 10

#### Page 244, self-check A:

It would just stay where it was. Plugging  $v = 0$  into eq. [1] would give  $F = 0$ , so it would not accelerate from rest, and would never fall into the sun. No astronomer had ever observed an object that did that!

#### Page 245, self-check B:

$$F \propto mr/T^2 \propto mr/(r^{3/2})^2 \propto mr/r^3 = m/r^2$$

#### Page 248, self-check C:

The equal-area law makes equally good sense in the case of a hyperbolic orbit (and observations verify it). The elliptical orbit law had to be generalized by Newton to include hyperbolas. The law of periods doesn't make sense in the case of a hyperbolic orbit, because a hyperbola never closes back on itself, so the motion never repeats.

#### Page 253, self-check D:

Above you there is a small part of the shell, comprising only a tiny fraction of the earth's mass. This part pulls you up, while the whole remainder of the shell pulls you down. However, the part above you is extremely close, so it makes sense that its force on you would be far out of proportion to its small mass.

### Answers to self-checks for chapter 11

#### Page 280, self-check A:

(1) A spring-loaded toy gun can cause a bullet to move, so the spring is capable of storing energy and then converting it into kinetic energy. (2) The amount of energy stored in the spring relates to the amount of compression, which can be measured with a ruler.

### Answers to self-checks for chapter 12

#### Page 300, self-check A:

Both balls start from the same height and end at the same height, so they have the same  $\Delta y$ . This implies that their losses in potential energy are the same, so they must both have gained the same amount of kinetic energy.

### Answers to self-checks for chapter 13

#### Page 308, self-check A:

Work is defined as the transfer of energy, so like energy it is a scalar with units of joules.

#### Page 311, self-check B:

Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the



ball is doing negative work (extracting energy).

**Page 314, self-check C:**

(a) No. The pack is moving at constant velocity, so its kinetic energy is staying the same. It is only moving horizontally, so its gravitational potential energy is also staying the same. No energy transfer is occurring. (b) No. The horse's upward force on the pack forms a 90-degree angle with the direction of motion, so  $\cos \theta = 0$ , and no work is done.

**Page 317, self-check D:**

Only in (a) can we use  $Fd$  to calculate work. In (b) and (c), the force is changing as the distance changes.

**Answers to self-checks for chapter 15**

**Page 380, self-check A:**

1, 2, and 4 all have the same sign, because they are trying to twist the wrench clockwise. The sign of torque 3 is opposite to the signs of the others. The magnitude of torque 3 is the greatest, since it has a large  $r$ , and the force is nearly all perpendicular to the wrench. Torques 1 and 2 are the same because they have the same values of  $r$  and  $F_{\perp}$ . Torque 4 is the smallest, due to its small  $r$ .

**Answers to self-checks for chapter 16**

**Page 405, self-check A:**

Solids can exert shear forces. A solid could be in an equilibrium in which the shear forces were canceling the forces due to unequal pressures on the sides of the cube.

**Answers to self-checks for chapter 18**

**Page 442, self-check A:**

The horizontal axis is a time axis, and the period of the vibrations is independent of amplitude. Shrinking the amplitude does not make the cycles faster.

**Page 443, self-check B:**

Energy is proportional to the square of the amplitude, so its energy is four times smaller after every cycle. It loses three quarters of its energy with each cycle.

**Page 449, self-check C:**

She should tap the wine glasses she finds in the store and look for one with a high  $Q$ , i.e., one whose vibrations die out very slowly. The one with the highest  $Q$  will have the highest-amplitude response to her driving force, making it more likely to break.

**Answers to self-checks for chapter 19**

**Page 467, self-check A:**

The leading edge is moving up, the trailing edge is moving down, and the top of the hump is motionless for one instant.

**Page 474, self-check B:**

(a) It doesn't have  $w$  or  $h$  in it. (b) Inertia is measured by  $\mu$ , tightness by  $T$ . (c) Inertia would be measured by the density of the metal, tightness by its resistance to compression. Lead is more dense than aluminum, and this would tend to make the speed of the waves lower in lead. Lead is also softer, so it probably has less resistance to compression, and we would expect this to provide an additional effect in the same direction. Compressional waves will definitely be slower in lead than in aluminum.

## Answers to self-checks for chapter 20

### Page 493, self-check A:

The energy of a wave is usually proportional to the square of its amplitude. Squaring a negative number gives a positive result, so the energy is the same.

### Page 493, self-check B:

A substance is invisible to sonar if the speed of sound waves in it is the same as in water. Reflections only occur at boundaries between media in which the wave speed is different.

### Page 495, self-check C:

No. A material object that loses kinetic energy slows down, but a wave is not a material object. The velocity of a wave ordinarily only depends on the medium, not the amplitude. The speed of a soft sound, for example, is the same as the speed of a loud sound.

### Page 504, self-check D:


1. No. To get the best possible interference, the thickness of the coating must be such that the second reflected wave train lags behind the first by an integer number of wavelengths. Optimal performance can therefore only be produced for one specific color of light. The typical greenish color of the coatings shows that they do the worst job for green light.

2. Light can be reflected either from the outer surface of the film or from the inner surface, and there can be either constructive or destructive interference between the two reflections. We see a pattern that varies across the surface because its thickness isn't constant. We see rainbow colors because the condition for destructive or constructive interference depends on wavelength. White light is a mixture of all the colors of the rainbow, and at a particular place on the soap bubble, part of that mixture, say red, may be reflected strongly, while another part, blue for example, is almost entirely transmitted.

### Page 505, self-check E:

The period is the time required to travel a distance  $2L$  at speed  $v$ ,  $T = 2L/v$ . The frequency is  $f = 1/T = v/2L$ .

### Page 510, self-check F:

The wave pattern will look like this: . Three quarters of a wavelength fit in the tube, so the wavelength is three times shorter than that of the lowest-frequency mode, in which one quarter of a wave fits. Since the wavelength is smaller by a factor of three, the frequency is three times higher. Instead of  $f_o, 2f_o, 3f_o, 4f_o, \dots$ , the pattern of wave frequencies of this air column goes  $f_o, 3f_o, 5f_o, 7f_o, \dots$

## Answers for volume 1

### Answers for chapter 1

#### Page 61, problem 23:

Check: The actual number of species of lupine occurring in the San Gabriels is 22. You should find that your answer comes out in the same ballpark as this figure, but not exactly the same, of course, because the scaling rule is only a generalization.

### Answers for chapter 16

#### Page 421, problem 10:

(a)  $\sim 2 - 10\%$  (b)  $5\%$  (c) The high end for the body's actual efficiency is higher than the limit imposed by the laws of thermodynamics. However, the high end of the 1-5 watt range quoted in

the problem probably includes large people who aren't just lying around. Still, it's impressive that the human body comes so close to the thermodynamic limit.

### **Answers for chapter 20**

#### **Page 512, problem 3:**

Check: The actual length of a flute is about 66 cm.

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# **Relativity and electromagnetism**





## Chapter 21

# Electricity and circuits

Where the telescope ends, the microscope begins. Which of the two has the grander view?

*Victor Hugo*

His father died during his mother's pregnancy. Rejected by her as a boy, he was packed off to boarding school when she remarried. He himself never married, but in middle age he formed an intense relationship with a much younger man, a relationship that he terminated when he underwent a psychotic break. Following his early scientific successes, he spent the rest of his professional life mostly in frustration over his inability to unlock the secrets of alchemy.

The man being described is Isaac Newton, but not the triumphant Newton of the standard textbook hagiography. Why dwell on the sad side of his life? To the modern science educator, Newton's life-long obsession with alchemy may seem an embarrassment, a distraction from his main achievement, the creation the modern science of mechanics. To Newton, however, his alchemical researches were naturally related to his investigations of force and motion. What was radical about Newton's analysis of motion was its universality: it succeeded in describing both the heavens and the earth with the same equations, whereas previously it had been assumed that the sun, moon, stars, and planets were fundamentally different from earthly objects. But Newton realized that if science was to describe all of nature in a unified way, it was not enough to unite the human scale with the scale of the universe: he would not be satisfied until



he fit the microscopic universe into the picture as well.

It should not surprise us that Newton failed. Although he was a firm believer in the existence of atoms, there was no more experimental evidence for their existence than there had been when the ancient Greeks first posited them on purely philosophical grounds. Alchemy labored under a tradition of secrecy and mysticism. Newton had already almost single-handedly transformed the fuzzyheaded field of “natural philosophy” into something we would recognize as the modern science of physics, and it would be unjust to criticize him for failing to change alchemy into modern chemistry as well. The time was not ripe. The microscope was a new invention, and it was cutting-edge science when Newton’s contemporary Hooke discovered that living things were made out of cells.

## 21.1 The quest for the atomic force

Newton was not the first of the age of reason. He was the last of the magicians.

*John Maynard Keynes*

Nevertheless it will be instructive to pick up Newton’s train of thought and see where it leads us with the benefit of modern hindsight. In uniting the human and cosmic scales of existence, he had reimagined both as stages on which the actors were objects (trees and houses, planets and stars) that interacted through attractions and repulsions. He was already convinced that the objects inhabiting the microworld were atoms, so it remained only to determine what kinds of forces they exerted on each other.

His next insight was no less brilliant for his inability to bring it to fruition. He realized that the many human-scale forces — friction, sticky forces, the normal forces that keep objects from occupying the same space, and so on — must all simply be expressions of a more fundamental force acting between atoms. Tape sticks to paper because the atoms in the tape attract the atoms in the paper. My house doesn’t fall to the center of the earth because its atoms repel the atoms of the dirt under it.

Here he got stuck. It was tempting to think that the atomic force was a form of gravity, which he knew to be universal, fundamental, and mathematically simple. Gravity, however, is always attractive, so how could he use it to explain the existence of both attractive and repulsive atomic forces? The gravitational force between objects of ordinary size is also extremely small, which is why we never notice cars and houses attracting us gravitationally. It would be hard to understand how gravity could be responsible for anything as vigorous as the beating of a heart or the explosion of gunpowder. Newton went on to write a million words of alchemical notes filled with speculation about some other force, perhaps a “divine force” or “vegetative force” that would for example be carried by the sperm

to the egg.

Luckily, we now know enough to investigate a different suspect as a candidate for the atomic force: electricity. Electric forces are often observed between objects that have been prepared by rubbing (or other surface interactions), for instance when clothes rub against each other in the dryer. A useful example is shown in figure a/1: stick two pieces of tape on a tabletop, and then put two more pieces on top of them. Lift each pair from the table, and then separate them. The two top pieces will then repel each other, a/2, as will the two bottom pieces. A bottom piece will attract a top piece, however, a/3. Electrical forces like these are similar in certain ways to gravity, the other force that we already know to be fundamental:

- Electrical forces are *universal*. Although some substances, such as fur, rubber, and plastic, respond more strongly to electrical preparation than others, all matter participates in electrical forces to some degree. There is no such thing as a “nonelectric” substance. Matter is both inherently gravitational and inherently electrical.
- Experiments show that the electrical force, like the gravitational force, is an *inverse square* force. That is, the electrical force between two spheres is proportional to  $1/r^2$ , where  $r$  is the center-to-center distance between them.

Furthermore, electrical forces make more sense than gravity as candidates for the fundamental force between atoms, because we have observed that they can be either attractive or repulsive.

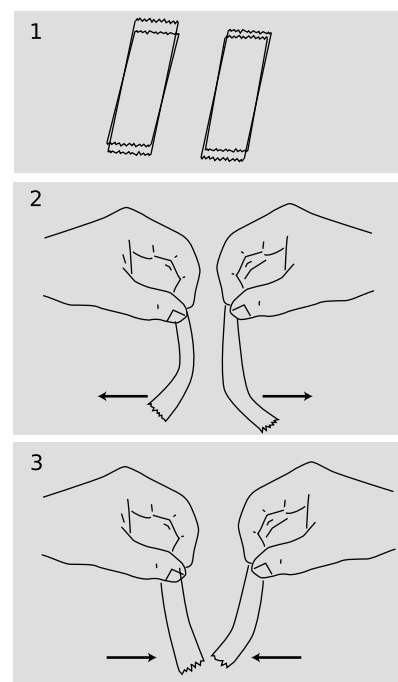
## 21.2 Electrical forces

### Charge

“Charge” is the technical term used to indicate that an object has been prepared so as to participate in electrical forces. This is to be distinguished from the common usage, in which the term is used indiscriminately for anything electrical. For example, although we speak colloquially of “charging” a battery, you may easily verify that a battery has no charge in the technical sense, e.g., it does not exert any electrical force on a piece of tape that has been prepared as described in the previous section.

#### *Two types of charge*

We can easily collect reams of data on electrical forces between different substances that have been charged in different ways. We find for example that cat fur prepared by rubbing against rabbit fur will attract glass that has been rubbed on silk. How can we make any sense of all this information? A vast simplification is



a / Four pieces of tape are prepared, 1, as described in the text. Depending on which combination is tested, the interaction can be either repulsive, 2, or attractive, 3.

achieved by noting that there are really only two types of charge. Suppose we pick cat fur rubbed on rabbit fur as a representative of type A, and glass rubbed on silk for type B. We will now find that there is no “type C.” Any object electrified by any method is either A-like, attracting things A attracts and repelling those it repels, or B-like, displaying the same attractions and repulsions as B. The two types, A and B, always display opposite interactions. If A displays an attraction with some charged object, then B is guaranteed to undergo repulsion with it, and vice-versa.

### *The coulomb*

Although there are only two types of charge, each type can come in different amounts. The metric unit of charge is the coulomb (rhymes with “drool on”), defined as follows:

One Coulomb (C) is the amount of charge such that a force of  $9.0 \times 10^9$  N occurs between two pointlike objects with charges of 1 C separated by a distance of 1 m.

The notation for an amount of charge is  $q$ . The numerical factor in the definition is historical in origin, and is not worth memorizing. The definition is stated for pointlike, i.e., very small, objects, because otherwise different parts of them would be at different distances from each other.

### *A model of two types of charged particles*

Experiments show that all the methods of rubbing or otherwise charging objects involve two objects, and both of them end up getting charged. If one object acquires a certain amount of one type of charge, then the other ends up with an equal amount of the other type. Various interpretations of this are possible, but the simplest is that the basic building blocks of matter come in two flavors, one with each type of charge. Rubbing objects together results in the transfer of some of these particles from one object to the other. In this model, an object that has not been electrically prepared may actually possess a great deal of *both* types of charge, but the amounts are equal and they are distributed in the same way throughout it. Since type A repels anything that type B attracts, and vice versa, the object will make a total force of zero on any other object. The rest of this chapter fleshes out this model and discusses how these mysterious particles can be understood as being internal parts of atoms.

### *Use of positive and negative signs for charge*

Because the two types of charge tend to cancel out each other’s forces, it makes sense to label them using positive and negative signs, and to discuss the *total* charge of an object. It is entirely arbitrary which type of charge to call negative and which to call positive. Benjamin Franklin decided to describe the one we’ve been calling

“A” as negative, but it really doesn’t matter as long as everyone is consistent with everyone else. An object with a total charge of zero (equal amounts of both types) is referred to as electrically *neutral*.

**self-check A**

Criticize the following statement: “There are two types of charge, attractive and repulsive.”

▷ Answer, p.

977

**Coulomb’s law**

A large body of experimental observations can be summarized as follows:

Coulomb’s law: The magnitude of the force acting between point-like charged objects at a center-to-center distance  $r$  is given by the equation

$$|\mathbf{F}| = k \frac{|q_1||q_2|}{r^2},$$

where the constant  $k$  equals  $9.0 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2$ . The force is attractive if the charges are of different signs, and repulsive if they have the same sign.

Clever modern techniques have allowed the  $1/r^2$  form of Coulomb’s law to be tested to incredible accuracy, showing that the exponent is in the range from 1.9999999999999998 to 2.0000000000000002.

Note that Coulomb’s law is closely analogous to Newton’s law of gravity, where the magnitude of the force is  $Gm_1m_2/r^2$ , except that there is only one type of mass, not two, and gravitational forces are never repulsive. Because of this close analogy between the two types of forces, we can recycle a great deal of our knowledge of gravitational forces. For instance, there is an electrical equivalent of the shell theorem: the electrical forces exerted externally by a uniformly charged spherical shell are the same as if all the charge was concentrated at its center, and the forces exerted internally are zero.

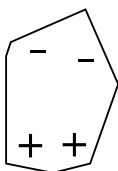
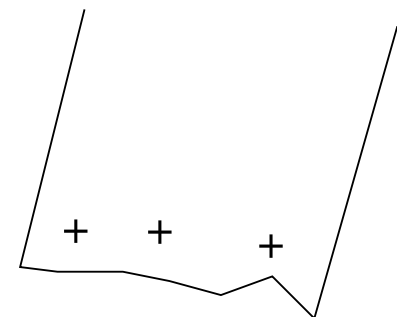
**Conservation of charge**

An even more fundamental reason for using positive and negative signs for electrical charge is that experiments show that charge is conserved according to this definition: in any closed system, the total amount of charge is a constant. This is why we observe that rubbing initially uncharged substances together always has the result that one gains a certain amount of one type of charge, while the other acquires an equal amount of the other type. Conservation of charge seems natural in our model in which matter is made of positive and negative particles. If the charge on each particle is a fixed property of that type of particle, and if the particles themselves

can be neither created nor destroyed, then conservation of charge is inevitable.



b / A charged piece of tape attracts uncharged pieces of paper from a distance, and they leap up to it.



c / The paper has zero total charge, but it does have charged particles in it that can move.

## Electrical forces involving neutral objects

As shown in figure b, an electrically charged object can attract objects that are uncharged. How is this possible? The key is that even though each piece of paper has a total charge of zero, it has at least some charged particles in it that have some freedom to move. Suppose that the tape is positively charged, c. Mobile particles in the paper will respond to the tape's forces, causing one end of the paper to become negatively charged and the other to become positive. The attraction between the paper and the tape is now stronger than the repulsion, because the negatively charged end is closer to the tape.

### self-check B

What would have happened if the tape was negatively charged? ▷

Answer, p. 977

## Discussion questions

**A** If the electrical attraction between two pointlike objects at a distance of 1 m is  $9 \times 10^9$  N, why can't we infer that their charges are +1 and -1 C? What further observations would we need to do in order to prove this?

**B** An electrically charged piece of tape will be attracted to your hand. Does that allow us to tell whether the mobile charged particles in your hand are positive or negative, or both?

## 21.3 Current

### Unity of all types of electricity

We are surrounded by things we have been *told* are “electrical,” but it's far from obvious what they have in common to justify being grouped together. What relationship is there between the way socks cling together and the way a battery lights a lightbulb? We have been told that both an electric eel and our own brains are somehow electrical in nature, but what do they have in common?

British physicist Michael Faraday (1791-1867) set out to address this problem. He investigated electricity from a variety of sources — including electric eels! — to see whether they could all produce the same effects, such as shocks and sparks, attraction and repulsion. “Heating” refers, for example, to the way a lightbulb filament gets hot enough to glow and emit light. Magnetic induction is an effect discovered by Faraday himself that connects electricity and magnetism. We will not study this effect, which is the basis for the electric generator, in detail until later in the book.

source	effect			
	shocks	sparks	attraction and repulsion	heating
rubbing	✓	✓	✓	✓
battery	✓	✓	✓	✓
animal	✓	✓	(✓)	✓
magnetically induced	✓	✓	✓	✓

The table shows a summary of some of Faraday's results. Check marks indicate that Faraday or his close contemporaries were able to verify that a particular source of electricity was capable of producing a certain effect. (They evidently failed to demonstrate attraction and repulsion between objects charged by electric eels, although modern workers have studied these species in detail and been able to understand all their electrical characteristics on the same footing as other forms of electricity.)

Faraday's results indicate that there is nothing fundamentally different about the types of electricity supplied by the various sources. They are all able to produce a wide variety of identical effects. Wrote Faraday, "The general conclusion which must be drawn from this collection of facts is that electricity, whatever may be its source, is identical in its nature."

If the types of electricity are the same thing, what thing is that? The answer is provided by the fact that all the sources of electricity can cause objects to repel or attract each other. We use the word "charge" to describe the property of an object that allows it to participate in such electrical forces, and we have learned that charge is present in matter in the form of nuclei and electrons. Evidently all these electrical phenomena boil down to the motion of charged particles in matter.

### Electric current

If the fundamental phenomenon is the motion of charged particles, then how can we define a useful numerical measurement of it? We might describe the flow of a river simply by the velocity of the water, but velocity will not be appropriate for electrical purposes because we need to take into account how much charge the moving particles have, and in any case there are no practical devices sold at Radio Shack that can tell us the velocity of charged particles. Experiments show that the intensity of various electrical effects is related to a different quantity: the number of coulombs of charge that pass by a certain point per second. By analogy with the flow of water, this quantity is called the electric *current*,  $I$ . Its units



d / Michael Faraday (1791-1867) was the son of a poor blacksmith.



e / *Gymnotus carapo*, a knifefish, uses electrical signals to sense its environment and to communicate with others of its species.



f / André Marie Ampère (1775-1836).

of coulombs/second are more conveniently abbreviated as amperes,  $1 \text{ A} = 1 \text{ C/s}$ . (In informal speech, one usually says “amps.”)

The main subtlety involved in this definition is how to account for the two types of charge. The stream of water coming from a hose is made of atoms containing charged particles, but it produces none of the effects we associate with electric currents. For example, you do not get an electrical shock when you are sprayed by a hose. This type of experiment shows that the effect created by the motion of one type of charged particle can be canceled out by the motion of the opposite type of charge in the same direction. In water, every oxygen atom with a charge of  $+8e$  is surrounded by eight electrons with charges of  $-e$ , and likewise for the hydrogen atoms.

We therefore refine our definition of current as follows:

#### **definition of electric current**

When charged particles are exchanged between regions of space A and B, the electric current flowing from A to B is

$$I = \frac{\Delta q}{\Delta t},$$

where  $\Delta q$  is the change in region B’s total charge occurring over a period of time  $\Delta t$ .

In the garden hose example, your body picks up equal amounts of positive and negative charge, resulting in no change in your total charge, so the electrical current flowing into you is zero.

#### *Interpretation of $\Delta q/\Delta t$*

#### *example 1*

▷ How should the expression  $\Delta q/\Delta t$  be interpreted when the current isn’t constant?

▷ You’ve seen lots of equations of this form before:  $v = \Delta x/\Delta t$ ,  $F = \Delta p/\Delta t$ , etc. These are all descriptions of rates of change, and they all require that the rate of change be constant. If the rate of change isn’t constant, you instead have to use the slope of the tangent line on a graph. The slope of a tangent line is equivalent to a derivative in calculus; applications of calculus are discussed in section 21.7.

#### *Ions moving across a cell membrane*

#### *example 2*

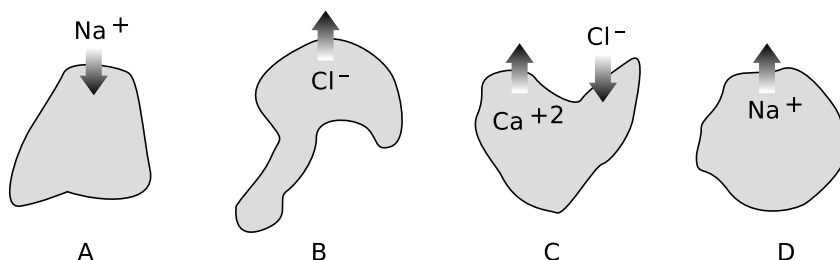
▷ Figure g shows ions, labeled with their charges, moving in or out through the membranes of four cells. If the ions all cross the membranes during the same interval of time, how would the currents into the cells compare with each other?

▷ Cell A has positive current going into it because its charge is increased, i.e., has a positive value of  $\Delta q$ .

Cell B has the same current as cell A, because by losing one unit of negative charge it also ends up increasing its own total charge by one unit.

Cell C's total charge is reduced by three units, so it has a large negative current going into it.

Cell D loses one unit of charge, so it has a small negative current into it.



g / Example 2

It may seem strange to say that a negatively charged particle going one way creates a current going the other way, but this is quite ordinary. As we will see, currents flow through metal wires via the motion of electrons, which are negatively charged, so the direction of motion of the electrons in a circuit is always opposite to the direction of the current. Of course it would have been convenient of Benjamin Franklin had defined the positive and negative signs of charge the opposite way, since so many electrical devices are based on metal wires.

*Number of electrons flowing through a lightbulb*      *example 3*

▷ If a lightbulb has 1.0 A flowing through it, how many electrons will pass through the filament in 1.0 s?

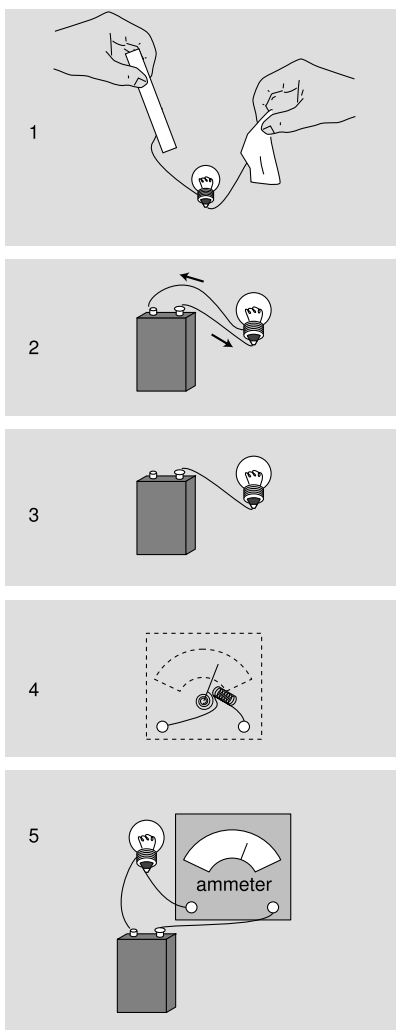
▷ We are only calculating the number of electrons that flow, so we can ignore the positive and negative signs. Solving for  $\Delta q = I\Delta t$  gives a charge of 1.0 C flowing in this time interval. The number of electrons is

$$\begin{aligned} \text{number of electrons} &= \text{coulombs} \times \frac{\text{electrons}}{\text{coulomb}} \\ &= \text{coulombs} / \frac{\text{coulombs}}{\text{electron}} \\ &= 1.0 \text{ C} / e \\ &= 6.2 \times 10^{18} \end{aligned}$$

## 21.4 Circuits

How can we put electric currents to work? The only method of controlling electric charge we have studied so far is to charge different substances, e.g., rubber and fur, by rubbing them against each other. Figure h/1 shows an attempt to use this technique to light a lightbulb. This method is unsatisfactory. True, current will flow through the bulb, since electrons can move through metal wires, and





h / 1. Static electricity runs out quickly. 2. A practical circuit. 3. An open circuit. 4. How an ammeter works. 5. Measuring the current with an ammeter.

the excess electrons on the rubber rod will therefore come through the wires and bulb due to the attraction of the positively charged fur and the repulsion of the other electrons. The problem is that after a zillionth of a second of current, the rod and fur will both have run out of charge. No more current will flow, and the lightbulb will go out.

Figure h/2 shows a setup that works. The battery pushes charge through the circuit, and recycles it over and over again. (We will have more to say later in this chapter about how batteries work.) This is called a *complete circuit*. Today, the electrical use of the word “circuit” is the only one that springs to mind for most people, but the original meaning was to travel around and make a round trip, as when a circuit court judge would ride around the boondocks, dispensing justice in each town on a certain date.

Note that an example like h/3 does not work. The wire will quickly begin acquiring a net charge, because it has no way to get rid of the charge flowing into it. The repulsion of this charge will make it more and more difficult to send any more charge in, and soon the electrical forces exerted by the battery will be canceled out completely. The whole process would be over so quickly that the filament would not even have enough time to get hot and glow. This is known as an *open circuit*. Exactly the same thing would happen if the complete circuit of figure h/2 was cut somewhere with a pair of scissors, and in fact that is essentially how an ordinary light switch works: by opening up a gap in the circuit.

The definition of electric current we have developed has the great virtue that it is easy to measure. In practical electrical work, one almost always measures current, not charge. The instrument used to measure current is called an *ammeter*. A simplified ammeter, h/4, simply consists of a coiled-wire magnet whose force twists an iron needle against the resistance of a spring. The greater the current, the greater the force. Although the construction of ammeters may differ, their use is always the same. We break into the path of the electric current and interpose the meter like a tollbooth on a road, h/5. There is still a complete circuit, and as far as the battery and bulb are concerned, the ammeter is just another segment of wire.

Does it matter where in the circuit we place the ammeter? Could we, for instance, have put it in the left side of the circuit instead of the right? Conservation of charge tells us that this can make no difference. Charge is not destroyed or “used up” by the lightbulb, so we will get the same current reading on either side of it. What is “used up” is energy stored in the battery, which is being converted into heat and light energy.

## 21.5 Voltage

### The volt unit

Electrical circuits can be used for sending signals, storing information, or doing calculations, but their most common purpose by far is to manipulate energy, as in the battery-and-bulb example of the previous section. We know that lightbulbs are rated in units of watts, i.e., how many joules per second of energy they can convert into heat and light, but how would this relate to the flow of charge as measured in amperes? By way of analogy, suppose your friend, who didn't take physics, can't find any job better than pitching bales of hay. The number of calories he burns per hour will certainly depend on how many bales he pitches per minute, but it will also be proportional to how much mechanical work he has to do on each bale. If his job is to toss them up into a hayloft, he will get tired a lot more quickly than someone who merely tips bales off a loading dock into trucks. In metric units,

$$\frac{\text{joules}}{\text{second}} = \frac{\text{haybales}}{\text{second}} \times \frac{\text{joules}}{\text{haybale}} \quad .$$

Similarly, the rate of energy transformation by a battery will not just depend on how many coulombs per second it pushes through a circuit but also on how much mechanical work it has to do on each coulomb of charge:

$$\frac{\text{joules}}{\text{second}} = \frac{\text{coulombs}}{\text{second}} \times \frac{\text{joules}}{\text{coulomb}}$$

or

$$\text{power} = \text{current} \times \text{work per unit charge} \quad .$$

Units of joules per coulomb are abbreviated as *volts*, 1 V=1 J/C, named after the Italian physicist Alessandro Volta. Everyone knows that batteries are rated in units of volts, but the voltage concept is more general than that; it turns out that voltage is a property of every point in space. To gain more insight, let's think more carefully about what goes on in the battery and bulb circuit.

### The voltage concept in general

To do work on a charged particle, the battery apparently must be exerting forces on it. How does it do this? Well, the only thing that can exert an electrical force on a charged particle is another charged particle. It's as though the haybales were pushing and pulling each other into the hayloft! This is potentially a horribly complicated situation. Even if we knew how much excess positive or negative charge there was at every point in the circuit (which realistically we don't) we would have to calculate zillions of forces using Coulomb's law, perform all the vector additions, and finally calculate how much work was being done on the charges as they moved along. To make



i / Alessandro Volta (1745-1827).

things even more scary, there is more than one type of charged particle that moves: electrons are what move in the wires and the bulb's filament, but ions are the moving charge carriers inside the battery. Luckily, there are two ways in which we can simplify things:

**The situation is unchanging.** Unlike the imaginary setup in which we attempted to light a bulb using a rubber rod and a piece of fur, this circuit maintains itself in a steady state (after perhaps a microsecond-long period of settling down after the circuit is first assembled). The current is steady, and as charge flows out of any area of the circuit it is replaced by the same amount of charge flowing in. The amount of excess positive or negative charge in any part of the circuit therefore stays constant. Similarly, when we watch a river flowing, the water goes by but the river doesn't disappear.

**Force depends only on position.** Since the charge distribution is not changing, the total electrical force on a charged particle depends only on its own charge and on its location. If another charged particle of the same type visits the same location later on, it will feel exactly the same force.

The second observation tells us that there is nothing all that different about the experience of one charged particle as compared to another's. If we single out one particle to pay attention to, and figure out the amount of work done on it by electrical forces as it goes from point A to point B along a certain path, then this is the same amount of work that will be done on any other charged particles of the same type as it follows the same path. For the sake of visualization, let's think about the path that starts at one terminal of the battery, goes through the light bulb's filament, and ends at the other terminal. When an object experiences a force that depends only on its position (and when certain other, technical conditions are satisfied), we can define an electrical energy associated with the position of that object. The amount of work done on the particle by electrical forces as it moves from A to B equals the drop in electrical energy between A and B. This electrical energy is what is being converted into other forms of energy such as heat and light. We therefore define voltage in general as electrical energy per unit charge:

**definition of voltage difference**

The difference in voltage between two points in space is defined as

$$\Delta V = \Delta PE_{elec}/q \quad ,$$

where  $\Delta PE_{elec}$  is the change in the electrical energy of a particle with charge  $q$  as it moves from the initial point to the final point.

The amount of power dissipated (i.e., rate at which energy is transformed by the flow of electricity) is then given by the

equation

$$P = I\Delta V$$

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*Energy stored in a battery*

*example 4*

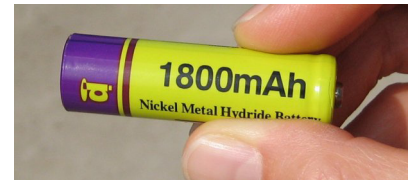
▷ The 1.2 V rechargeable battery in figure j is labeled 1800 milliamp-hours. What is the maximum amount of energy the battery can store?

▷ An ampere-hour is a unit of current multiplied by a unit of time. Current is charge per unit time, so an ampere-hour is in fact a funny unit of *charge*:

$$\begin{aligned}(1 \text{ A})(1 \text{ hour}) &= (1 \text{ C/s})(3600 \text{ s}) \\ &= 3600 \text{ C}\end{aligned}$$

1800 milliamp-hours is therefore  $1800 \times 10^{-3} \times 3600 \text{ C} = 6.5 \times 10^3 \text{ C}$ . That's a huge number of charged particles, but the total loss of electrical energy will just be their total charge multiplied by the voltage difference across which they move:

$$\begin{aligned}\Delta PE_{elec} &= q\Delta V \\ &= (6.5 \times 10^3 \text{ C})(1.2 \text{ V}) \\ &= 7.8 \text{ kJ}\end{aligned}$$



j / Example 4.

---

*Units of volt-amps*

*example 5*

▷ Doorbells are often rated in volt-amps. What does this combination of units mean?

▷ Current times voltage gives units of power,  $P = I\Delta V$ , so volt-amps are really just a nonstandard way of writing watts. They are telling you how much power the doorbell requires.

---

*Power dissipated by a battery and bulb*

*example 6*

▷ If a 9.0-volt battery causes 1.0 A to flow through a lightbulb, how much power is dissipated?

▷ The voltage rating of a battery tells us what voltage difference  $\Delta V$  it is designed to maintain between its terminals.

$$\begin{aligned}P &= I\Delta V \\ &= 9.0 \text{ A} \cdot \text{V} \\ &= 9.0 \frac{\text{C}}{\text{s}} \cdot \frac{\text{J}}{\text{C}} \\ &= 9.0 \text{ J/s} \\ &= 9.0 \text{ W}\end{aligned}$$

The only nontrivial thing in this problem was dealing with the units. One quickly gets used to translating common combinations like  $\text{A} \cdot \text{V}$  into simpler terms.

Here are a few questions and answers about the voltage concept.

*Question:* OK, so what *is* voltage, really?

*Answer:* A device like a battery has positive and negative charges inside it that push other charges around the outside circuit. A higher-voltage battery has denser charges in it, which will do more work on each charged particle that moves through the outside circuit.

To use a gravitational analogy, we can put a paddlewheel at the bottom of either a tall waterfall or a short one, but a kg of water that falls through the greater gravitational energy difference will have more energy to give up to the paddlewheel at the bottom.

*Question:* Why do we define voltage as electrical energy divided by charge, instead of just defining it as electrical energy?

*Answer:* One answer is that it's the only definition that makes the equation  $P = I\Delta V$  work. A more general answer is that we want to be able to define a voltage difference between any two points in space without having to know in advance how much charge the particles moving between them will have. If you put a nine-volt battery on your tongue, then the charged particles that move across your tongue and give you that tingly sensation are not electrons but ions, which may have charges of  $+e$ ,  $-2e$ , or practically anything. The manufacturer probably expected the battery to be used mostly in circuits with metal wires, where the charged particles that flowed would be electrons with charges of  $-e$ . If the ones flowing across your tongue happen to have charges of  $-2e$ , the electrical energy difference for them will be twice as much, but dividing by their charge of  $-2e$  in the definition of voltage will still give a result of 9 V.

*Question:* Are there two separate roles for the charged particles in the circuit, a type that sits still and exerts the forces, and another that moves under the influence of those forces?

*Answer:* No. Every charged particle simultaneously plays both roles. Newton's third law says that any particle that has an electrical force acting on it must also be exerting an electrical force back on the other particle. There are no "designated movers" or "designated force-makers."

*Question:* Why does the definition of voltage only refer to voltage *differences*?

*Answer:* It's perfectly OK to define voltage as  $V = PE_{elec}/q$ . But recall that it is only *differences* in interaction energy,  $U$ , that have direct physical meaning in physics. Similarly, voltage differences are really more useful than absolute voltages. A voltmeter measures voltage differences, not absolute voltages.

### Discussion questions

**A** A roller coaster is sort of like an electric circuit, but it uses gravitational forces on the cars instead of electric ones. What would a high-voltage roller coaster be like? What would a high-current roller coaster be like?

**B** Criticize the following statements:

“He touched the wire, and 10000 volts went through him.”

“That battery has a charge of 9 volts.”

“You used up the charge of the battery.”

**C** When you touch a 9-volt battery to your tongue, both positive and negative ions move through your saliva. Which ions go which way?

**D** I once touched a piece of physics apparatus that had been wired incorrectly, and got a several-thousand-volt voltage difference across my hand. I was not injured. For what possible reason would the shock have had insufficient power to hurt me?

## 21.6 Resistance

### Resistance

So far we have simply presented it as an observed fact that a battery-and-bulb circuit quickly settles down to a steady flow, but why should it? Newton's second law,  $a = F/m$ , would seem to predict that the steady forces on the charged particles should make them whip around the circuit faster and faster. The answer is that as charged particles move through matter, there are always forces, analogous to frictional forces, that resist the motion. These forces need to be included in Newton's second law, which is really  $a = F_{\text{total}}/m$ , not  $a = F/m$ . If, by analogy, you push a crate across the floor at constant speed, i.e., with zero acceleration, the total force on it must be zero. After you get the crate going, the floor's frictional force is exactly canceling out your force. The chemical energy stored in your body is being transformed into heat in the crate and the floor, and no longer into an increase in the crate's kinetic energy. Similarly, the battery's internal chemical energy is converted into heat, not into perpetually increasing the charged particles' kinetic energy. Changing energy into heat may be a nuisance in some circuits, such as a computer chip, but it is vital in a lightbulb, which must get hot enough to glow. Whether we like it or not, this kind of heating effect is going to occur any time charged particles move through matter.

What determines the amount of heating? One flashlight bulb designed to work with a 9-volt battery might be labeled 1.0 watts, another 5.0. How does this work? Even without knowing the details of this type of friction at the atomic level, we can relate the heat dissipation to the amount of current that flows via the equation  $P = I\Delta V$ . If the two flashlight bulbs can have two different values of  $P$  when used with a battery that maintains the same  $\Delta V$ , it must be that the 5.0-watt bulb allows five times more current to flow through it.

For many substances, including the tungsten from which lightbulb filaments are made, experiments show that the amount of current that will flow through it is directly proportional to the voltage difference placed across it. For an object made of such a substance, we define its electrical *resistance* as follows:

#### definition of resistance

If an object inserted in a circuit displays a current flow proportional to the voltage difference across it, then we define its resistance as the constant ratio

$$R = \Delta V/I$$



k / Georg Simon Ohm (1787-1854).

The units of resistance are volts/ampere, usually abbreviated as ohms, symbolized with the capital Greek letter omega,  $\Omega$ .

▷ A flashlight bulb powered by a 9-volt battery has a resistance of  $10\ \Omega$ . How much current will it draw?

▷ Solving the definition of resistance for  $I$ , we find

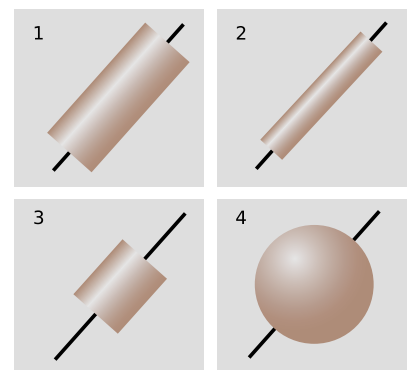
$$\begin{aligned} I &= \Delta V / R \\ &= 0.9\ \text{V} / \Omega \\ &= 0.9\ \text{V} / (\text{V/A}) \\ &= 0.9\ \text{A} \end{aligned}$$

Ohm's law states that many substances, including many solids and some liquids, display this kind of behavior, at least for voltages that are not too large. The fact that Ohm's law is called a "law" should not be taken to mean that all materials obey it, or that it has the same fundamental importance as Newton's laws, for example. Materials are called *ohmic* or *nonohmic*, depending on whether they obey Ohm's law. Although we will concentrate on ohmic materials in this book, it's important to keep in mind that a great many materials are nonohmic, and devices made from them are often very important. For instance, a transistor is a nonohmic device that can be used to amplify a signal (as in a guitar amplifier) or to store and manipulate the ones and zeroes in a computer chip.

If objects of the same size and shape made from two different ohmic materials have different resistances, we can say that one material is more resistive than the other, or equivalently that it is less conductive. Materials, such as metals, that are very conductive are said to be good *conductors*. Those that are extremely poor conductors, for example wood or rubber, are classified as *insulators*. There is no sharp distinction between the two classes of materials. Some, such as silicon, lie midway between the two extremes, and are called *semiconductors*.

On an intuitive level, we can understand the idea of resistance by making the sounds "hhhhhh" and "fffff." To make air flow out of your mouth, you use your diaphragm to compress the air in your chest. The pressure difference between your chest and the air outside your mouth is analogous to a voltage difference. When you make the "h" sound, you form your mouth and throat in a way that allows air to flow easily. The large flow of air is like a large current. Dividing by a large current in the definition of resistance means that we get a small resistance. We say that the small resistance of your mouth and throat allows a large current to flow. When you make the "f" sound, you increase the resistance and cause a smaller current to flow.

Note that although the resistance of an object depends on the substance it is made of, we cannot speak simply of the "resistance of gold" or the "resistance of wood." Figure 1 shows four examples of



1 / Four objects made of the same substance have different resistances.



objects that have had wires attached at the ends as electrical connections. If they were made of the same substance, they would all nevertheless have different resistances because of their different sizes and shapes. A more detailed discussion will be more natural in the context of the following chapter, but it should not be too surprising that the resistance of  $l/2$  will be greater than that of  $l/1$  — the image of water flowing through a pipe, however incorrect, gives us the right intuition. Object  $l/3$  will have a smaller resistance than  $l/1$  because the charged particles have less of it to get through.

## Superconductors

All materials display some variation in resistance according to temperature (a fact that is used in thermostats to make a thermometer that can be easily interfaced to an electric circuit). More spectacularly, most metals have been found to exhibit a sudden change to *zero* resistance when cooled to a certain critical temperature. They are then said to be superconductors. Currently, the most important practical application of superconductivity is in medical MRI (magnetic resonance imaging) scanners. The mechanism of MRI is explained on p. 451, but the important point for now is that when your body is inserted into one of these devices, you are being immersed in an extremely strong magnetic field produced by electric currents flowing through the coiled wires of an electromagnet. If these wires were not superconducting, they would instantly burn up because of the heat generated by their resistance.

There are many other potential applications for superconductors, but most of these, such as power transmission, are not currently economically feasible because of the extremely low temperatures required for superconductivity to occur.

However, it was discovered in 1986 that certain ceramics are superconductors at less extreme temperatures. The technological barrier is now in finding practical methods for making wire out of these brittle materials. Wall Street is currently investing billions of dollars in developing superconducting devices for cellular phone relay stations based on these materials.

There is currently no satisfactory theory of superconductivity in general, although superconductivity in metals is understood fairly well. Unfortunately I have yet to find a fundamental explanation of superconductivity in metals that works at the introductory level.



m / A medical MRI scanner, which uses superconductors.

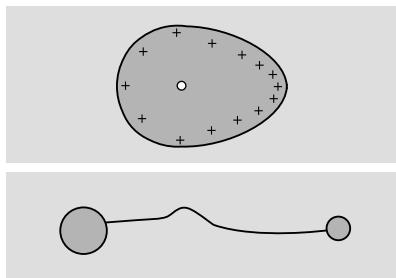
## Constant voltage throughout a conductor

The idea of a superconductor leads us to the question of how we should expect an object to behave if it is made of a very good conductor. Superconductors are an extreme case, but often a metal wire can be thought of as a perfect conductor, for example if the parts of the circuit other than the wire are made of much less conductive materials. What happens if  $R$  equals zero in the equation  $R = \Delta V/I$ ? The result of dividing two numbers can only be zero if the number on top equals zero. This tells us that if we pick any two points in a perfect conductor, the voltage difference between them must be zero. In other words, the entire conductor must be at the same voltage.

Constant voltage means that no work would be done on a charge as it moved from one point in the conductor to another. If zero work was done only along a certain path between two specific points, it might mean that positive work was done along part of the path and negative work along the rest, resulting in a cancellation. But there is no way that the work could come out to be zero for all possible paths unless the electrical force on a charge was in fact zero at every point. Suppose, for example, that you build up a static charge by scuffing your feet on a carpet, and then you deposit some of that charge onto a doorknob, which is a good conductor. How can all that charge be in the doorknob without creating any electrical force at any point inside it? The only possible answer is that the charge moves around until it has spread itself into just the right configuration so that the forces exerted by all the little bits of excess surface charge on any charged particle within the doorknob exactly canceled out.

We can explain this behavior if we assume that the charge placed on the doorknob eventually settles down into a stable equilibrium. Since the doorknob is a conductor, the charge is free to move through it. If it was free to move and any part of it did experience a nonzero total force from the rest of the charge, then it would move, and we would not have an equilibrium.

It also turns out that charge placed on a conductor, once it reaches its equilibrium configuration, is entirely on the surface, not on the interior. We will not prove this fact formally, but it is intuitively reasonable. Suppose, for instance, that the net charge on the conductor is negative, i.e., it has an excess of electrons. These electrons all repel each other, and this repulsion will tend to push them onto the surface, since being on the surface allows them to be as far apart as possible.



n / foo

### The lightning rod

example 8

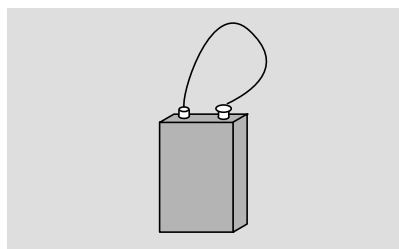
Suppose you have a teardrop-shaped conductor like the one shown in the top panel of figure n. Since the teardrop is a conductor, there are free charges everywhere inside it, but consider a free charged particle at the location shown with a white circle. It seems that we must have a higher density of charge at the tip, since the greater distance from the white charge would otherwise lead to a force too small to cancel with the force from the flatter left-hand end. Experiments show that this is indeed the case, but the reasoning above is a little sloppy. For example, if we had chosen to draw the white charge nearer to the right side of the teardrop, we might have convinced ourselves of the opposite result — which is not what experiments tell us really happens! The difficulty is that these forces are all vectors, and we haven't properly taken into account the angles at which the various forces act.

To make this more rigorous, consider the bottom panel of figure n. Two conducting spheres are connected by a conducting wire. Since the whole apparatus is conducting, it must all be at one voltage. As shown in problem 43 on p. 588, the density of charge is greater on the smaller sphere. This is an example of a more general fact, which is that the charge on a conductor packs itself more densely in areas that are more sharply curved.

Similar reasoning shows why Benjamin Franklin used a sharp tip when he invented the lightning rod. The charged stormclouds induce positive and negative charges to move to opposite ends of the rod. At the pointed upper end of the rod, the charge tends to concentrate at the point, and this charge attracts the lightning. The same effect can sometimes be seen when a scrap of aluminum foil is inadvertently put in a microwave oven. Modern experiments (Moore *et al.*, Journal of Applied Meteorology 39 (1999) 593) show that although a sharp tip is best at starting a spark, a more moderate curve, like the right-hand tip of the teardrop in this problem, is better at successfully sustaining the spark for long enough to connect a discharge to the clouds.

### Short circuits

So far we have been assuming a perfect conductor. What if it is a good conductor, but not a perfect one? Then we can solve for  $\Delta V = IR$ . An ordinary-sized current will make a very small result when we multiply it by the resistance of a good conductor such as a metal wire. The voltage throughout the wire will then be nearly constant. If, on the other hand, the current is extremely large, we can have a significant voltage difference. This is what happens in a *short-circuit*: a circuit in which a low-resistance pathway connects the two sides of a voltage source. Note that this is much more specific than the popular use of the term to indicate any electrical



o / Short-circuiting a battery. Warning: you can burn yourself this way or start a fire! If you want to try this, try making the connection only very briefly, use a low-voltage battery, and avoid touching the battery or the wire, both of which will get hot.

malfunction at all. If, for example, you short-circuit a 9-volt battery as shown in figure o, you will produce perhaps a thousand amperes of current, leading to a very large value of  $P = I\Delta V$ . The wire gets hot!

#### self-check C

What would happen to the battery in this kind of short circuit? ▷

Answer, p. 977

## Resistors

Inside any electronic gadget you will see quite a few little circuit elements like the one shown in the photo. These *resistors* are simply a cylinder of ohmic material with wires attached to the end.

At this stage, most students have a hard time understanding why resistors would be used inside a radio or a computer. We obviously want a lightbulb or an electric stove to have a circuit element that resists the flow of electricity and heats up, but heating is undesirable in radios and computers. Without going too far afield, let's use a mechanical analogy to get a general idea of why a resistor would be used in a radio.

The main parts of a radio receiver are an antenna, a tuner for selecting the frequency, and an amplifier to strengthen the signal sufficiently to drive a speaker. The tuner resonates at the selected frequency, just as in the examples of mechanical resonance discussed in chapter 18. The behavior of a mechanical resonator depends on three things: its inertia, its stiffness, and the amount of friction or damping. The first two parameters locate the peak of the resonance curve, while the damping determines the width of the resonance. In the radio tuner we have an electrically vibrating system that resonates at a particular frequency. Instead of a physical object moving back and forth, these vibrations consist of electrical currents that flow first in one direction and then in the other. In a mechanical system, damping means taking energy out of the vibration in the form of heat, and exactly the same idea applies to an electrical system: the resistor supplies the damping, and therefore controls the width of the resonance. If we set out to eliminate all resistance in the tuner circuit, by not building in a resistor and by somehow getting rid of all the inherent electrical resistance of the wires, we would have a useless radio. The tuner's resonance would be so narrow that we could never get close enough to the right frequency to bring in the station. The roles of inertia and stiffness are played by other circuit elements we have not discussed (a capacitor and a coil).

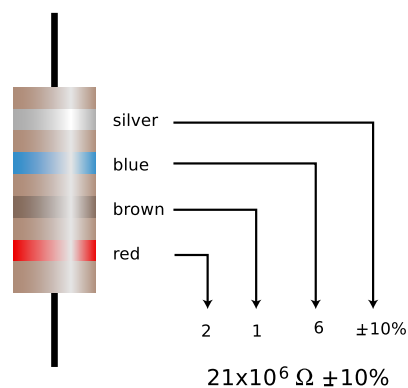
Many electrical devices are based on electrical resistance and Ohm's law, even if they do not have little components in them that look like the usual resistor. The following are some examples.



Resistors.



p / The symbol used in schematics to represent a resistor.



q / An example of a resistor with a color code.

color	meaning
black	0
brown	1
red	2
orange	3
yellow	4
green	5
blue	6
violet	7
gray	8
white	9
silver	±10%
gold	±5%

r / Color codes used on resistors.

### *Lightbulb*

There is nothing special about a lightbulb filament — you can easily make a lightbulb by cutting a narrow waist into a metallic gum wrapper and connecting the wrapper across the terminals of a 9-volt battery. The trouble is that it will instantly burn out. Edison solved this technical challenge by encasing the filament in an evacuated bulb, which prevented burning, since burning requires oxygen.

### *Polygraph*

The polygraph, or “lie detector,” is really just a set of meters for recording physical measures of the subject’s psychological stress, such as sweating and quickened heartbeat. The real-time sweat measurement works on the principle that dry skin is a good insulator, but sweaty skin is a conductor. Of course a truthful subject may become nervous simply because of the situation, and a practiced liar may not even break a sweat. The method’s practitioners claim that they can tell the difference, but you should think twice before allowing yourself to be polygraph tested. Most U.S. courts exclude all polygraph evidence, but some employers attempt to screen out dishonest employees by polygraph testing job applicants, an abuse that ranks with such pseudoscience as handwriting analysis.

### *Fuse*

A fuse is a device inserted in a circuit tollbooth-style in the same manner as an ammeter. It is simply a piece of wire made of metals having a relatively low melting point. If too much current passes through the fuse, it melts, opening the circuit. The purpose is to make sure that the building’s wires do not carry so much current that they themselves will get hot enough to start a fire. Most modern houses use circuit breakers instead of fuses, although fuses are still common in cars and small devices. A circuit breaker is a switch operated by a coiled-wire magnet, which opens the circuit when enough current flows. The advantage is that once you turn off some of the appliances that were sucking up too much current, you can immediately flip the switch closed. In the days of fuses, one might get caught without a replacement fuse, or even be tempted to stuff aluminum foil in as a replacement, defeating the safety feature.

### *Voltmeter*

A voltmeter is nothing more than an ammeter with an additional high-value resistor through which the current is also forced to flow. Ohm’s law relates the current through the resistor is related directly to the voltage difference across it, so the meter can be calibrated in units of volts based on the known value of the resistor. The voltmeter’s two probes are touched to the two locations in a circuit between which we wish to measure the voltage difference,  $s/2$ . Note how cumbersome this type of drawing is, and how difficult it can be to tell what is connected to what. This is why electrical drawing

are usually shown in schematic form. Figure s/3 is a schematic representation of figure s/2.

The setups for measuring current and voltage are different. When we are measuring current, we are finding “how much stuff goes through,” so we place the ammeter where all the current is forced to go through it. Voltage, however, is not “stuff that goes through,” it is a measure of electrical energy. If an ammeter is like the meter that measures your water use, a voltmeter is like a measuring stick that tells you how high a waterfall is, so that you can determine how much energy will be released by each kilogram of falling water. We do not want to force the water to go through the measuring stick! The arrangement in figure s/3 is a *parallel* circuit: one in there are “forks in the road” where some of the current will flow one way and some will flow the other. Figure s/4 is said to be wired in *series*: all the current will visit all the circuit elements one after the other. We will deal with series and parallel circuits in more detail in the following chapter.

If you inserted a voltmeter incorrectly, in series with the bulb and battery, its large internal resistance would cut the current down so low that the bulb would go out. You would have severely disturbed the behavior of the circuit by trying to measure something about it.

Incorrectly placing an ammeter in parallel is likely to be even more disconcerting. The ammeter has nothing but wire inside it to provide resistance, so given the choice, most of the current will flow through it rather than through the bulb. So much current will flow through the ammeter, in fact, that there is a danger of burning out the battery or the meter or both! For this reason, most ammeters have fuses or circuit breakers inside. Some models will trip their circuit breakers and make an audible alarm in this situation, while others will simply blow a fuse and stop working until you replace it.

### Discussion questions

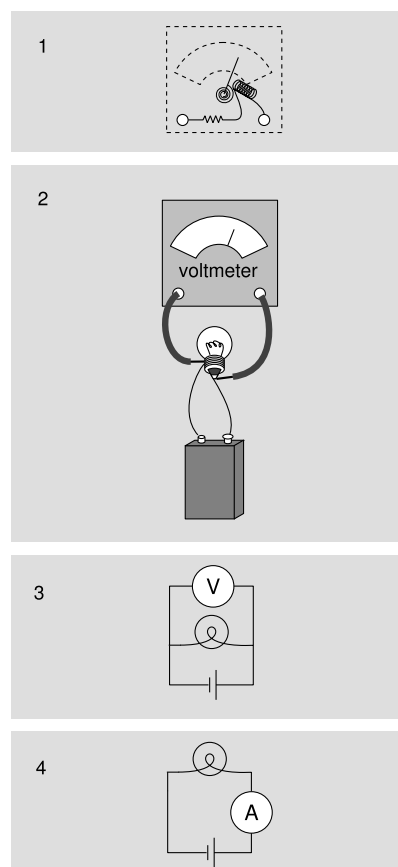
**A** In figure s/1, would it make any difference in the voltage measurement if we touched the voltmeter’s probes to different points along the same segments of wire?

**B** Explain why it would be incorrect to define resistance as the amount of charge the resistor allows to flow.

## 21.7 ∫ Applications of calculus

As discussed in example 1 on page 548, the definition of current as the rate of change of charge with respect to time must be reexpressed as a derivative in the case where the rate of change is not constant,

$$I = \frac{dq}{dt} \quad .$$



s / 1. A simplified diagram of how a voltmeter works. 2. Measuring the voltage difference across a lightbulb. 3. The same setup drawn in schematic form. 4. The setup for measuring current is different.

▷ A charged balloon falls to the ground, and its charge begins leaking off to the Earth. Suppose that the charge on the balloon is given by  $q = ae^{-bt}$ . Find the current as a function of time, and interpret the answer.

▷ Taking the derivative, we have

$$\begin{aligned} I &= \frac{dq}{dt} \\ &= -abe^{-bt} \end{aligned}$$

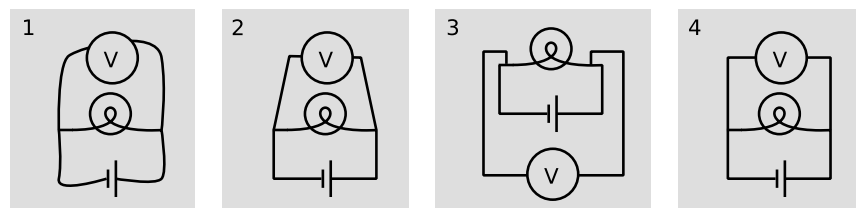
An exponential function approaches zero as the exponent gets more and more negative. This means that both the charge and the current are decreasing in magnitude with time. It makes sense that the charge approaches zero, since the balloon is losing its charge. It also makes sense that the current is decreasing in magnitude, since charge cannot flow at the same rate forever without overshooting zero.

## 21.8 Series and parallel circuits

### Schematics

I see a chess position; Kasparov sees an interesting Ruy Lopez variation. To the uninitiated a schematic may look as unintelligible as Mayan hieroglyphs, but even a little bit of eye training can go a long way toward making its meaning leap off the page. A schematic is a stylized and simplified drawing of a circuit. The purpose is to eliminate as many irrelevant features as possible, so that the relevant ones are easier to pick out.

1. Wrong: The shapes of the wires are irrelevant. 2. Wrong: Right angles should be used. 3. Wrong: A simple pattern is made to look unfamiliar and complicated. 4. Right.



An example of an irrelevant feature is the physical shape, length, and diameter of a wire. In nearly all circuits, it is a good approximation to assume that the wires are perfect conductors, so that any piece of wire uninterrupted by other components has constant voltage throughout it. Changing the length of the wire, for instance, does not change this fact. (Of course if we used miles and miles of wire, as in a telephone line, the wire's resistance would start to add up, and its length would start to matter.) The shapes of the wires are likewise irrelevant, so we draw them with standardized, stylized shapes made only of vertical and horizontal lines with right-angle

bends in them. This has the effect of making similar circuits look more alike and helping us to recognize familiar patterns, just as words in a newspaper are easier to recognize than handwritten ones. Figure t shows some examples of these concepts.

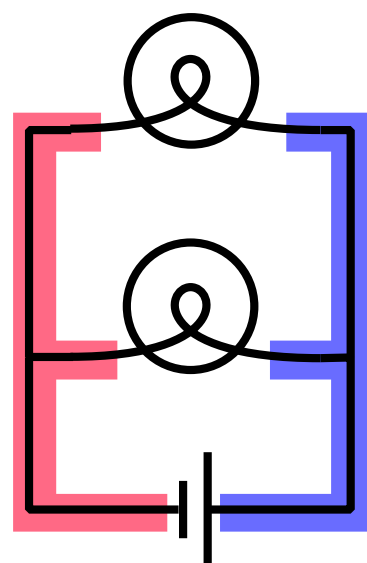
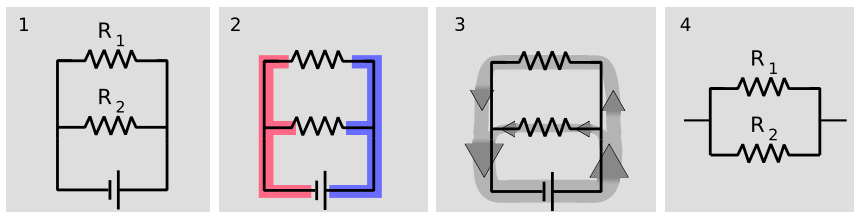
The most important first step in learning to read schematics is to learn to recognize contiguous pieces of wire which must have constant voltage throughout. In figure u, for example, the two shaded E-shaped pieces of wire must each have constant voltage. This focuses our attention on two of the main unknowns we'd like to be able to predict: the voltage of the left-hand E and the voltage of the one on the right.

### Parallel resistances and the junction rule

One of the simplest examples to analyze is the parallel resistance circuit, of which figure u was an example. In general we may have unequal resistances  $R_1$  and  $R_2$ , as in v/1. Since there are only two constant-voltage areas in the circuit,  $v/2$ , all three components have the same voltage difference across them. A battery normally succeeds in maintaining the voltage differences across itself for which it was designed, so the voltage drops  $\Delta V_1$  and  $\Delta V_2$  across the resistors must both equal the voltage of the battery:

$$\Delta V_1 = \Delta V_2 = \Delta V_{\text{battery}} \quad .$$

Each resistance thus feels the same voltage difference as if it was the only one in the circuit, and Ohm's law tells us that the amount of current flowing through each one is also the same as it would have been in a one-resistor circuit. This is why household electrical circuits are wired in parallel. We want every appliance to work the same, regardless of whether other appliances are plugged in or unplugged, turned on or switched off. (The electric company doesn't use batteries of course, but our analysis would be the same for any device that maintains a constant voltage.)



u / The two shaded areas shaped like the letter “E” are both regions of constant voltage.

v / 1. Two resistors in parallel. 2. There are two constant-voltage areas. 3. The current that comes out of the battery splits between the two resistors, and later reunites. 4. The two resistors in parallel can be treated as a single resistor with a smaller resistance value.

Of course the electric company can tell when we turn on every light in the house. How do they know? The answer is that we draw more current. Each resistance draws a certain amount of current, and



the amount that has to be supplied is the sum of the two individual currents. The current is like a river that splits in half,  $v/3$ , and then reunites. The total current is

$$I_{total} = I_1 + I_2 \quad .$$

This is an example of a general fact called the junction rule:

**the junction rule**

In any circuit that is not storing or releasing charge, conservation of charge implies that the total current flowing out of any junction must be the same as the total flowing in.

Coming back to the analysis of our circuit, we apply Ohm's law to each resistance, resulting in

$$\begin{aligned} I_{total} &= \Delta V/R_1 + \Delta V/R_2 \\ &= \Delta V \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad . \end{aligned}$$

As far as the electric company is concerned, your whole house is just one resistor with some resistance  $R$ , called the *equivalent resistance*. They would write Ohm's law as

$$I_{total} = \Delta V/R \quad ,$$

from which we can determine the equivalent resistance by comparison with the previous expression:

$$\begin{aligned} 1/R &= \frac{1}{R_1} + \frac{1}{R_2} \\ R &= \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \end{aligned}$$

[equivalent resistance of two resistors in parallel]

Two resistors in parallel,  $v/4$ , are equivalent to a single resistor with a value given by the above equation.

---

*Two lamps on the same household circuit* *example 10*

▷ You turn on two lamps that are on the same household circuit. Each one has a resistance of 1 ohm. What is the equivalent resistance, and how does the power dissipation compare with the case of a single lamp?

▷ The equivalent resistance of the two lamps in parallel is

$$\begin{aligned} R &= \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \\ &= \left( \frac{1}{1 \, \Omega} + \frac{1}{1 \, \Omega} \right)^{-1} \\ &= \left( 1 \, \Omega^{-1} + 1 \, \Omega^{-1} \right)^{-1} \\ &= \left( 2 \, \Omega^{-1} \right)^{-1} \\ &= 0.5 \, \Omega \end{aligned}$$

The voltage difference across the whole circuit is always the 110 V set by the electric company (it's alternating current, but that's irrelevant). The resistance of the whole circuit has been cut in half by turning on the second lamp, so a fixed amount of voltage will produce twice as much current. Twice the current flowing across the same voltage difference means twice as much power dissipation, which makes sense.

The cutting in half of the resistance surprises many students, since we are “adding more resistance” to the circuit by putting in the second lamp. Why does the equivalent resistance come out to be less than the resistance of a single lamp? This is a case where purely verbal reasoning can be misleading. A resistive circuit element, such as the filament of a lightbulb, is neither a perfect insulator nor a perfect conductor. Instead of analyzing this type of circuit in terms of “resistors,” i.e., partial insulators, we could have spoken of “conductors.” This example would then seem reasonable, since we “added more conductance,” but one would then have the incorrect expectation about the case of resistors in series, discussed in the following section.

Perhaps a more productive way of thinking about it is to use mechanical intuition. By analogy, your nostrils resist the flow of air through them, but having two nostrils makes it twice as easy to breathe.

### Three resistors in parallel

### example 11

▷ What happens if we have three or more resistors in parallel?

▷ This is an important example, because the solution involves an important technique for understanding circuits: breaking them down into smaller parts and then simplifying those parts. In the circuit 21.8.2/1, with three resistors in parallel, we can think of two of the resistors as forming a single resistor, 21.8.2/2, with equivalent resistance

$$R_{12} = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}.$$

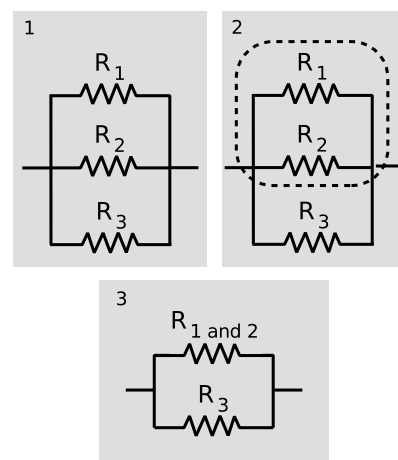
We can then simplify the circuit as shown in 21.8.2/3, so that it contains only two resistances. The equivalent resistance of the whole circuit is then given by

$$R_{123} = \left( \frac{1}{R_{12}} + \frac{1}{R_3} \right)^{-1}.$$

Substituting for  $R_{12}$  and simplifying, we find the result

$$R_{123} = \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1},$$

which you probably could have guessed. The interesting point here is the divide-and-conquer concept, not the mathematical result.



Example 11.

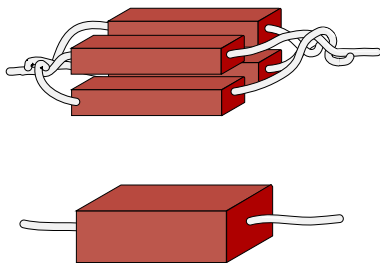
*An arbitrary number of identical resistors in parallel example 12*

- ▷ What is the resistance of  $N$  identical resistors in parallel?
- ▷ Generalizing the results for two and three resistors, we have

$$R_N = \left( \frac{1}{R_1} + \frac{1}{R_2} + \dots \right)^{-1},$$

where “...” means that the sum includes all the resistors. If all the resistors are identical, this becomes

$$\begin{aligned} R_N &= \left( \frac{N}{R} \right)^{-1} \\ &= \frac{R}{N} \end{aligned}$$



Example 13: Uniting four resistors in parallel is equivalent to making a single resistor with the same length but four times the cross-sectional area. The result is to make a resistor with one quarter the resistance.

*Dependence of resistance on cross-sectional area example 13*

We have alluded briefly to the fact that an object's electrical resistance depends on its size and shape, but now we are ready to begin making more mathematical statements about it. As suggested by figure 13, increasing a resistor's cross-sectional area is equivalent to adding more resistors in parallel, which will lead to an overall decrease in resistance. Any real resistor with straight, parallel sides can be sliced up into a large number of pieces, each with cross-sectional area of, say,  $1 \mu\text{m}^2$ . The number,  $N$ , of such slices is proportional to the total cross-sectional area of the resistor, and by application of the result of the previous example we therefore find that the resistance of an object is inversely proportional to its cross-sectional area.



A fat pipe has less resistance than a skinny pipe.

An analogous relationship holds for water pipes, which is why high-flow trunk lines have to have large cross-sectional areas. To make lots of water (current) flow through a skinny pipe, we'd need an impractically large pressure (voltage) difference.

### Incorrect readings from a voltmeter

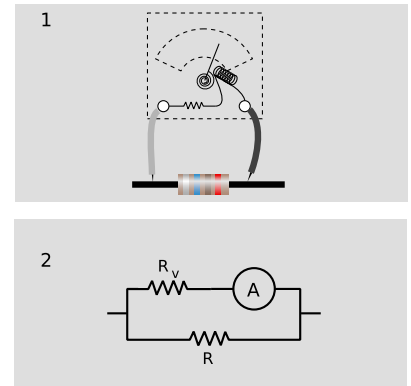
### example 14

A voltmeter is really just an ammeter with an internal resistor, and we use a voltmeter in parallel with the thing that we're trying to measure the voltage difference across. This means that any time we measure the voltage drop across a resistor, we're essentially putting two resistors in parallel. The ammeter inside the voltmeter can be ignored for the purpose of analyzing how current flows in the circuit, since it is essentially just some coiled-up wire with a very low resistance.

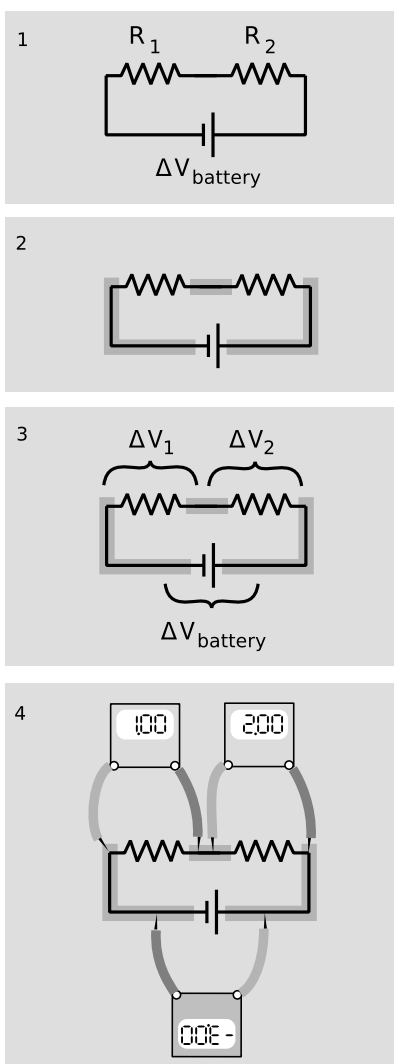
Now if we are carrying out this measurement on a resistor that is part of a larger circuit, we have changed the behavior of the circuit through our act of measuring. It is as though we had modified the circuit by replacing the resistance  $R$  with the smaller equivalent resistance of  $R$  and  $R_v$  in parallel. It is for this reason that voltmeters are built with the largest possible internal resistance. As a numerical example, if we use a voltmeter with an internal resistance of  $1\text{ M}\Omega$  to measure the voltage drop across a one-ohm resistor, the equivalent resistance is  $0.999999\ \Omega$ , which is not different enough to make any difference. But if we tried to use the same voltmeter to measure the voltage drop across a  $2\text{ M}\Omega$  resistor, we would be reducing the resistance of that part of the circuit by a factor of three, which would produce a drastic change in the behavior of the whole circuit.

This is the reason why you can't use a voltmeter to measure the voltage difference between two different points in mid-air, or between the ends of a piece of wood. This is by no means a stupid thing to want to do, since the world around us is not a constant-voltage environment, the most extreme example being when an electrical storm is brewing. But it will not work with an ordinary voltmeter because the resistance of the air or the wood is many gigaohms. The effect of waving a pair of voltmeter probes around in the air is that we provide a reuniting path for the positive and negative charges that have been separated — through the voltmeter itself, which is a good conductor compared to the air. This reduces to zero the voltage difference we were trying to measure.

In general, a voltmeter that has been set up with an open circuit (or a very large resistance) between its probes is said to be “floating.” An old-fashioned analog voltmeter of the type described here will read zero when left floating, the same as when it was sitting on the shelf. A floating digital voltmeter usually shows an error message.



x / A voltmeter is really an ammeter with an internal resistor. When we measure the voltage difference across a resistor, 1, we are really constructing a parallel resistance circuit, 2.



y / 1. A battery drives current through two resistors in series. 2. There are three constant-voltage regions. 3. The three voltage differences are related. 4. If the meter crab-walks around the circuit without flipping over or crossing its legs, the resulting voltages have plus and minus signs that make them add up to zero.

## Series resistances

The two basic circuit layouts are parallel and series, so a pair of resistors in series, y/1, is another of the most basic circuits we can make. By conservation of charge, all the current that flows through one resistor must also flow through the other (as well as through the battery):

$$I_1 = I_2 \quad .$$

The only way the information about the two resistance values is going to be useful is if we can apply Ohm's law, which will relate the resistance of each resistor to the current flowing through it and the voltage difference across it. Figure y/2 shows the three constant-voltage areas. Voltage differences are more physically significant than voltages, so we define symbols for the voltage differences across the two resistors in figure y/3.

We have three constant-voltage areas, with symbols for the difference in voltage between every possible pair of them. These three voltage differences must be related to each other. It is as though I tell you that Fred is a foot taller than Ginger, Ginger is a foot taller than Sally, and Fred is two feet taller than Sally. The information is redundant, and you really only needed two of the three pieces of data to infer the third. In the case of our voltage differences, we have

$$|\Delta V_1| + |\Delta V_2| = |\Delta V_{battery}| \quad .$$

The absolute value signs are because of the ambiguity in how we define our voltage differences. If we reversed the two probes of the voltmeter, we would get a result with the opposite sign. Digital voltmeters will actually provide a minus sign on the screen if the wire connected to the "V" plug is lower in voltage than the one connected to the "COM" plug. Analog voltmeters pin the needle against a peg if you try to use them to measure negative voltages, so you have to fiddle to get the leads connected the right way, and then supply any necessary minus sign yourself.

Figure y/4 shows a standard way of taking care of the ambiguity in signs. For each of the three voltage measurements around the loop, we keep the same probe (the darker one) on the clockwise side. It is as though the voltmeter was sidling around the circuit like a crab, without ever "crossing its legs." With this convention, the relationship among the voltage drops becomes

$$\Delta V_1 + \Delta V_2 = -\Delta V_{battery} \quad ,$$

or, in more symmetrical form,

$$\Delta V_1 + \Delta V_2 + \Delta V_{battery} = 0 \quad .$$

More generally, this is known as the loop rule for analyzing circuits:

**the loop rule**

Assuming the standard convention for plus and minus signs, the sum of the voltage drops around any closed loop in a circuit must be zero.

Looking for an exception to the loop rule would be like asking for a hike that would be downhill all the way and that would come back to its starting point!

For the circuit we set out to analyze, the equation

$$\Delta V_1 + \Delta V_2 + \Delta V_{\text{battery}} = 0$$

can now be rewritten by applying Ohm's law to each resistor:

$$I_1 R_1 + I_2 R_2 + \Delta V_{\text{battery}} = 0 \quad .$$

The currents are the same, so we can factor them out:

$$I (R_1 + R_2) + \Delta V_{\text{battery}} = 0 \quad ,$$

and this is the same result we would have gotten if we had been analyzing a one-resistor circuit with resistance  $R_1 + R_2$ . Thus the equivalent resistance of resistors in series equals the sum of their resistances.

*Two lightbulbs in series*

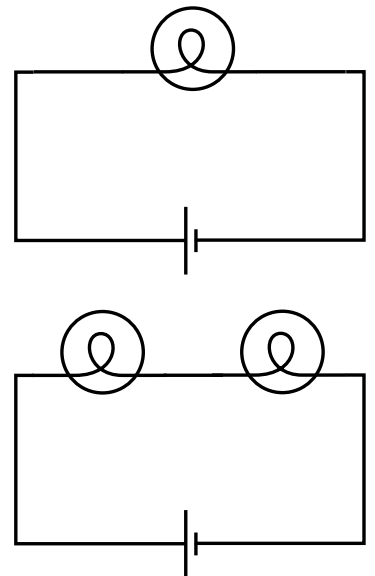
*example 15*

▷ If two identical lightbulbs are placed in series, how do their brightnesses compare with the brightness of a single bulb?

▷ Taken as a whole, the pair of bulbs act like a doubled resistance, so they will draw half as much current from the wall. Each bulb will be dimmer than a single bulb would have been.

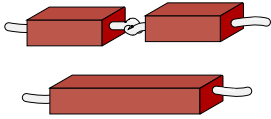
The total power dissipated by the circuit is  $I\Delta V$ . The voltage drop across the whole circuit is the same as before, but the current is halved, so the two-bulb circuit draws half as much total power as the one-bulb circuit. Each bulb draws one-quarter of the normal power.

Roughly speaking, we might expect this to result in one quarter the light being produced by each bulb, but in reality lightbulbs waste quite a high percentage of their power in the form of heat and wavelengths of light that are not visible (infrared and ultraviolet). Less light will be produced, but it's hard to predict exactly how much less, since the efficiency of the bulbs will be changed by operating them under different conditions.



Example 15.

*More than two equal resistances in series* *example 16*  
 By straightforward application of the divide-and-conquer technique discussed in the previous section, we find that the equivalent resistance of  $N$  identical resistances  $R$  in series will be  $NR$ .



Example 17. Doubling the length of a resistor is like putting two resistors in series. The resistance is doubled.

*Dependence of resistance on length* *example 17*  
 In the previous section, we proved that resistance is inversely proportional to cross-sectional area. By equivalent reason about resistances in series, we find that resistance is proportional to length. Analogously, it is harder to blow through a long straw than through a short one.

Combining the results of examples 13 and 17, we find that the resistance of an object with straight, parallel sides is given by

$$R = (\text{constant}) \cdot L/A$$

The proportionality constant is called the resistivity, and it depends only on the substance of which the object is made. A resistivity measurement could be used, for instance, to help identify a sample of an unknown substance.

*Choice of high voltage for power lines* *example 18*  
 Thomas Edison got involved in a famous technological controversy over the voltage difference that should be used for electrical power lines. At this time, the public was unfamiliar with electricity, and easily scared by it. The president of the United States, for instance, refused to have electrical lighting in the White House when it first became commercially available because he considered it unsafe, preferring the known fire hazard of oil lamps to the mysterious dangers of electricity. Mainly as a way to overcome public fear, Edison believed that power should be transmitted using small voltages, and he publicized his opinion by giving demonstrations at which a dog was lured into position to be killed by a large voltage difference between two sheets of metal on the

ground. (Edison's opponents also advocated alternating current rather than direct current, and AC is more dangerous than DC as well. As we will discuss later, AC can be easily stepped up and down to the desired voltage level using a device called a transformer.)

Now if we want to deliver a certain amount of power  $P_L$  to a load such as an electric lightbulb, we are constrained only by the equation  $P_L = I\Delta V_L$ . We can deliver any amount of power we wish, even with a low voltage, if we are willing to use large currents. Modern electrical distribution networks, however, use dangerously high voltage differences of tens of thousands of volts. Why did Edison lose the debate?

It boils down to money. The electric company must deliver the amount of power  $P_L$  desired by the customer through a transmission line whose resistance  $R_T$  is fixed by economics and geography. The same current flows through both the load and the transmission line, dissipating power usefully in the former and wastefully in the latter. The efficiency of the system is

$$\begin{aligned}\text{efficiency} &= \frac{\text{power paid for by the customer}}{\text{power paid for by the utility}} \\ &= \frac{P_L}{P_L + P_T} \\ &= \frac{1}{1 + P_T/P_L}\end{aligned}$$

Putting ourselves in the shoes of the electric company, we wish to get rid of the variable  $P_T$ , since it is something we control only indirectly by our choice of  $\Delta V_T$  and  $I$ . Substituting  $P_T = I\Delta V_T$ , we find

$$\text{efficiency} = \frac{1}{1 + \frac{I\Delta V_T}{P_L}}$$

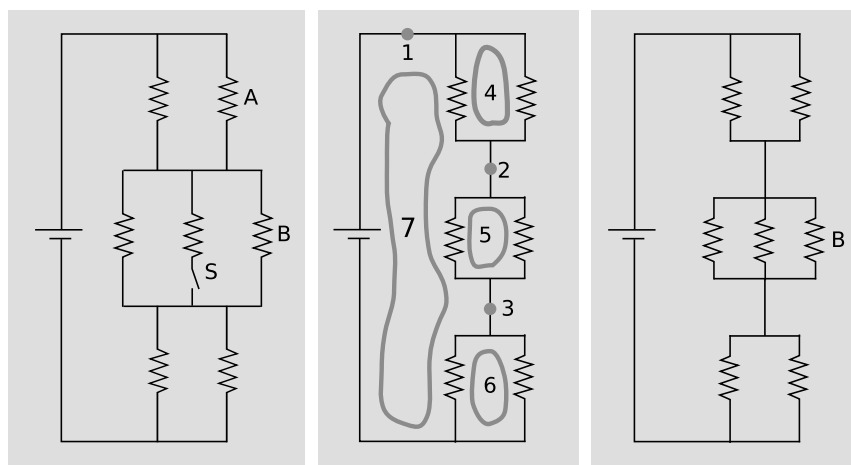
We assume the transmission line (but not necessarily the load) is ohmic, so substituting  $\Delta V_T = IR_T$  gives

$$\text{efficiency} = \frac{1}{1 + \frac{I^2 R_T}{P_L}}$$

This quantity can clearly be maximized by making  $I$  as small as possible, since we will then be dividing by the smallest possible quantity on the bottom of the fraction. A low-current circuit can only deliver significant amounts of power if it uses high voltages, which is why electrical transmission systems use dangerous high voltages.



Example 19.



### A complicated circuit

example 19

▷ All seven resistors in the left-hand panel of figure z are identical. Initially, the switch  $S$  is open as shown in the figure, and the current through resistor  $A$  is  $I_0$ . The switch is then closed. Find the current through resistor  $B$ , after the switch is closed, in terms of  $I_0$ .

▷ The second panel shows the circuit redrawn for simplicity, in the initial condition with the switch open. When the switch is open, no current can flow through the central resistor, so we may as well ignore it. I've also redrawn the junctions, without changing what's connected to what. This is the kind of mental rearranging that you'll eventually learn to do automatically from experience with analyzing circuits. The redrawn version makes it easier to see what's happening with the current. Charge is conserved, so any charge that flows past point 1 in the circuit must also flow past points 2 and 3. This would have been harder to reason about by applying the junction rule to the original version, which appears to have nine separate junctions.

In the new version, it's also clear that the circuit has a great deal of symmetry. We could flip over each parallel pair of identical resistors without changing what's connected to what, so that makes it clear that the voltage drops and currents must be equal for the members of each pair. We can also prove this by using the loop rule. The loop rule says that the two voltage drops in loop 4 must be equal, and similarly for loops 5 and 6. Since the resistors obey Ohm's law, equal voltage drops across them also imply equal currents. That means that when the current at point 1 comes to the top junction, exactly half of it goes through each resistor. Then the current reunites at 2, splits between the next pair, and so on. We conclude that each of the six resistors in the circuit experiences the same voltage drop and the same current. Applying the

loop rule to loop 7, we find that the sum of the three voltage drops across the three left-hand resistors equals the battery's voltage,  $V$ , so each resistor in the circuit experiences a voltage drop  $V/3$ . Letting  $R$  stand for the resistance of one of the resistors, we find that the current through resistor B, which is the same as the currents through all the others, is given by  $I_0 = V/3R$ .

We now pass to the case where the switch is closed, as shown in the third panel. The battery's voltage is the same as before, and each resistor's resistance is the same, so we can still use the same symbols  $V$  and  $R$  for them. It is no longer true, however, that each resistor feels a voltage drop  $V/3$ . The equivalent resistance of the whole circuit is  $R/2 + R/3 + R/2 = 4R/3$ , so the total current drawn from the battery is  $3V/4R$ . In the middle group of resistors, this current is split three ways, so the new current through B is  $(1/3)(3V/4R) = V/4R = 3I_0/4$ .

Interpreting this result, we see that it comes from two effects that partially cancel. Closing the switch reduces the equivalent resistance of the circuit by giving charge another way to flow, and increases the amount of current drawn from the battery. Resistor B, however, only gets a  $1/3$  share of this greater current, not  $1/2$ . The second effect turns out to be bigger than first, and therefore the current through resistor B is lessened over all.

#### *Getting killed by your ammeter*

#### *example 20*

As with a voltmeter, an ammeter can give erroneous readings if it is used in such a way that it changes the behavior the circuit. An ammeter is used in series, so if it is used to measure the current through a resistor, the resistor's value will effectively be changed to  $R + R_a$ , where  $R_a$  is the resistance of the ammeter. Ammeters are designed with very low resistances in order to make it unlikely that  $R + R_a$  will be significantly different from  $R$ .

In fact, the real hazard is death, not a wrong reading! Virtually the only circuits whose resistances are significantly less than that of an ammeter are those designed to carry huge currents. An ammeter inserted in such a circuit can easily melt. When I was working at a laboratory funded by the Department of Energy, we got periodic bulletins from the DOE safety office about serious accidents at other sites, and they held a certain ghoulish fascination. One of these was about a DOE worker who was completely incinerated by the explosion created when he inserted an ordinary Radio Shack ammeter into a high-current circuit. Later estimates showed that the heat was probably so intense that the explosion was a ball of plasma — a gas so hot that its atoms have been ionized.

### Discussion question

**A** We have stated the loop rule in a symmetric form where a series of voltage drops adds up to zero. To do this, we had to define a standard way of connecting the voltmeter to the circuit so that the plus and minus signs would come out right. Suppose we wish to restate the junction rule in a similar symmetric way, so that instead of equating the current coming in to the current going out, it simply states that a certain sum of currents at a junction adds up to zero. What standard way of inserting the ammeter would we have to use to make this work?

## Summary

### Selected vocabulary

charge . . . . .	a numerical rating of how strongly an object participates in electrical forces
coulomb (C) . . .	the unit of electrical charge
current . . . . .	the rate at which charge crosses a certain boundary
ampere . . . . .	the metric unit of current, one coulomb per second; also “amp”
ammeter . . . . .	a device for measuring electrical current
circuit . . . . .	an electrical device in which charge can come back to its starting point and be recycled rather than getting stuck in a dead end
open circuit . . .	a circuit that does not function because it has a gap in it
short circuit . . .	a circuit that does not function because charge is given a low-resistance “shortcut” path that it can follow, instead of the path that makes it do something useful
voltage . . . . .	electrical potential energy per unit charge that will be possessed by a charged particle at a certain point in space
volt . . . . .	the metric unit of voltage, one joule per coulomb
voltmeter . . . . .	a device for measuring voltage differences
ohmic . . . . .	describes a substance in which the flow of current between two points is proportional to the voltage difference between them
resistance . . . .	the ratio of the voltage difference to the current in an object made of an ohmic substance
ohm . . . . .	the metric unit of electrical resistance, one volt per ampere

### Notation

$q$ . . . . .	charge
$I$ . . . . .	current
$A$ . . . . .	units of amperes
$V$ . . . . .	voltage
$V$ . . . . .	units of volts
$R$ . . . . .	resistance
$\Omega$ . . . . .	units of ohms

### Other terminology and notation

electric potential	rather than the more informal “voltage” used here; despite the misleading name, it is not the same as electric potential energy
eV . . . . .	a unit of energy, equal to $e$ multiplied by 1 volt; $1.6 \times 10^{-19}$ joules

## Summary

All the forces we encounter in everyday life boil down to two basic types: gravitational forces and electrical forces. A force such as friction or a “sticky force” arises from electrical forces between individual atoms.

Just as we use the word “mass” to describe how strongly an object participates in gravitational forces, we use the word “charge” for the intensity of its electrical forces. There are two types of charge. Two charges of the same type repel each other, but objects whose charges are different attract each other. Charge is measured in units of coulombs (C).

Mobile charged particle model: A great many phenomena are easily understood if we imagine matter as containing two types of charged particles, which are at least partially able to move around.

Positive and negative charge: Ordinary objects that have not been specially prepared have both types of charge spread evenly throughout them in equal amounts. The object will then tend not to exert electrical forces on any other object, since any attraction due to one type of charge will be balanced by an equal repulsion from the other. (We say “tend not to” because bringing the object near an object with unbalanced amounts of charge could cause its charges to separate from each other, and the force would no longer cancel due to the unequal distances.) It therefore makes sense to describe the two types of charge using positive and negative signs, so that an unprepared object will have zero *total* charge.

The Coulomb force law states that the magnitude of the electrical force between two charged particles is given by  $|\mathbf{F}| = k|q_1||q_2|/r^2$ .

Conservation of charge: An even more fundamental reason for using positive and negative signs for charge is that with this definition the total charge of a closed system is a conserved quantity.

All electrical phenomena are alike in that they arise from the presence or motion of charge. Most practical electrical devices are based on the motion of charge around a complete circuit, so that the charge can be recycled and does not hit any dead ends. The most useful measure of the flow of charge is current,  $I = \Delta q / \Delta t$ .

An electrical device whose job is to transform energy from one form into another, e.g., a lightbulb, uses power at a rate which depends both on how rapidly charge is flowing through it and on how much work is done on each unit of charge. The latter quantity is known as the voltage difference between the point where the current enters the device and the point where the current leaves it. Since there is a type of potential energy associated with electrical forces, the amount of work they do is equal to the difference in potential energy between the two points, and we therefore define voltage differences directly

in terms of potential energy,  $\Delta V = \Delta PE_{elec}/q$ . The rate of power dissipation is  $P = I\Delta V$ .

Many important electrical phenomena can only be explained if we understand the mechanisms of current flow at the atomic level. In metals, currents are carried by electrons, in liquids by ions. Gases are normally poor conductors unless their atoms are subjected to such intense electrical forces that the atoms become ionized.

Many substances, including all solids, respond to electrical forces in such a way that the flow of current between two points is proportional to the voltage difference between those points. Such a substance is called ohmic, and an object made out of an ohmic substance can be rated in terms of its resistance,  $R = \Delta V/I$ . An important corollary is that a perfect conductor, with  $R = 0$ , must have constant voltage everywhere within it.

A schematic is a drawing of a circuit that standardizes and stylizes its features to make it easier to understand. Any circuit can be broken down into smaller parts. For instance, one big circuit may be understood as two small circuits in series, another as three circuits in parallel. When circuit elements are combined in parallel and in series, we have two basic rules to guide us in understanding how the parts function as a whole:

**the junction rule:** In any circuit that is not storing or releasing charge, conservation of charge implies that the total current flowing out of any junction must be the same as the total flowing in.

**the loop rule:** Assuming the standard convention for plus and minus signs, the sum of the voltage drops around any closed loop in a circuit must be zero.

The simplest application of these rules is to pairs of resistors combined in series or parallel. In such cases, the pair of resistors acts just like a single unit with a certain resistance value, called their equivalent resistance. Resistances in series add to produce a larger equivalent resistance,

$$R_{series} = R_1 + R_2 \quad ,$$

because the current has to fight its way through both resistances. Parallel resistors combine to produce an equivalent resistance that is smaller than either individual resistance,

$$R_{parallel} = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \quad ,$$

because the current has two different paths open to it.

An important example of resistances in parallel and series is the use of voltmeters and ammeters in resistive circuits. A voltmeter acts

as a large resistance in parallel with the resistor across which the voltage drop is being measured. The fact that its resistance is not infinite means that it alters the circuit it is being used to investigate, producing a lower equivalent resistance. An ammeter acts as a small resistance in series with the circuit through which the current is to be determined. Its resistance is not quite zero, which leads to an increase in the resistance of the circuit being tested.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** The figure shows a neuron, which is the type of cell your nerves are made of. Neurons serve to transmit sensory information to the brain, and commands from the brain to the muscles. All this data is transmitted electrically, but even when the cell is resting and not transmitting any information, there is a layer of negative electrical charge on the inside of the cell membrane, and a layer of positive charge just outside it. This charge is in the form of various ions dissolved in the interior and exterior fluids. Why would the negative charge remain plastered against the inside surface of the membrane, and likewise why doesn't the positive charge wander away from the outside surface?

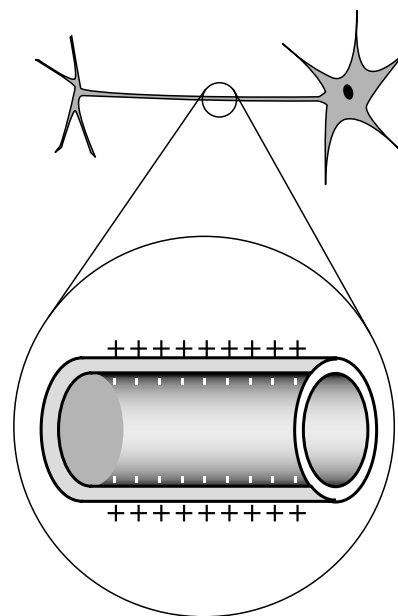
**2** A helium atom finds itself momentarily in this arrangement. Find the direction and magnitude of the force acting on the right-hand electron. The two protons in the nucleus are so close together ( $\sim 1$  fm) that you can consider them as being right on top of each other. As discussed in chapter 26, the charge of an electron is  $-e$ , and the charge of a proton  $e$ , where  $e = 1.60 \times 10^{-19}$  C. ✓

**3** The helium atom of problem 2 has some new experiences, goes through some life changes, and later on finds itself in the configuration shown here. What are the direction and magnitude of the force acting on the bottom electron? (Draw a sketch to make clear the definition you are using for the angle that gives direction.) ✓

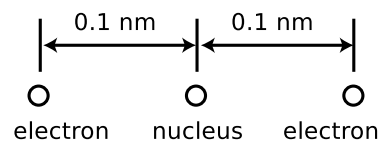
**4** [See chapter 1 for help on how to do order-of-magnitude estimates.] Suppose you are holding your hands in front of you, 10 cm apart.

- (a) Estimate the total number of electrons in each hand. ✓
- (b) Estimate the total repulsive force of all the electrons in one hand on all the electrons in the other. ✓
- (c) Why don't you feel your hands repelling each other?
- (d) Estimate how much the charge of a proton could differ in magnitude from the charge of an electron without creating a noticeable force between your hands.

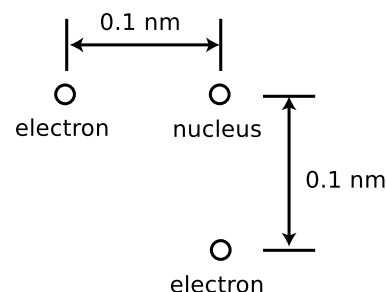
**5** Suppose that a proton in a lead nucleus wanders out to the surface of the nucleus, and experiences a strong nuclear force of about 8 kN from the nearby neutrons and protons pulling it back in. Compare this numerically to the repulsive electrical force from the other protons, and verify that the net force is attractive. A lead nucleus is very nearly spherical, is about 6.5 fm in radius, and contains 82 protons, each with a charge of  $+e$ , where  $e = 1.60 \times 10^{-19}$  C. ✓



**Problem 1.** Top: A realistic picture of a neuron. Bottom: A simplified diagram of one segment of the tail (axon).



**Problem 2.**



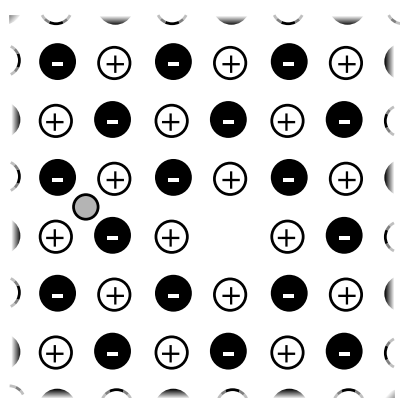
**Problem 3.**



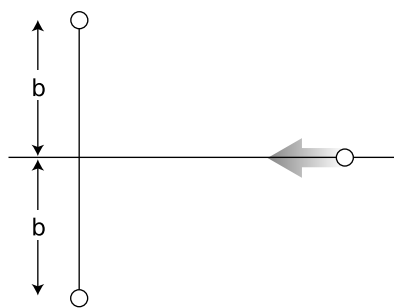
**6** The subatomic particles called muons behave exactly like electrons, except that a muon's mass is greater by a factor of 206.77. Muons are continually bombarding the Earth as part of the stream of particles from space known as cosmic rays. When a muon strikes an atom, it can displace one of its electrons. If the atom happens to be a hydrogen atom, then the muon takes up an orbit that is on the average 206.77 times closer to the proton than the orbit of the ejected electron. How many times greater is the electric force experienced by the muon than that previously felt by the electron? ✓

**7** The Earth and Moon are bound together by gravity. If, instead, the force of attraction were the result of each having a charge of the same magnitude but opposite in sign, find the quantity of charge that would have to be placed on each to produce the required force. ✓

**8** The figure shows one layer of the three-dimensional structure of a salt crystal. The atoms extend much farther off in all directions, but only a six-by-six square is shown here. The larger circles are the chlorine ions, which have charges of  $-e$ , where  $e = 1.60 \times 10^{-19}$  C. The smaller circles are sodium ions, with charges of  $+e$ . The center-to-center distance between neighboring ions is about 0.3 nm. Real crystals are never perfect, and the crystal shown here has two defects: a missing atom at one location, and an extra lithium atom, shown as a grey circle, inserted in one of the small gaps. If the lithium atom has a charge of  $+e$ , what is the direction and magnitude of the total force on it? Assume there are no other defects nearby in the crystal besides the two shown here. [Hints: The force on the lithium ion is the vector sum of all the forces of all the quadrillions of sodium and chlorine atoms, which would obviously be too laborious to calculate. Nearly all of these forces, however, are canceled by a force from an ion on the opposite side of the lithium.] ✓ ★



Problem 8.

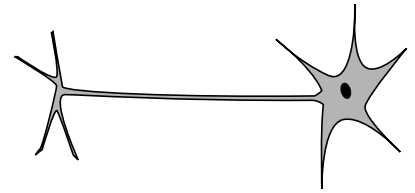


Problem 9.

**9** In the semifinals of an electrostatic croquet tournament, Jessica hits her positively charged ball, sending it across the playing field, rolling to the left along the  $x$  axis. It is repelled by two other positive charges. These two equal charges are fixed on the  $y$  axis at the locations shown in the figure. (a) Express the force on the ball in terms of the ball's position,  $x$ . (b) At what value of  $x$  does the ball experience the greatest deceleration? Express your answer in terms of  $b$ . [Based on a problem by Halliday and Resnick.] ✓ ∫

**10** In a wire carrying a current of 1.0 pA, how long do you have to wait, on the average, for the next electron to pass a given point? Express your answer in units of microseconds. The charge of an electron is  $-e = -1.60 \times 10^{-19}$  C. ▷ Solution, p. 973 ✓

**11** Referring back to our old friend the neuron from problem 1 on page 581, let's now consider what happens when the nerve is stimulated to transmit information. When the blob at the top (the cell body) is stimulated, it causes  $\text{Na}^+$  ions to rush into the top of the tail (axon). This electrical pulse will then travel down the axon, like a flame burning down from the end of a fuse, with the  $\text{Na}^+$  ions at each point first going out and then coming back in. If  $10^{10}$   $\text{Na}^+$  ions cross the cell membrane in 0.5 ms, what amount of current is created? The charge of a  $\text{Na}^+$  ion is  $+e = 1.60 \times 10^{-19}$  C. ✓



Problem 11.

**12** If a typical light bulb draws about 900 mA from a 110-V household circuit, what is its resistance? (Don't worry about the fact that it's alternating current.) ✓

**13** A resistor has a voltage difference  $\Delta V$  across it, causing a current  $I$  to flow.

(a) Find an equation for the power it dissipates as heat in terms of the variables  $I$  and  $R$  only, eliminating  $\Delta V$ . ✓

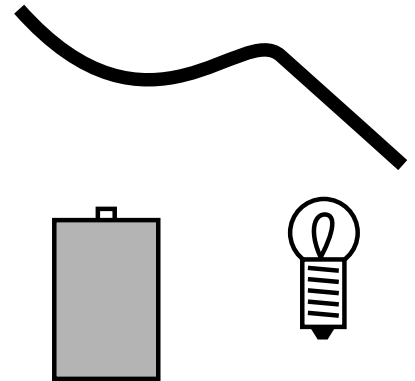
(b) If an electrical line coming to your house is to carry a given amount of current, interpret your equation from part a to explain whether the wire's resistance should be small, or large.

**14** (a) Express the power dissipated by a resistor in terms of  $R$  and  $\Delta V$  only, eliminating  $I$ . ✓

(b) Electrical receptacles in your home are mostly 110 V, but circuits for electric stoves, air conditioners, and washers and driers are usually 220 V. The two types of circuits have differently shaped receptacles. Suppose you rewire the plug of a drier so that it can be plugged in to a 110 V receptacle. The resistor that forms the heating element of the drier would normally draw 200 W. How much power does it actually draw now? ✓

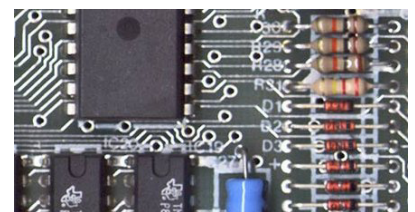
**15** Use the result of problem 42 on page 588 to find an equation for the voltage at a point in space at a distance  $r$  from a point charge  $Q$ . (Take your  $V = 0$  distance to be anywhere you like.) ✓

**16** You are given a battery, a flashlight bulb, and a single piece of wire. Draw at least two configurations of these items that would result in lighting up the bulb, and at least two that would not light it. (Don't draw schematics.) If you're not sure what's going on, borrow the materials from your instructor and try it. Note that the bulb has two electrical contacts: one is the threaded metal jacket, and the other is the tip (at the bottom in the figure). [Problem by Arnold Arons.]



Problem 16.

**17** You have to do different things with a circuit to measure current than to measure a voltage difference. Which would be more practical for a printed circuit board, in which the wires are actually strips of metal inlaid on the surface of the board?

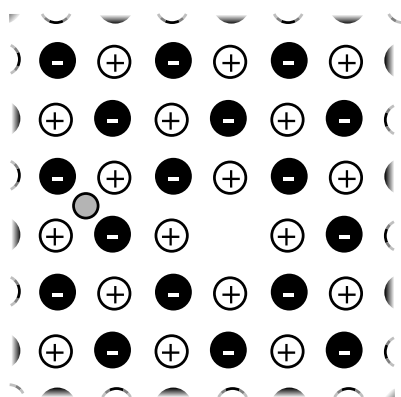


A printed circuit board, like the kind referred to in problem 17.

▷ Solution, p. 973



An LP record, problem 19.



Problem 21.

**18** Problem 18 has been replaced with problem 43.

**19** (a) You take an LP record out of its sleeve, and it acquires a static charge of 1 nC. You play it at the normal speed of  $33\frac{1}{3}$  r.p.m., and the charge moving in a circle creates an electric current. What is the current, in amperes? ✓

(b) Although the planetary model of the atom can be made to work with any value for the radius of the electrons' orbits, more advanced models that we will study later in this course predict definite radii. If the electron is imagined as circling around the proton at a speed of  $2.2 \times 10^6$  m/s, in an orbit with a radius of 0.05 nm, what electric current is created? The charge of an electron is  $-e = -1.60 \times 10^{-19}$  C. ✓ ★

**20** Three charges, each of strength  $Q$  ( $Q > 0$ ) form a fixed equilateral triangle with sides of length  $b$ . You throw a particle of mass  $m$  and positive charge  $q$  from far away, with an initial speed  $v$ . Your goal is to get the particle to go to the center of the triangle, your aim is perfect, and you are free to throw from any direction you like. What is the minimum possible value of  $v$ ? You will need the result of problem 42. ✓

**21** Referring back to problem 8 on page 582 about the sodium chloride crystal, suppose the lithium ion is going to jump from the gap it is occupying to one of the four closest neighboring gaps. Which one will it jump to, and if it starts from rest, how fast will it be going by the time it gets there? (It will keep on moving and accelerating after that, but that does not concern us.) You will need the result of problem 42. ✓ ★

**22** We have referred to resistors *dissipating* heat, i.e., we have assumed that  $P = I\Delta V$  is always greater than zero. Could  $I\Delta V$  come out to be negative for a resistor? If so, could one make a refrigerator by hooking up a resistor in such a way that it absorbed heat instead of dissipating it?

**23** Today, even a big luxury car like a Cadillac can have an electrical system that is relatively low in power, since it doesn't need to do much more than run headlights, power windows, etc. In the near future, however, manufacturers plan to start making cars with electrical systems about five times more powerful. This will allow certain energy-wasting parts like the water pump to be run on electrical motors and turned off when they're not needed — currently they're run directly on shafts from the motor, so they can't be shut off. It may even be possible to make an engine that can shut off at a stoplight and then turn back on again without cranking, since the valves can be electrically powered. Current cars' electrical systems have 12-volt batteries (with 14-volt chargers), but the new systems will have 36-volt batteries (with 42-volt chargers).

(a) Suppose the battery in a new car is used to run a device that requires the same amount of power as the corresponding device in

the old car. Based on the sample figures above, how would the currents handled by the wires in one of the new cars compare with the currents in the old ones?

(b) The real purpose of the greater voltage is to handle devices that need *more* power. Can you guess why they decided to change to 36-volt batteries rather than increasing the power without increasing the voltage? ✓

**24** (a) Many battery-operated devices take more than one battery. If you look closely in the battery compartment, you will see that the batteries are wired in series. Consider a flashlight circuit. What does the loop rule tell you about the effect of putting several batteries in series in this way?

(b) The cells of an electric eel's nervous system are not that different from ours — each cell can develop a voltage difference across it of somewhere on the order of one volt. How, then, do you think an electric eel can create voltages of thousands of volts between different parts of its body?

**25** The heating element of an electric stove is connected in series with a switch that opens and closes many times per second. When you turn the knob up for more power, the fraction of the time that the switch is closed increases. Suppose someone suggests a simpler alternative for controlling the power by putting the heating element in series with a variable resistor controlled by the knob. (With the knob turned all the way clockwise, the variable resistor's resistance is nearly zero, and when it's all the way counterclockwise, its resistance is essentially infinite.) (a) Draw schematics. (b) Why would the simpler design be undesirable?

**26** A  $1.0\ \Omega$  toaster and a  $2.0\ \Omega$  lamp are connected in parallel with the 110-V supply of your house. (Ignore the fact that the voltage is AC rather than DC.)

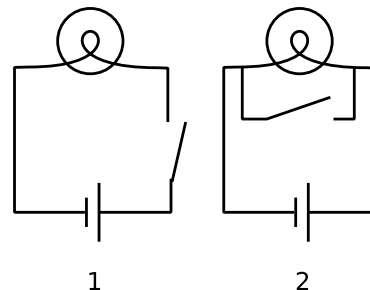
(a) Draw a schematic of the circuit.

(b) For each of the three components in the circuit, find the current passing through it and the voltage drop across it. ✓

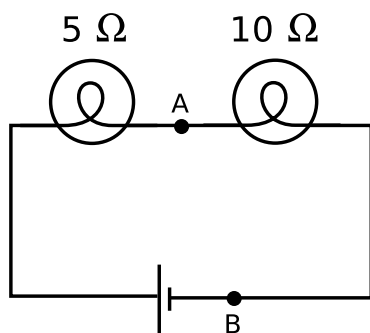
(c) Suppose they were instead hooked up in series. Draw a schematic and calculate the same things. ✓

**27** Wire is sold in a series of standard diameters, called “gauges.” The difference in diameter between one gauge and the next in the series is about 20%. How would the resistance of a given length of wire compare with the resistance of the same length of wire in the next gauge in the series? ✓

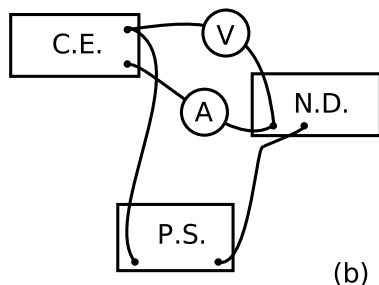
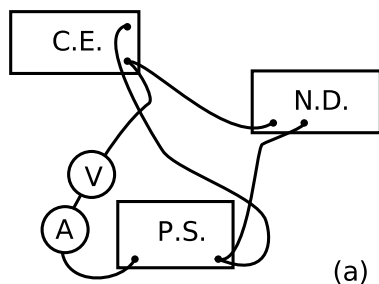
**28** The figure shows two possible ways of wiring a flashlight with a switch. Both will serve to turn the bulb on and off, although the switch functions in the opposite sense. Why is method 1 preferable?



Problem 28.



Problem 29.



Problem 31.

**29** In the figure, the battery is 9 V.

- What are the voltage differences across each light bulb? ✓
- What current flows through each of the three components of the circuit? ✓
- If a new wire is added to connect points A and B, how will the appearances of the bulbs change? What will be the new voltages and currents? ✓
- Suppose no wire is connected from A to B, but the two bulbs are switched. How will the results compare with the results from the original setup as drawn? ✓

**30** You have a circuit consisting of two unknown resistors in series, and a second circuit consisting of two unknown resistors in parallel.

- What, if anything, would you learn about the resistors in the series circuit by finding that the currents through them were equal?
- What if you found out the voltage differences across the resistors in the series circuit were equal?
- What would you learn about the resistors in the parallel circuit from knowing that the currents were equal?
- What if the voltages in the parallel circuit were equal?

**31** A student in a biology lab is given the following instructions: “Connect the cerebral eraser (C.E.) and the neural depolarizer (N.D.) in parallel with the power supply (P.S.). (Under no circumstances should you ever allow the cerebral eraser to come within 20 cm of your head.) Connect a voltmeter to measure the voltage across the cerebral eraser, and also insert an ammeter in the circuit so that you can make sure you don’t put more than 100 mA through the neural depolarizer.” The diagrams show two lab groups’ attempts to follow the instructions. (a) Translate diagram a into a standard-style schematic. What is correct and incorrect about this group’s setup? (b) Do the same for diagram b.

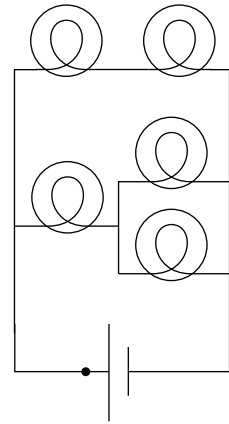
**32** How many different resistance values can be created by combining three unequal resistors? (Don’t count possibilities where not all the resistors are used.)

**33** A person in a rural area who has no electricity runs an extremely long extension cord to a friend’s house down the road so she can run an electric light. The cord is so long that its resistance,  $x$ , is not negligible. Show that the lamp’s brightness is greatest if its resistance,  $y$ , is equal to  $x$ . Explain physically why the lamp is dim for values of  $y$  that are too small or too large.  $\int$

**34** What resistance values can be created by combining a  $1\text{ k}\Omega$  resistor and a  $10\text{ k}\Omega$  resistor? ▷ Solution, p. 973

**35** Suppose six identical resistors, each with resistance  $R$ , are connected so that they form the edges of a tetrahedron (a pyramid with three sides in addition to the base, i.e., one less side than an Egyptian pyramid). What resistance value or values can be obtained by making connections onto any two points on this arrangement?

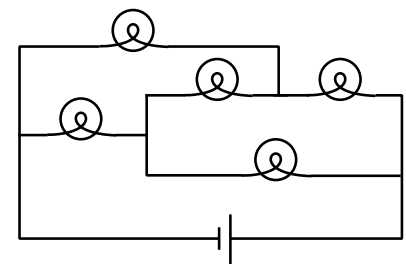
▷ Solution, p. 973 ★



Problems 36 and 37.

**36** The figure shows a circuit containing five lightbulbs connected to a battery. Suppose you're going to connect one probe of a voltmeter to the circuit at the point marked with a dot. How many unique, nonzero voltage differences could you measure by connecting the other probe to other wires in the circuit?

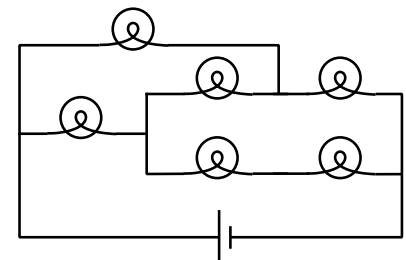
**37** The lightbulbs in the figure are all identical. If you were inserting an ammeter at various places in the circuit, how many unique currents could you measure? If you know that the current measurement will give the same number in more than one place, only count that as one unique current.



Problem 38.

**38** The bulbs are all identical. Which one doesn't light up? ★

**39** Each bulb has a resistance of one ohm. How much power is drawn from the one-volt battery? ✓ ★

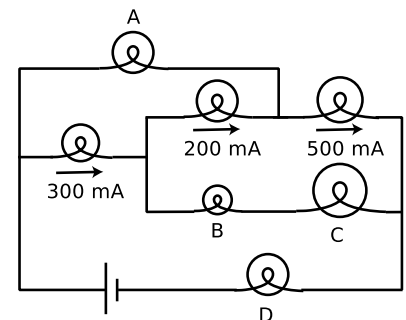


Problem 39.

**40** The bulbs all have unequal resistances. Given the three currents shown in the figure, find the currents through bulbs A, B, C, and D.

**41** A silk thread is uniformly charged by rubbing it with llama fur. The thread is then dangled vertically above a metal plate and released. As each part of the thread makes contact with the conducting plate, its charge is deposited onto the plate. Since the thread is accelerating due to gravity, the rate of charge deposition increases with time, and by time  $t$  the cumulative amount of charge is  $q = ct^2$ , where  $c$  is a constant. (a) Find the current flowing onto the plate. ✓  
(b) Suppose that the charge is immediately carried away through a resistance  $R$ . Find the power dissipated as heat. ✓

∫



Problem 40.

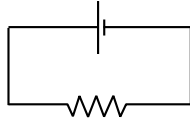
**42** (a) Recall from example 8 on p. 322 that the gravitational energy of two gravitationally interacting spheres is given by  $PE = -Gm_1m_2/r$ , where  $r$  is the center-to-center distance. Sketch a graph of  $PE$  as a function of  $r$ , making sure that your graph behaves properly at small values of  $r$ , where you're dividing by a small number, and at large ones, where you're dividing by a large one. Check that your graph behaves properly when a rock is dropped from a larger  $r$  to a smaller one; the rock should *lose* potential energy as it gains kinetic energy.

(b) Electrical forces are closely analogous to gravitational ones, since both depend on  $1/r^2$ . Since the forces are analogous, the potential energies should also behave analogously. Using this analogy, write down the expression for the electrical potential energy of two interacting charged particles. The main uncertainty here is the sign out in front. Like masses attract, but like charges repel. To figure out whether you have the right sign in your equation, sketch graphs in the case where both charges are positive, and also in the case where one is positive and one negative; make sure that in both cases, when the charges are released near one another, their motion causes them to lose PE while gaining KE. ✓

**43** In example 8 on p. 560, suppose that the larger sphere has radius  $a$ , the smaller one  $b$ . (a) Use the result of problem 15 show that the ratio of the charges on the two spheres is  $q_a/q_b = a/b$ . (b) Show that the density of charge (charge per unit area) is the other way around: the charge density on the smaller sphere is *greater* than that on the larger sphere in the ratio  $a/b$ .

## Exercise 21A: Electrical measurements

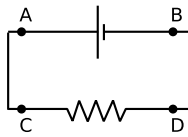
1. How many different currents could you measure in this circuit? Make a prediction, and then try it.



What do you notice? How does this make sense in terms of the roller coaster metaphor introduced in discussion question 21.5A on p. 555?

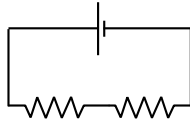
What is being *used up* in the resistor?

2. By connecting probes to these points, how many ways could you measure a voltage? How many of them would be different numbers? Make a prediction, and then do it.



What do you notice? Interpret this using the roller coaster metaphor, and color in parts of the circuit that represent constant voltages.

3. The resistors are unequal. How many *different* voltages and currents can you measure? Make a prediction, and then try it.



What do you notice? Interpret this using the roller coaster metaphor, and color in parts of the circuit that represent constant voltages.



## Exercise 21B: Voltage and current

This exercise is based on one created by Virginia Roundy.

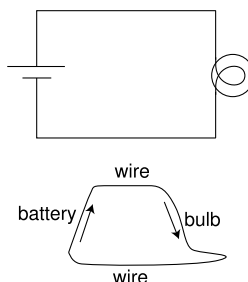
Apparatus:

- DC power supply
- 1.5 volt batteries
- lightbulbs and holders
- wire
- highlighting pens, 3 colors

When you first glance at this exercise, it may look scary and intimidating — all those circuits! However, all those wild-looking circuits can be analyzed using the following four guides to thinking:

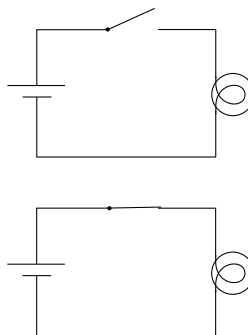
1. *A circuit has to be complete*, i.e., it must be possible for charge to get recycled as it goes around the circuit. If it's not complete, then charge will build up at a dead end. This built-up charge will repel any other charge that tries to get in, and everything will rapidly grind to a stop.
2. *There is constant voltage everywhere along a piece of wire*. To apply this rule during this lab, I suggest you use the colored highlighting pens to mark the circuit. For instance, if there's one whole piece of the circuit that's all at the same voltage, you could highlight it in yellow. A second piece of the circuit, at some other voltage, could be highlighted in blue.
3. *Charge is conserved*, so charge can't "get used up."

4. You can draw a *rollercoaster diagram*, like the one shown below. On this kind of diagram, height corresponds to voltage — that's why the wires are drawn as horizontal tracks.



### A Bulb and a Switch

Look at circuit 1, and try to predict what will happen when the switch is open, and what will happen when it's closed. Write both your predictions in the table on the following page before you build the circuit. When you build the circuit, you don't need an actual switch like a light switch; just connect and disconnect the banana plugs. Use one of the 1.5 volt batteries as your voltage source.

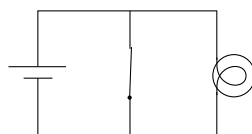
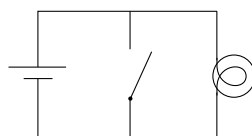


Circuit 1

	<i>switch open</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>switch closed</i>
prediction	
explanation	
observation	
explanation (if different)	

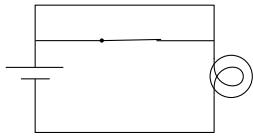
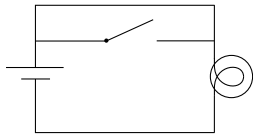
Did it work the way you expected? If not, try to figure it out with the benefit of hindsight, and write your explanation in the table above.



Circuit 2 (Don't leave the switch closed for a long time!)

	<i>switch open</i>
prediction	
explanation	
observation	
explanation (if different)	

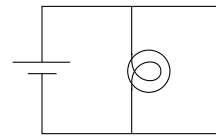
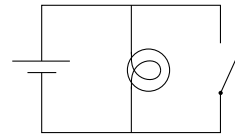
	<i>switch closed</i>
prediction	
explanation	
observation	
explanation (if different)	



Circuit 3

	<i>switch open</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>switch closed</i>
prediction	
explanation	
observation	
explanation (if different)	



Circuit 4

	<i>switch open</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>switch closed</i>
prediction	
explanation	
observation	
explanation (if different)	

### Two Bulbs

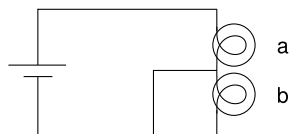
Instead of a battery, use the DC power supply, set to 2.4 volts, for circuits 5 and 6. Analyze this one both by highlighting and by drawing a rollercoaster diagram.



Circuit 5

	<i>bulb a</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>bulb b</i>
prediction	
explanation	
observation	
explanation (if different)	



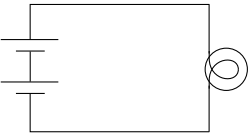
Circuit 6

	<i>bulb a</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>bulb b</i>
prediction	
explanation	
observation	
explanation (if different)	

Two Batteries

Use batteries for circuits 7-9. Circuits 7 and 8 are both good candidates for rollercoaster diagrams.



Circuit 7

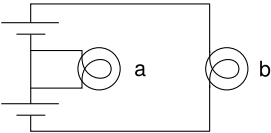
prediction	
explanation	
observation	
explanation (if different)	



Circuit 8

prediction	
explanation	
observation	
explanation (if different)	

A Final Challenge



Circuit 9

	<i>bulb a</i>
prediction	
explanation	
observation	
explanation (if different)	

	<i>bulb b</i>
prediction	
explanation	
observation	
explanation (if different)	

## Exercise 21C: Reasoning about circuits

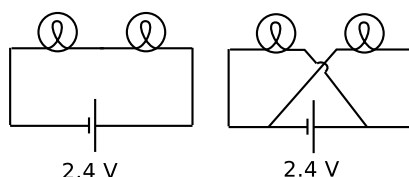
The questions in this exercise can all be solved using some combination of the following approaches:

- There is constant voltage throughout any conductor.
- Ohm's law can be applied to any *part* of a circuit.
- Apply the loop rule.
- Apply the junction rule.

In each case, discuss the question, decide what you think is the right answer, and then try the experiment.

If you've already done exercise 21B, skip number 1.

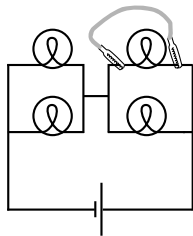
1. The series circuit is changed as shown.



Which reasoning is correct?

- Each bulb now has its sides connected to the two terminals of the battery, so each now has 2.4 V across it instead of 1.2 V. They get brighter.*
- Just as in the original circuit, the current goes through one bulb, then the other. It's just that now the current goes in a figure-8 pattern. The bulbs glow the same as before.*

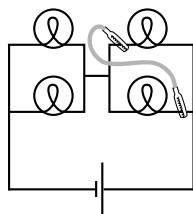
2. A wire is added as shown to the original circuit.



What is wrong with the following reasoning?

*The top right bulb will go out, because its two sides are now connected with wire, so there will be no voltage difference across it. The other three bulbs will not be affected.*

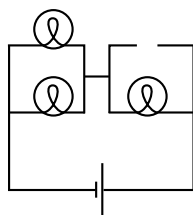
3. A wire is added as shown to the original circuit.



What is wrong with the following reasoning?

*The current flows out of the right side of the battery. When it hits the first junction, some of it will go left and some will keep going up. The part that goes up lights the top right bulb. The part that turns left then follows the path of least resistance, going through the new wire instead of the bottom bulb. The top bulb stays lit, the bottom one goes out, and others stay the same.*

4. What happens when one bulb is unscrewed, leaving an air gap?



5. This part is optional. You can do it if you finished early and would like an extra challenge.

Predict the voltage drop across each of the three bulbs in part 4, and also predict how the three currents will compare with one another. (You can't predict the currents in units of amperes, since you don't know the resistances of the bulbs.) Test your predictions. If your predictions are wrong, try to figure out what's going on.

## Chapter 22

# The nonmechanical universe

“Okay. Your duties are as follows: Get Breen. I don’t care how you get him, but get him soon. That faker! He posed for twenty years as a scientist without ever being apprehended. Well, I’m going to do some apprehending that’ll make all previous apprehending look like no apprehension at all. You with me?”

“Yes,” said Battle, very much confused. “What’s that thing you have?”

“Piggy-back heat-ray. You transpose the air in its path into an unstable isotope which tends to carry all energy as heat. Then you shoot your juice light, or whatever along the isotopic path and you burn whatever’s on the receiving end. You want a few?”

“No,” said Battle. “I have my gats. What else have you got for offense and defense?” Underbottom opened a cabinet and proudly waved an arm. “Everything,” he said.

“Disintegraters, heat-rays, bombs of every type. And impenetrable shields of energy, massive and portable. What more do I need?”<sup>1</sup>

Cutting-edge science readily infiltrates popular culture, though sometimes in garbled form. The Newtonian imagination populated the universe mostly with that nice solid stuff called matter, which was made of little hard balls called atoms. In the early twentieth century, consumers of pulp fiction and popularized science began to hear of a new image of the universe, full of x-rays, N-rays, and Hertzian waves. What they were beginning to soak up through their skins was a drastic revision of Newton’s concept of a universe made of chunks of matter which happened to interact via forces. In the newly emerging picture, the universe was *made* of force, or, to be more technically accurate, of ripples in universal fields of force. Unlike the average reader of *Cosmic Stories* in 1941, you now have enough technical background to understand what a “force field” really is.



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<sup>1</sup>From “The Reversible Revolutions,” by Cecil Corwin, *Cosmic Stories*, March 1941. Art by Morey, Bok, Kyle, Hunt, Forte. Copyright expired.



## 22.1 The stage and the actors

### Newton's instantaneous action at a distance

The Newtonian picture has particles interacting with each other by exerting forces from a distance, and these forces are imagined to occur without any time delay. For example, suppose that super-powerful aliens, angered when they hear disco music in our AM radio transmissions, come to our solar system on a mission to cleanse the universe of our aesthetic contamination. They apply a force to our sun, causing it to go flying out of the solar system at a gazillion miles an hour. According to Newton's laws, the gravitational force of the sun on the earth will *immediately* start dropping off. This will be detectable on earth, and since sunlight takes eight minutes to get from the sun to the earth, the change in gravitational force will, according to Newton, be the first way in which earthlings learn the bad news — the sun will not visibly start receding until a little later. Although this scenario is fanciful enough to be at home in the pages of *Cosmic Stories*, it shows a real feature of Newton's laws: that information can be transmitted from one place in the universe to another with zero time delay, so that transmission and reception occur at exactly the same instant.



a / This Global Positioning System (GPS) system, running on a smartphone attached to a bike's handlebar, depends on Einstein's theory of relativity. Time flows at a different rate aboard a GPS satellite than it does on the bike, and the GPS software has to take this into account.



b / The clock took up two seats, and two tickets were bought for it under the name of “Mr. Clock.”

Newton was sharp enough to realize that this required a nontrivial assumption, which was that there was some completely objective and well-defined way of saying whether two things *happened* at exactly the same instant. He stated this assumption explicitly: “Absolute, true, and mathematical time, of itself, and from its own nature flows at a constant rate without regard to anything external...”

### No absolute time

Ever since Einstein, we've known that this assumption was false. When Einstein first began to develop the theory of relativity, around 1905, the only real-world observations he could draw on were ambiguous and indirect. Today, the evidence is part of everyday life. For example, every time you use a GPS receiver, figure a, you're using Einstein's theory of relativity. Somewhere between 1905 and today, technology became good enough to allow conceptually *simple* experiments that students in the early 20th century could only discuss in terms like “Imagine that we could...” A good jumping-on point is 1971. In that year, J.C. Hafele and R.E. Keating brought atomic clocks aboard commercial airliners, b, and went around the world, once from east to west and once from west to east. Hafele and Keating observed that there was a discrepancy between the times measured by the traveling clocks and the times measured by similar clocks that stayed home at the U.S. Naval Observatory in Washington. The east-going clock lost time, ending up off by  $-59 \pm 10$  nanoseconds, while the west-going one gained  $273 \pm 7$  ns. Although this example is particularly dramatic, a large number of other ex-

periments have also confirmed that time is not absolute, as Newton had imagined.

Nevertheless, the effects that Hafele and Keating observed were small. This makes sense: Newton's laws have already been thoroughly tested by experiments under a wide variety of conditions, so a new theory like relativity must agree with Newton's to a good approximation, within the Newtonian theory's realm of applicability. This requirement of backward-compatibility is known as the *correspondence principle*.

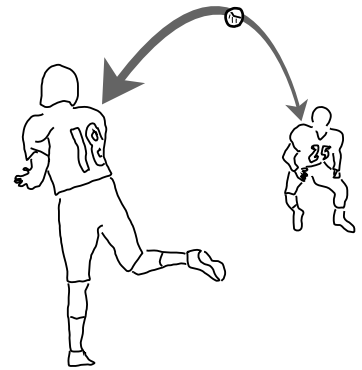
## Causality

It's also reassuring that the effects on time were small compared to the three-day lengths of the plane trips. There was therefore no opportunity for paradoxical scenarios such as one in which the east-going experimenter arrived back in Washington before he left and then convinced himself not to take the trip. A theory that maintains this kind of orderly relationship between cause and effect is said to satisfy causality.

Causality is like a water-hungry front-yard lawn in Los Angeles: we know we want it, but it's not easy to explain why. Even in plain old Newtonian physics, there is no clear distinction between past and future. In figure c, number 18 throws the football to number 25, and the ball obeys Newton's laws of motion. If we took a video of the pass and played it backward, we would see the ball flying from 25 to 18, and Newton's laws would still be satisfied. Nevertheless, we have a strong psychological impression that there is a forward arrow of time. I can remember what the stock market did last year, but I can't remember what it will do next year. Joan of Arc's military victories against England caused the English to burn her at the stake; it's hard to accept that Newton's laws provide an equally good description of a process in which her execution in 1431 caused her to win a battle in 1429. There is no consensus at this point among physicists on the origin and significance of time's arrow, and for our present purposes we don't need to solve this mystery. Instead, we merely note the empirical fact that, regardless of what causality really means and where it really comes from, its behavior is consistent. Specifically, experiments show that if an observer in a certain frame of reference observes that event A causes event B, then observers in other frames agree that A causes B, not the other way around. This is merely a generalization about a large body of experimental results, not a logically necessary assumption. If Keating had gone around the world and arrived back in Washington before he left, it would have disproved this statement about causality.

## Time delays in forces exerted at a distance

Relativity is closely related to electricity and magnetism, and we will go into relativity in more detail in chapters 24-27. What we



c / Newton's laws do not distinguish past from future. The football could travel in either direction while obeying Newton's laws.

care about for now is that relativity forbids Newton's instantaneous action at a distance. For suppose that instantaneous action at a distance existed. It would then be possible to send signals from one place in the universe to another without any time lag. This would allow perfect synchronization of all clocks. But the Hafele-Keating experiment demonstrates that clocks A and B that have been initially synchronized will drift out of sync if one is in motion relative to the other.<sup>2</sup> With instantaneous transmission of signals, we could determine, without having to wait for A and B to be reunited, which was ahead and which was behind. Since they don't need to be reunited, neither one needs to undergo any acceleration; each clock can fix an inertial frame of reference, with a velocity vector that changes neither its direction nor its magnitude. But this violates the principle that constant-velocity motion is relative, because each clock can be considered to be at rest, in its own frame of reference. Since no experiment has ever detected any violation of the relativity of motion, we conclude that instantaneous action at a distance is impossible.

Since forces can't be transmitted instantaneously, it becomes natural to imagine force-effects spreading outward from their source like ripples on a pond, and we then have no choice but to impute some physical reality to these ripples. We call them fields, and they have their own independent existence. Chapters 22-24 are mainly about the electric and magnetic fields, although we'll also talk about the gravitational field. Ripples of the electric and magnetic fields turn out to be light waves. Fields don't have to wiggle; they can hold still as well. The earth's magnetic field, for example, is nearly constant, which is why we can use it for direction-finding.

Even empty space, then, is not perfectly featureless. It has measurable properties. For example, we can drop a rock in order to measure the direction of the gravitational field, or use a magnetic compass to find the direction of the magnetic field. This concept made a deep impression on Einstein as a child. He recalled that as a five-year-old, the gift of a magnetic compass convinced him that there was "something behind things, something deeply hidden."

### **More evidence that fields of force are real: they carry energy.**

The smoking-gun argument for this strange notion of traveling force ripples comes from the fact that they carry energy. In figure d/1, Alice and Betty hold positive charges A and B at some distance from one another. If Alice chooses to move her charge closer to Betty's,  $d/2$ , Alice will have to do some mechanical work against the electrical repulsion, burning off some of the calories from that chocolate cheesecake she had at lunch. This reduction in her body's chemical energy is offset by a corresponding increase in the electrical potential

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<sup>2</sup>As discussed in ch. 24, there are actually two different effects here, one due to motion and one due to gravity.

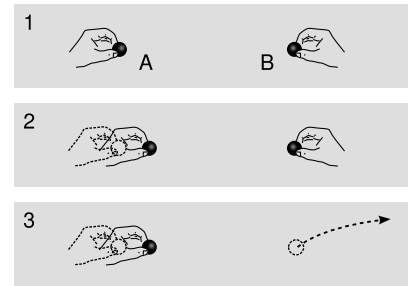
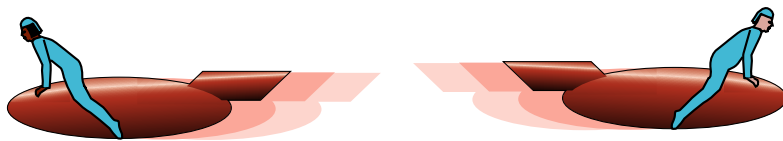
energy  $q\Delta V$ . Not only that, but Alice feels the resistance stiffen as the charges get closer together and the repulsion strengthens. She has to do a little extra work, but this is all properly accounted for in the electrical potential energy.

But now suppose, d/3, that Betty decides to play a trick on Alice by tossing charge B far away just as Alice is getting ready to move charge A. We have already established that Alice can't feel charge B's motion instantaneously, so the electric forces must actually be propagated by an electric *field*. Of course this experiment is utterly impractical, but suppose for the sake of argument that the time it takes the change in the electric field to propagate across the diagram is long enough so that Alice can complete her motion before she feels the effect of B's disappearance. She is still getting stale information about B's position. As she moves A to the right, she feels a repulsion, because the field in her region of space is still the field caused by B in its *old* position. She has burned some chocolate cheesecake calories, and it appears that conservation of energy has been violated, because these calories can't be properly accounted for by any interaction with B, which is long gone.

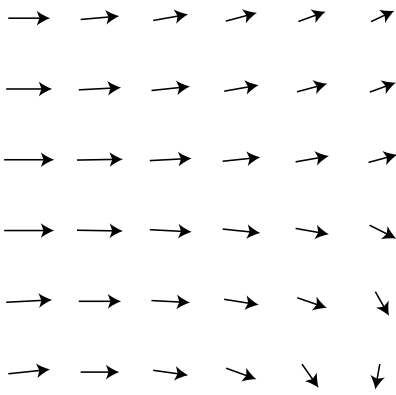
If we hope to preserve the law of conservation of energy, then the only possible conclusion is that the electric field itself carries away the cheesecake energy. In fact, this example represents an impractical method of transmitting radio waves. Alice does work on charge A, and that energy goes into the radio waves. Even if B had never existed, the radio waves would still have carried energy, and Alice would still have had to do work in order to create them.

### Discussion questions

**A** Amy and Bill are flying on spaceships in opposite directions at such high velocities that the relativistic effect on time's rate of flow is easily noticeable. Motion is relative, so Amy considers herself to be at rest and Bill to be in motion. She says that time is flowing normally for her, but Bill is slow. But Bill can say exactly the same thing. How can they *both* think the other is slow? Can they settle the disagreement by getting on the radio and seeing whose voice is normal and whose sounds slowed down and Darth-Vadery?



d / Fields carry energy.



e / The wind patterns in a certain area of the ocean could be charted in a “sea of arrows” representation like this. Each arrow represents both the wind’s strength and its direction at a certain location.

## 22.2 The gravitational field

Given that fields of force are real, how do we define, measure, and calculate them? A fruitful metaphor will be the wind patterns experienced by a sailing ship. Wherever the ship goes, it will feel a certain amount of force from the wind, and that force will be in a certain direction. The weather is ever-changing, of course, but for now let’s just imagine steady wind patterns. Definitions in physics are operational, i.e., they describe how to measure the thing being defined. The ship’s captain can measure the wind’s “field of force” by going to the location of interest and determining both the direction of the wind and the strength with which it is blowing. Charting all these measurements on a map leads to a depiction of the field of wind force like the one shown in the figure. This is known as the “sea of arrows” method of visualizing a field.

Now let’s see how these concepts are applied to the fundamental force fields of the universe. We’ll start with the gravitational field, which is the easiest to understand. As with the wind patterns, we’ll start by imagining gravity as a static field, even though the existence of the tides proves that there are continual changes in the gravity field in our region of space. Defining the direction of the gravitational field is easy enough: we simply go to the location of interest and measure the direction of the gravitational force on an object, such as a weight tied to the end of a string.

But how should we define the strength of the gravitational field? Gravitational forces are weaker on the moon than on the earth, but we cannot specify the strength of gravity simply by giving a certain number of newtons. The number of newtons of gravitational force depends not just on the strength of the local gravitational field but also on the mass of the object on which we’re testing gravity, our “test mass.” A boulder on the moon feels a stronger gravitational force than a pebble on the earth. We can get around this problem by defining the strength of the gravitational field as the force acting on an object, *divided by the object’s mass*.

### definition of the gravitational field

The gravitational field vector,  $\mathbf{g}$ , at any location in space is found by placing a test mass  $m_t$  at that point. The field vector is then given by  $\mathbf{g} = \mathbf{F}/m_t$ , where  $\mathbf{F}$  is the gravitational force on the test mass.

The magnitude of the gravitational field near the surface of the earth is about 9.8 N/kg, and it’s no coincidence that this number looks familiar, or that the symbol  $\mathbf{g}$  is the same as the one for gravitational acceleration. The force of gravity on a test mass will equal  $m_t\mathbf{g}$ , where  $\mathbf{g}$  is the gravitational acceleration. Dividing by  $m_t$  simply gives the gravitational acceleration. Why define a new name and new units for the same old quantity? The main reason is

that it prepares us with the right approach for defining other fields.

The most subtle point about all this is that the gravitational field tells us about what forces *would* be exerted on a test mass by the earth, sun, moon, and the rest of the universe, *if* we inserted a test mass at the point in question. The field still exists at all the places where we didn't measure it.

#### Gravitational field of the earth

#### example 1

▷ What is the magnitude of the earth's gravitational field, in terms of its mass,  $M$ , and the distance  $r$  from its center?

▷ Substituting  $|\mathbf{F}| = GMm_t/r^2$  into the definition of the gravitational field, we find  $|\mathbf{g}| = GM/r^2$ . This expression could be used for the field of any spherically symmetric mass distribution, since the equation we assumed for the gravitational force would apply in any such case.

### Sources and sinks

If we make a sea-of-arrows picture of the gravitational fields surrounding the earth, f, the result is evocative of water going down a drain. For this reason, anything that creates an inward-pointing field around itself is called a sink. The earth is a gravitational sink. The term “source” can refer specifically to things that make outward fields, or it can be used as a more general term for both “outies” and “innies.” However confusing the terminology, we know that gravitational fields are only attractive, so we will never find a region of space with an outward-pointing field pattern.

Knowledge of the field is interchangeable with knowledge of its sources (at least in the case of a static, unchanging field). If aliens saw the earth's gravitational field pattern they could immediately infer the existence of the planet, and conversely if they knew the mass of the earth they could predict its influence on the surrounding gravitational field.

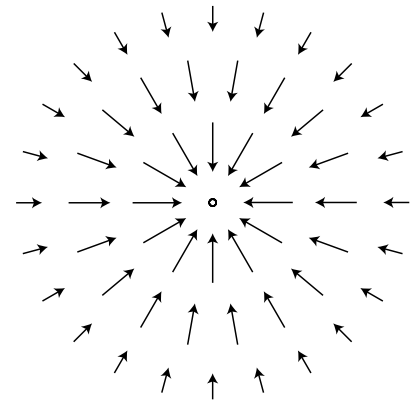
### Superposition of fields

A very important fact about all fields of force is that when there is more than one source (or sink), the fields add according to the rules of vector addition. The gravitational field certainly will have this property, since it is defined in terms of the force on a test mass, and forces add like vectors. Superposition is an important characteristics of waves, so the superposition property of fields is consistent with the idea that disturbances can propagate outward as waves in a field.

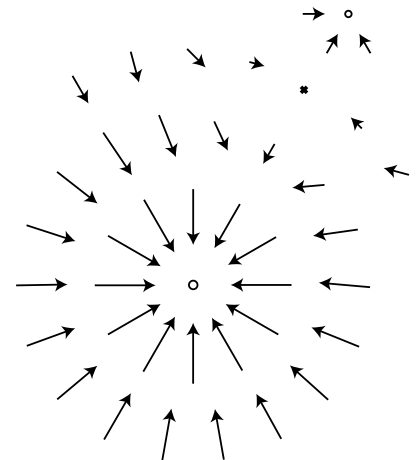
#### Reduction in gravity on Io due to Jupiter's gravity

#### example 2

▷ The average gravitational field on Jupiter's moon Io is 1.81 N/kg. By how much is this reduced when Jupiter is directly overhead? Io's orbit has a radius of  $4.22 \times 10^8$  m, and Jupiter's mass is  $1.899 \times 10^{27}$  kg.



f / The gravitational field surrounding a clump of mass such as the earth.



g / The gravitational fields of the earth and moon superpose. Note how the fields cancel at one point, and how there is no boundary between the interpenetrating fields surrounding the two bodies.

▷ By the shell theorem, we can treat the Jupiter as if its mass was all concentrated at its center, and likewise for Io. If we visit Io and land at the point where Jupiter is overhead, we are on the same line as these two centers, so the whole problem can be treated one-dimensionally, and vector addition is just like scalar addition. Let's use positive numbers for downward fields (toward the center of Io) and negative for upward ones. Plugging the appropriate data into the expression derived in example 1, we find that the Jupiter's contribution to the field is  $-0.71 \text{ N/kg}$ . Superposition says that we can find the actual gravitational field by adding up the fields created by Io and Jupiter:  $1.81 - 0.71 \text{ N/kg} = 1.1 \text{ N/kg}$ . You might think that this reduction would create some spectacular effects, and make Io an exciting tourist destination. Actually you would not detect any difference if you flew from one side of Io to the other. This is because your body and Io both experience Jupiter's gravity, so you follow the same orbital curve through the space around Jupiter.

### Gravitational waves

Looking back at the argument given on p. 600 for the existence of energy-bearing ripples in the electric field, we see that nowhere was it necessary to appeal to any specific properties of the electrical interaction. We therefore expect energy-carrying gravitational waves to exist, and Einstein's general theory of relativity does describe such waves and their properties. J.H. Taylor and R.A. Hulse were awarded the Nobel Prize in 1993 for giving indirect evidence to confirm Einstein's prediction. They discovered a pair of exotic, ultra-dense stars called neutron stars orbiting one another very closely, and showed that they were losing orbital energy at the rate predicted by general relativity.

A Caltech-MIT collaboration has built a pair of gravitational wave detectors called LIGO to search for more direct evidence of gravitational waves. Since they are essentially the most sensitive vibration detectors ever made, they are located in quiet rural areas, and signals will be compared between them to make sure that they were not due to passing trucks. The project began operating at full sensitivity in 2005, and is now able to detect a vibration that causes a change of  $10^{-18} \text{ m}$  in the distance between the mirrors at the ends of the 4-km vacuum tunnels. This is a thousand times less than the size of an atomic nucleus! There is only enough funding to keep the detectors operating for a few more years, so the physicists can only hope that during that time, somewhere in the universe, a sufficiently violent cataclysm will occur to make a detectable gravitational wave. (More accurately, they want the wave to arrive in our solar system during that time, although it will have been produced millions of years before.)



h / The part of the LIGO gravity wave detector at Hanford Nuclear Reservation, near Richland, Washington. The other half of the detector is in Louisiana.

## 22.3 The electric field

### Definition

The definition of the electric field is directly analogous to, and has the same motivation as, the definition of the gravitational field:

#### definition of the electric field

The electric field vector,  $\mathbf{E}$ , at any location in space is found by placing a test charge  $q_t$  at that point. The electric field vector is then given by  $\mathbf{E} = \mathbf{F}/q_t$ , where  $\mathbf{F}$  is the electric force on the test charge.

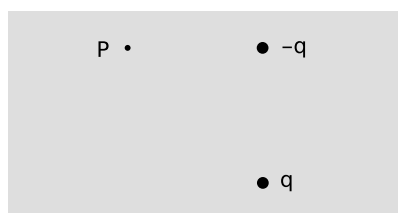
Charges are what create electric fields. Unlike gravity, which is always attractive, electricity displays both attraction and repulsion. A positive charge is a source of electric fields, and a negative one is a sink.

The most difficult point about the definition of the electric field is that the force on a negative charge is in the opposite direction compared to the field. This follows from the definition, since dividing a vector by a negative number reverses its direction. It's as though we had some objects that fell upward instead of down.

#### self-check A

Find an equation for the magnitude of the field of a single point charge  $Q$ .  
 ▷ Answer, p. 977





i / Example 3.

### Superposition of electric fields

example 3

▷ Charges  $q$  and  $-q$  are at a distance  $b$  from each other, as shown in the figure. What is the electric field at the point P, which lies at a third corner of the square?

▷ The field at P is the vector sum of the fields that would have been created by the two charges independently. Let positive  $x$  be to the right and let positive  $y$  be up.

Negative charges have fields that point at them, so the charge  $-q$  makes a field that points to the right, i.e., has a positive  $x$  component. Using the answer to the self-check, we have

$$\begin{aligned} E_{-q,x} &= \frac{kq}{b^2} \\ E_{-q,y} &= 0 \end{aligned} .$$

Note that if we had blindly ignored the absolute value signs and plugged in  $-q$  to the equation, we would have incorrectly concluded that the field went to the left.

By the Pythagorean theorem, the positive charge is at a distance  $\sqrt{2}b$  from P, so the magnitude of its contribution to the field is  $E = kq/2b^2$ . Positive charges have fields that point away from them, so the field vector is at an angle of  $135^\circ$  counterclockwise from the  $x$  axis.

$$\begin{aligned} E_{q,x} &= \frac{kq}{2b^2} \cos 135^\circ \\ &= -\frac{kq}{2^{3/2}b^2} \\ E_{q,y} &= \frac{kq}{2b^2} \sin 135^\circ \\ &= \frac{kq}{2^{3/2}b^2} \end{aligned}$$

The total field is

$$\begin{aligned} E_x &= \left(1 - 2^{-3/2}\right) \frac{kq}{b^2} \\ E_y &= \frac{kq}{2^{3/2}b^2} \end{aligned} .$$

### A line of charge

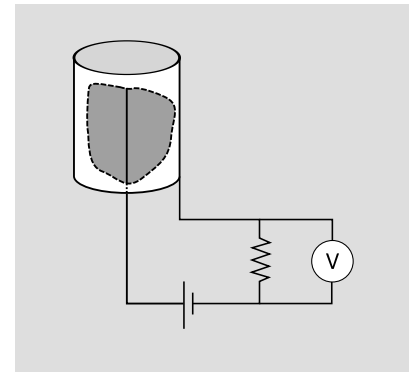
### example 4

In a complete circuit, there is typically no net charge on any of the wires. However, there are some devices in which a circuit is intentionally left open in order to produce a nonzero net charge on a wire. One example is a type of radiation detector called a Geiger-Müller tube, figure j. A high high voltage is applied between the outside of the cylinder and the wire that runs along the central axis. A net positive charge builds up on the wire and a negative one on the cylinder's wall. Electric fields originate from the wire, spread outward from the axis, and terminate on the wall. The cylinder is filled with a low-pressure inert gas. An incoming particle of radioactivity strikes an atom of the gas, *ionizing* it, i.e., splitting it into positively and negatively charged parts, known as *ions*. These ions then accelerate in opposite directions, since the force exerted by an electric field on a charged particle flips directions when the charge is reversed. The ions accelerate up to speeds at which they are capable of ionizing other atoms when they collide with them. The result is an electrical avalanche that causes a disturbance on the voltmeter.

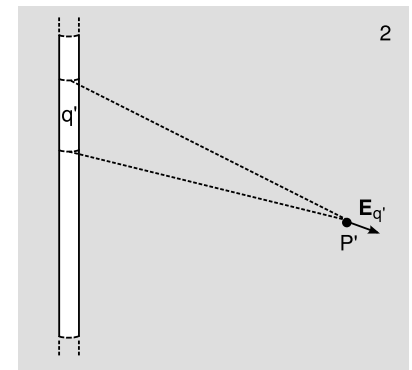
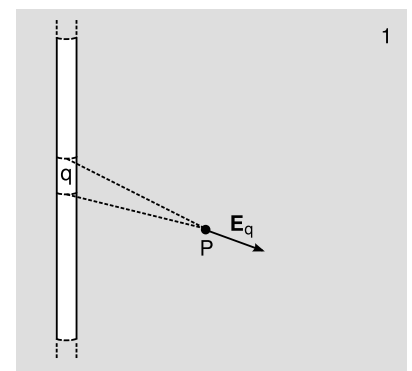
Motivated by this example, we would like to find how the field of a long, uniformly charged wire varies with distance. In figure k/1, the point P experiences a field that is the vector sum of contributions such as the one coming from the segment  $q$ . The field  $\mathbf{E}_q$  arising from this segment has to be added to similar contributions from all other segments of the wire. By symmetry, the total field will end up pointing at a right angle to the wire. We now consider point P', figure k/2, at twice the distance from the wire. If we reproduce all the angles from k/1, then the new triangle is simply a copy of the old one that has been scaled up by a factor of two. The left side's length has doubled, so  $q' = 2q$ , and this would tend to make  $\mathbf{E}_{q'}$  twice as big. But all the distances have also been doubled, and the  $1/r^2$  in Coulomb's law therefore contributes an additional factor of  $1/4$ . Combining these two factors, we find  $\mathbf{E}_{q'} = \mathbf{E}_q/2$ . The total field is the sum of contributions such as  $\mathbf{E}_{q'}$ , so if all of these have been weakened by a factor of two, the same must apply to the total as well. There was nothing special about the number 2, so we conclude that in general the electric field of a line of charge is proportional to  $1/r$ .

Applying this to the Geiger-Müller tube, we can see the reason why the device is built with a wire. When  $r$  is small,  $1/r$  is big, and the field is very strong. Therefore the device can be sensitive enough to trigger an avalanche in the gas when only a single atom has been ionized.

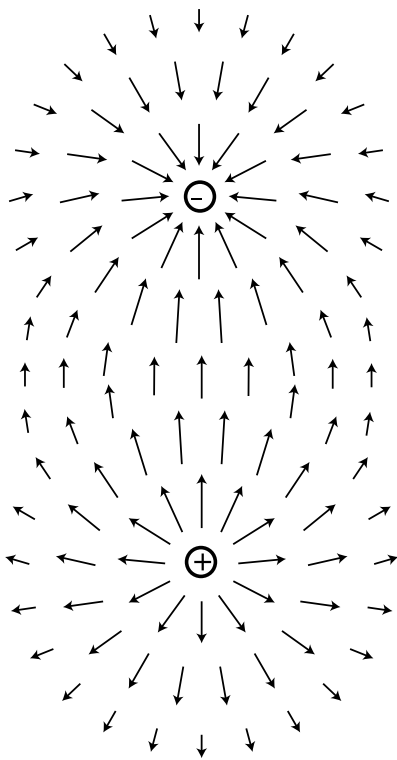
We have only shown that the field is proportional to  $1/r$ , but we haven't filled in the other factors in the equation. This is done in example 14 on p. 619.



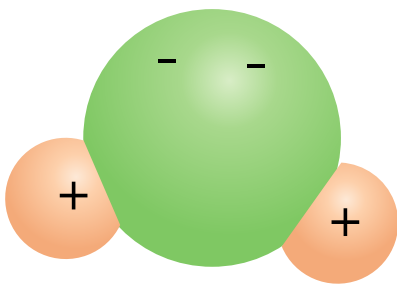
j / A Geiger-Müller tube.



k / Proof that the electric field of a line of charge is proportional to  $1/r$ , example 4.



m / A dipole field. Electric fields diverge from a positive charge and converge on a negative charge.



n / A water molecule is a dipole.

## Dipoles

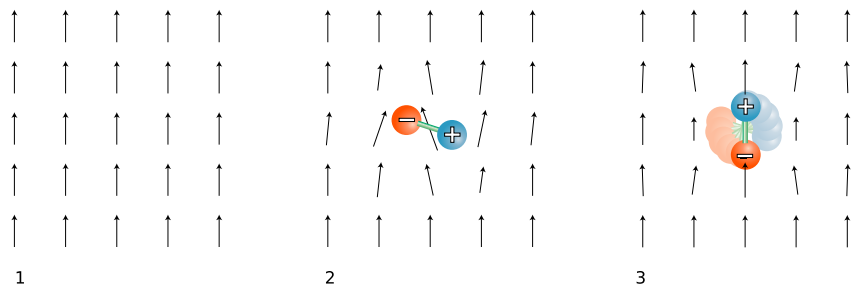
The simplest set of sources that can occur with electricity but not with gravity is the *dipole*, consisting of a positive charge and a negative charge with equal magnitudes. More generally, an electric dipole can be any object with an imbalance of positive charge on one side and negative on the other. A water molecule, n, is a dipole because the electrons tend to shift away from the hydrogen atoms and onto the oxygen atom.

Your microwave oven acts on water molecules with electric fields. Let us imagine what happens if we start with a uniform electric field, l/1, made by some external charges, and then insert a dipole, l/2, consisting of two charges connected by a rigid rod. The dipole disturbs the field pattern, but more important for our present purposes is that it experiences a torque. In this example, the positive charge feels an upward force, but the negative charge is pulled down. The result is that the dipole wants to align itself with the field, l/3. The microwave oven heats food with electrical (and magnetic) waves. The alternation of the torque causes the molecules to wiggle and increase the amount of random motion. The slightly vague definition of a dipole given above can be improved by saying that a dipole is any object that experiences a torque in an electric field.

What determines the torque on a dipole placed in an externally created field? Torque depends on the force, the distance from the axis at which the force is applied, and the angle between the force and the line from the axis to the point of application. Let a dipole consisting of charges  $+q$  and  $-q$  separated by a distance  $\ell$  be placed in an external field of magnitude  $|\mathbf{E}|$ , at an angle  $\theta$  with respect to the field. The total torque on the dipole is

$$\begin{aligned}\tau &= \frac{\ell}{2}q|\mathbf{E}|\sin\theta + \frac{\ell}{2}q|\mathbf{E}|\sin\theta \\ &= \ell q|\mathbf{E}|\sin\theta.\end{aligned}$$

(Note that even though the two forces are in opposite directions, the torques do not cancel, because they are both trying to twist the dipole in the same direction.) The quantity  $\ell q$  is called the dipole



l / 1. A uniform electric field created by some charges “off-stage.”  
2. A dipole is placed in the field. 3. The dipole aligns with the field.

moment, notated  $D$ . (More complex dipoles can also be assigned a dipole moment — they are defined as having the same dipole moment as the two-charge dipole that would experience the same torque.)

---

*Dipole moment of a molecule of NaCl gas* *example 5*

▷ In a molecule of NaCl gas, the center-to-center distance between the two atoms is about 0.6 nm. Assuming that the chlorine completely steals one of the sodium's electrons, compute the magnitude of this molecule's dipole moment.

▷ The total charge is zero, so it doesn't matter where we choose the origin of our coordinate system. For convenience, let's choose it to be at one of the atoms, so that the charge on that atom doesn't contribute to the dipole moment. The magnitude of the dipole moment is then

$$\begin{aligned} D &= (6 \times 10^{-10} \text{ m})(e) \\ &= (6 \times 10^{-10} \text{ m})(1.6 \times 10^{-19} \text{ C}) \\ &= 1 \times 10^{-28} \text{ C} \cdot \text{m} \end{aligned}$$

### Alternative definition of the electric field

The behavior of a dipole in an externally created field leads us to an alternative definition of the electric field:

#### alternative definition of the electric field

The electric field vector,  $E$ , at any location in space is defined by observing the torque exerted on a test dipole  $D_t$  placed there. The direction of the field is the direction in which the field tends to align a dipole (from  $-$  to  $+$ ), and the field's magnitude is  $|\mathbf{E}| = \tau / D_t \sin \theta$ .

The main reason for introducing a second definition for the same concept is that the magnetic field is most easily defined using a similar approach.

### Voltage related to electric field

Voltage is potential energy per unit charge, and electric field is force per unit charge. We can therefore relate voltage and field if we start from the relationship between potential energy and force,

$$\Delta PE = -Fd \quad , \quad \begin{array}{l} \text{[assuming constant force and} \\ \text{motion parallel to the force]} \end{array}$$

and divide by charge,

$$\frac{\Delta PE}{q} = -\frac{F}{q}d \quad ,$$

giving

$$\Delta V = -Ed \quad , \quad \begin{array}{l} \text{[assuming constant force and} \\ \text{motion parallel to the force]} \end{array}$$

In other words, the difference in voltage between two points equals the electric field strength multiplied by the distance between them. The interpretation is that a strong electric field is a region of space where the voltage is rapidly changing. By analogy, a steep hillside is a place on the map where the altitude is rapidly changing.

*Field generated by an electric eel* example 6

▷ Suppose an electric eel is 1 m long, and generates a voltage difference of 1000 volts between its head and tail. What is the electric field in the water around it?

▷ We are only calculating the amount of field, not its direction, so we ignore positive and negative signs. Subject to the possibly inaccurate assumption of a constant field parallel to the eel's body, we have

$$|E| = \frac{\Delta V}{\Delta x} = 1000 \text{ V/m} \quad .$$



o / Example 7.

*The hammerhead shark* example 7

One of the reasons hammerhead sharks have their heads shaped the way they do is that, like quite a few other fish, they can sense electric fields as a way of finding prey, which may for example be hidden in the sand. From the equation  $E = \Delta V / \Delta x$ , we can see that if the shark is sensing the voltage difference between two points, it will be able to detect smaller electric fields if those two points are farther apart. The shark has a network of sensory organs, called the ampullae of Lorenzini, on the skin of its head. Since the network is spread over a wider head, the  $\Delta x$  is larger. Some sharks can detect electric fields as weak as 50 *picovolts* per meter!

*Relating the units of electric field and voltage* example 8

From our original definition of the electric field, we expect it to have units of newtons per coulomb, N/C. The example above, however, came out in volts per meter, V/m. Are these inconsistent? Let's reassure ourselves that this all works. In this kind of situation, the best strategy is usually to simplify the more complex units so that they involve only mks units and coulombs. Since voltage is defined as electrical energy per unit charge, it has units of J/C:

$$\frac{\text{V}}{\text{m}} = \frac{\text{J/C}}{\text{m}} = \frac{\text{J}}{\text{C} \cdot \text{m}} \quad .$$

To connect joules to newtons, we recall that work equals force times distance, so  $J = N \cdot m$ , so

$$\frac{\text{V}}{\text{m}} = \frac{\text{N} \cdot \text{m}}{\text{C} \cdot \text{m}} = \frac{\text{N}}{\text{C}}$$

As with other such difficulties with electrical units, one quickly begins to recognize frequently occurring combinations.

## Discussion questions

**A** In the definition of the electric field, does the test charge need to be 1 coulomb? Does it need to be positive?

**B** Does a charged particle such as an electron or proton feel a force from its own electric field?

**C** Is there an electric field surrounding a wall socket that has nothing plugged into it, or a battery that is just sitting on a table?

**D** In a flashlight powered by a battery, which way do the electric fields point? What would the fields be like inside the wires? Inside the filament of the bulb?

**E** Criticize the following statement: "An electric field can be represented by a sea of arrows showing how current is flowing."

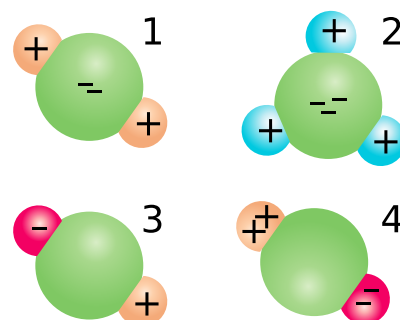
**F** The field of a point charge,  $|\mathbf{E}| = kQ/r^2$ , was derived in the self-check above. How would the field pattern of a uniformly charged sphere compare with the field of a point charge?

**G** The interior of a perfect electrical conductor in equilibrium must have zero electric field, since otherwise the free charges within it would be drifting in response to the field, and it would not be in equilibrium. What about the field right at the surface of a perfect conductor? Consider the possibility of a field perpendicular to the surface or parallel to it.

**H** Compare the dipole moments of the molecules and molecular ions shown in the figure.

**I** Small pieces of paper that have not been electrically prepared in any way can be picked up with a charged object such as a charged piece of tape. In our new terminology, we could describe the tape's charge as inducing a dipole moment in the paper. Can a similar technique be used to induce not just a dipole moment but a charge?

**J** The earth and moon are fairly uneven in size and far apart, like a baseball and a ping-pong ball held in your outstretched arms. Imagine instead a planetary system with the character of a double planet: two planets of equal size, close together. Sketch a sea of arrows diagram of their gravitational field.



p / Discussion

question H.

## 22.4 Calculating energy in fields

We found on p. 600 that fields have energy, and we now know as well that fields act like vectors. Presumably there is a relationship between the strength of a field and its energy density. Flipping the direction of the field can't change the density of energy, which is a scalar and therefore has no direction in space. We therefore expect that the energy density to be proportional to the square of the field, so that changing  $\mathbf{E}$  to  $-\mathbf{E}$  has no effect on the result. This is exactly what we've already learned to expect for waves: the energy depends on the square of the amplitude. The relevant equations for the

gravitational and electric fields are as follows:

$$\begin{aligned}\text{energy stored in the gravitational field per m}^3 &= -\frac{1}{8\pi G}|\mathbf{g}|^2 \\ \text{energy stored in the electric field per m}^3 &= \frac{1}{8\pi k}|\mathbf{E}|^2\end{aligned}$$

A similar expression is given on p. 659 for the magnetic field.

Although funny factors of  $8\pi$  and the plus and minus signs may have initially caught your eye, they are not the main point. The important idea is that the energy density is proportional to the square of the field strength in all cases. We first give a simple numerical example and work a little on the concepts, and then turn our attention to the factors out in front.

In chapter 22 when we discussed the original reason for introducing the concept of a field of force, a prime motivation was that otherwise there was no way to account for the energy transfers involved when forces were delayed by an intervening distance. We used to think of the universe's energy as consisting of

kinetic energy  
+gravitational potential energy based on the distances between  
objects that interact gravitationally  
+electric potential energy based on the distances between  
objects that interact electrically  
+magnetic potential energy based on the distances between  
objects that interact magnetically ,

but in nonstatic situations we must use a different method:

kinetic energy  
+gravitational potential energy stored in gravitational fields  
+electric potential energy stored in electric fields  
+magnetic potential stored in magnetic fields

Surprisingly, the new method still gives the same answers for the static cases.

---

*Energy stored in a capacitor* *example 9*

A pair of parallel metal plates, seen from the side in figure q, can be used to store electrical energy by putting positive charge on one side and negative charge on the other. Such a device is called a capacitor. (We have encountered such an arrangement previously, but there its purpose was to deflect a beam of electrons, not to store energy.)

In the old method of describing potential energy, 1, we think in terms of the mechanical work that had to be done to separate the positive and negative charges onto the two plates, working against their electrical attraction. The new description, 2, attributes the storage of energy to the newly created electric field occupying the volume between the plates. Since this is a static case, both methods give the same, correct answer.

*Potential energy of a pair of opposite charges*      *example 10*  
Imagine taking two opposite charges,  $r$ , that were initially far apart and allowing them to come together under the influence of their electrical attraction.

According to the old method, potential energy is lost because the electric force did positive work as it brought the charges together. (This makes sense because as they come together and accelerate it is their potential energy that is being lost and converted to kinetic energy.)

By the new method, we must ask how the energy stored in the electric field has changed. In the region indicated approximately by the shading in the figure, the superposing fields of the two charges undergo partial cancellation because they are in opposing directions. The energy in the shaded region is reduced by this effect. In the unshaded region, the fields reinforce, and the energy is increased.

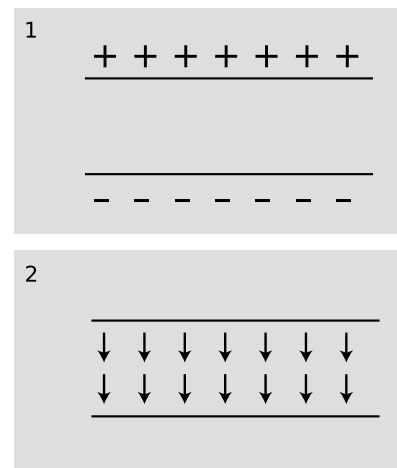
It would be quite a project to do an actual numerical calculation of the energy gained and lost in the two regions (this is a case where the old method of finding energy gives greater ease of computation), but it is fairly easy to convince oneself that the energy is less when the charges are closer. This is because bringing the charges together shrinks the high-energy unshaded region and enlarges the low-energy shaded region.

*Energy transmitted by ripples in the electric and magnetic fields*      *example 11*

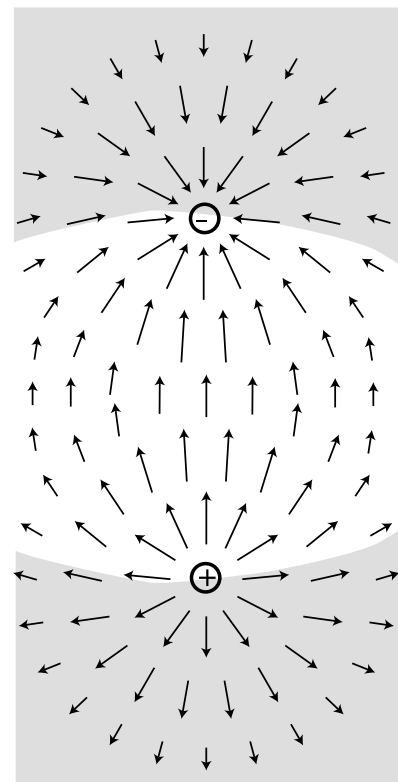
We'll see in chapter 24 that phenomena like light, radio waves, and x-rays are all ripples in the electric and magnetic fields. The old method would give zero energy for a region of space containing a light wave but no charges. That would be wrong! We can only use the old method in static cases.

Now let's give at least some justification for the other features of the expressions for energy density,  $-\frac{1}{8\pi G}|\mathbf{g}|^2$  and  $\frac{1}{8\pi k}|\mathbf{E}|^2$ , besides the proportionality to the square of the field strength.

First, why the different plus and minus signs? The basic idea is that the signs have to be opposite in the gravitational and electric cases because there is an attraction between two positive masses (which are the only kind that exist), but two positive charges would repel. Since we've already seen examples where the positive sign in the



q / Example 9.



r / Example 10.



electric energy makes sense, the gravitational energy equation must be the one with the minus sign.

It may also seem strange that the constants  $G$  and  $k$  are in the denominator. They tell us how strong the three different forces are, so shouldn't they be on top? No. Consider, for instance, an alternative universe in which gravity is twice as strong as in ours. The numerical value of  $G$  is doubled. Because  $G$  is doubled, all the gravitational field strengths are doubled as well, which quadruples the quantity  $|\mathbf{g}|^2$ . In the expression  $-\frac{1}{8\pi G}|\mathbf{g}|^2$ , we have quadrupled something on top and doubled something on the bottom, which makes the energy twice as big. That makes perfect sense.

### Discussion questions

**A** The figure shows a positive charge in the gap between two capacitor plates. First make a large drawing of the field pattern that would be formed by the capacitor itself, without the extra charge in the middle. Next, show how the field pattern changes when you add the particle at these two positions. Compare the energy of the electric fields in the two cases. Does this agree with what you would have expected based on your knowledge of electrical forces?

**B** Criticize the following statement: "A solenoid makes a charge in the space surrounding it, which dissipates when you release the energy."

**C** In example 10, I argued that the fields surrounding a positive and negative charge contain less energy when the charges are closer together. Perhaps a simpler approach is to consider the two extreme possibilities: the case where the charges are infinitely far apart, and the one in which they are at zero distance from each other, i.e., right on top of each other. Carry out this reasoning for the case of (1) a positive charge and a negative charge of equal magnitude, (2) two positive charges of equal magnitude, (3) the gravitational energy of two equal masses.

## 22.5 $\int$ Voltage for nonuniform fields

The calculus-savvy reader will have no difficulty generalizing the field-voltage relationship to the case of a varying field. The potential energy associated with a varying force is

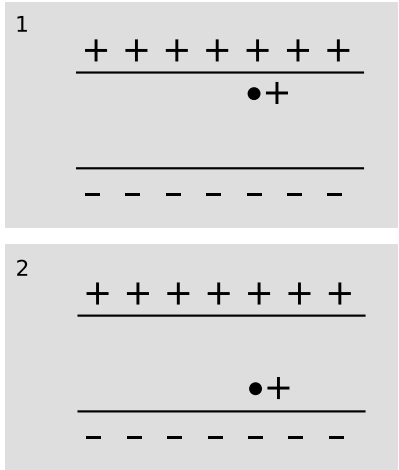
$$\Delta PE = - \int F \, dx \quad , \quad [\text{one dimension}]$$

so for electric fields we divide by  $q$  to find

$$\Delta V = - \int E \, dx \quad , \quad [\text{one dimension}]$$

Applying the fundamental theorem of calculus yields

$$E = - \frac{dV}{dx} \quad . \quad [\text{one dimension}]$$



s / Discussion question A.

- ▷ What is the voltage associated with a point charge?
- ▷ As derived previously in self-check A on page 605, the field is

$$|\mathbf{E}| = \frac{kQ}{r^2}$$

The difference in voltage between two points on the same radius line is

$$\begin{aligned}\Delta V &= \int dV \\ &= - \int E_x dx\end{aligned}$$

In the general discussion above,  $x$  was just a generic name for distance traveled along the line from one point to the other, so in this case  $x$  really means  $r$ .

$$\begin{aligned}\Delta V &= - \int_{r_1}^{r_2} E_r dr \\ &= - \int_{r_1}^{r_2} \frac{kQ}{r^2} dr \\ &= \left[ \frac{kQ}{r} \right]_{r_1}^{r_2} \\ &= \frac{kQ}{r_2} - \frac{kQ}{r_1} .\end{aligned}$$

The standard convention is to use  $r_1 = \infty$  as a reference point, so that the voltage at any distance  $r$  from the charge is

$$V = \frac{kQ}{r} .$$

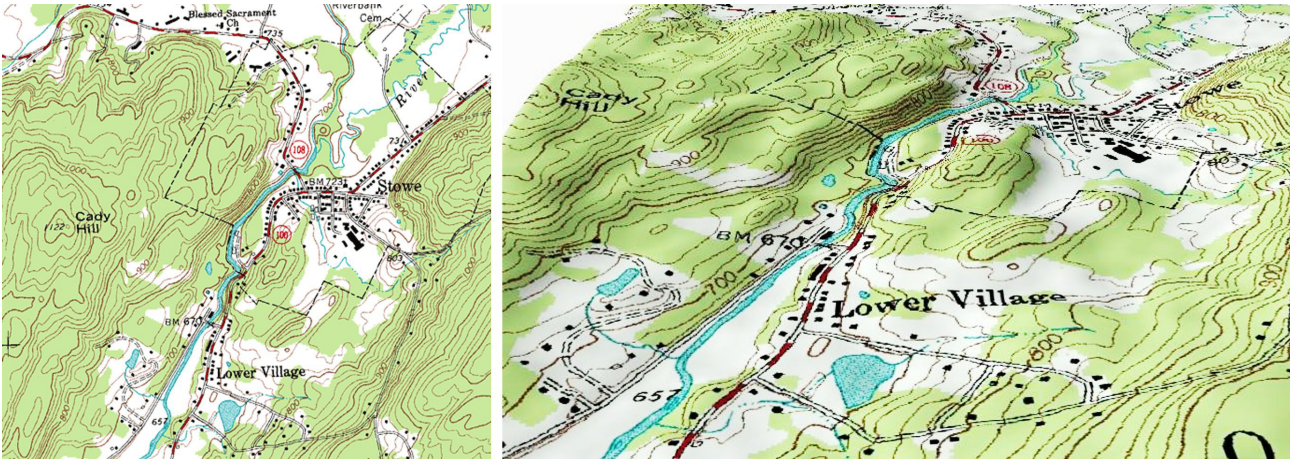
The interpretation is that if you bring a positive test charge closer to a positive charge, its electrical energy is increased; if it was released, it would spring away, releasing this as kinetic energy.

#### self-check B

Show that you can recover the expression for the field of a point charge by evaluating the derivative  $E_x = -dV/dx$ . ▷ Answer, p. 977

## 22.6 Two or three dimensions

The topographical map shown in figure t suggests a good way to visualize the relationship between field and voltage in two dimensions. Each contour on the map is a line of constant height; some of these are labeled with their elevations in units of feet. Height is related to gravitational potential energy, so in a gravitational analogy, we can



t / Left: A topographical map of Stowe, Vermont. From one constant-height line to the next is a height difference of 200 feet. Lines far apart, as in the lower village, indicate relatively flat terrain, while lines close together, like the ones to the west of the main town, represent a steep slope. Streams flow downhill, perpendicular to the constant-height lines. Right: The same map has been redrawn in perspective, with shading to suggest relief.

think of height as representing voltage. Where the contour lines are far apart, as in the town, the slope is gentle. Lines close together indicate a steep slope.

If we walk along a straight line, say straight east from the town, then height (voltage) is a function of the east-west coordinate  $x$ . Using the usual mathematical definition of the slope, and writing  $V$  for the height in order to remind us of the electrical analogy, the slope along such a line is  $\Delta V/\Delta x$ . If the slope isn't constant, we either need to use the slope of the  $V - x$  graph, or use calculus and talk about the derivative  $dV/dx$ .

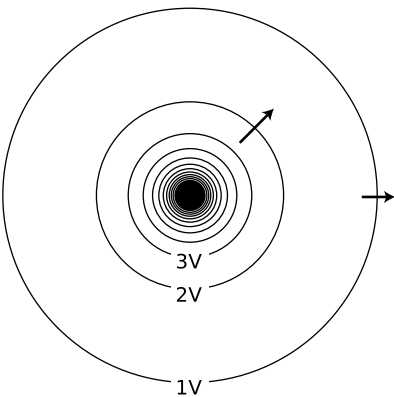
What if everything isn't confined to a straight line? Water flows downhill. Notice how the streams on the map cut perpendicularly through the lines of constant height.

It is possible to map voltages in the same way, as shown in figure u. The electric field is strongest where the constant-voltage curves are closest together, and the electric field vectors always point perpendicular to the constant-voltage curves.

Figure w shows some examples of ways to visualize field and voltage patterns.

Mathematically, the calculus of section 22.5 generalizes to three dimensions as follows:

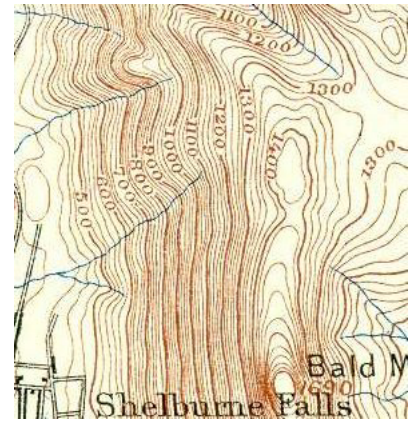
$$\begin{aligned} E_x &= -dV/dx \\ E_y &= -dV/dy \\ E_z &= -dV/dz \end{aligned}$$



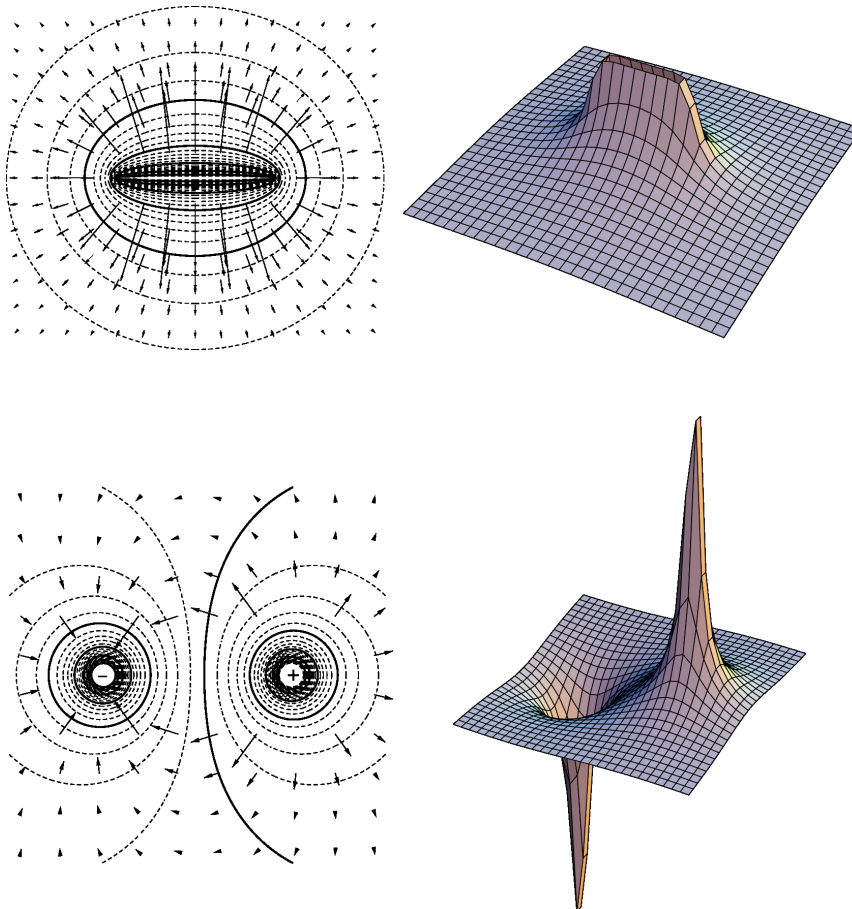
u / The constant-voltage curves surrounding a point charge. Near the charge, the curves are so closely spaced that they blend together on this drawing due to the finite width with which they were drawn. Some electric fields are shown as arrows.

### self-check C

Imagine that the topographical map in figure v represents voltage rather than height. (a) Consider the stream that starts near the center of the map. Determine the positive and negative signs of  $dV/dx$  and  $dV/dy$ , and relate these to the direction of the force that is pushing the current forward against the resistance of friction. (b) If you wanted to find a lot of electric charge on this map, where would you look? ▷ Answer, p. 978



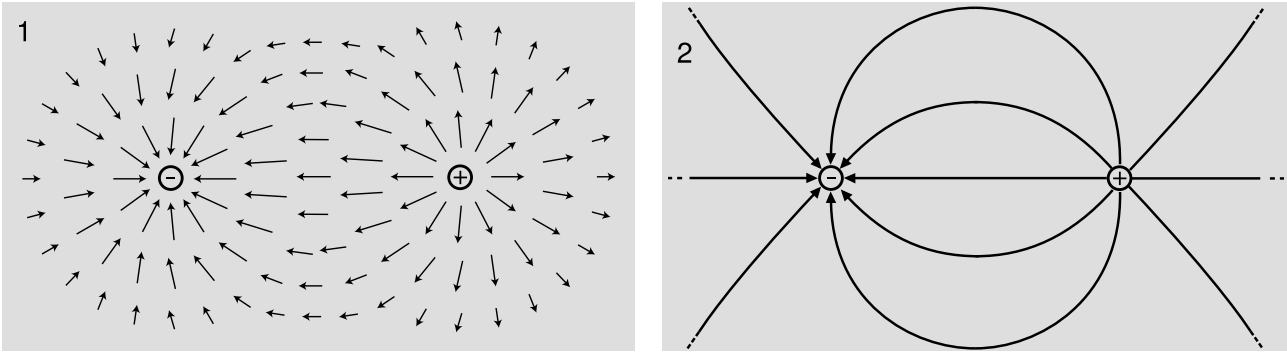
v / Self-check C.



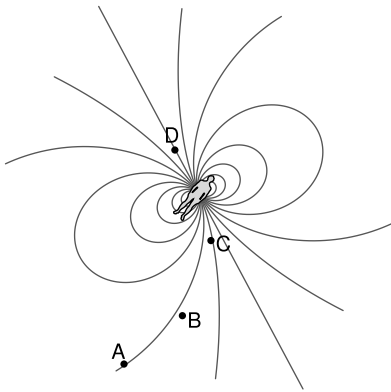
w / Two-dimensional field and voltage patterns. Top: A uniformly charged rod. Bottom: A dipole. In each case, the diagram on the left shows the field vectors and constant-voltage curves, while the one on the right shows the voltage (up-down coordinate) as a function of  $x$  and  $y$ . Interpreting the field diagrams: Each arrow represents the field at the point where its tail has been positioned. For clarity, some of the arrows in regions of very strong field strength are not shown — they would be too long to show. Interpreting the constant-voltage curves: In regions of very strong fields, the curves are not shown because they would merge together to make solid black regions. Interpreting the perspective plots: Keep in mind that even though we're visualizing things in three dimensions, these are really two-dimensional voltage patterns being represented. The third (up-down) dimension represents voltage, not position.

## 22.7 ★ Field lines and Gauss's law

When we look at the “sea of arrows” representation of a field,  $x/1$ , there is a natural visual tendency to imagine connecting the arrows as in  $x/2$ . The curves formed in this way are called field lines, and they have a direction, shown by the arrowheads.



x / Two different representations of an electric field.



y / Example 13.

### *An avalanche transceiver*

### *example 13*

The dipole field pattern is universal: it always has the same mathematical shape, and this holds regardless of whether the field is electric or magnetic, static or oscillating. One of its universal properties is that the field lines always pass through the source, and this is taken advantage of in a device called an avalanche transceiver used by backcountry skiers and hikers in winter. The device is worn on the body and is left in transmitting mode while the user is traveling. It creates a dipole pattern of electric and magnetic fields, oscillating at 457 kHz. If a member of the party is buried in an avalanche, his companions must find and locate him within about 30-90 minutes, or he will suffocate. The rescuers switch their own transceivers to receive mode, allowing them to determine the direction of the field line; when the antenna is parallel to the field line, the oscillating electric field drives electrons up and down it, creating the maximum possible current.

In the figure, the rescuer moves along the field line from A to B. At B, she verifies that the strength of the signal has increased. (If it hadn't, she would have had to turn around and go in the opposite direction.) She redetermines the direction of the field and continues in the new direction. Continuing the process, she proceeds along an approximation to the field line. At D she finds that the field strength has fallen off again, so she knows that she has just passed over the victim's position.

Electric field lines originate from positive charges and terminate on negative ones. We can choose a constant of proportionality that fixes how coarse or fine the “grain of the wood” is, but once this choice is made the strength of each charge is shown by the number of lines

that begin or end on it. For example, figure x/2 shows eight lines at each charge, so we know that  $q_1/q_2 = (-8)/8 = -1$ . Because lines never begin or end except on a charge, we can always find the total charge inside any given region by subtracting the number of lines that go in from the number that come out and multiplying by the appropriate constant of proportionality. Ignoring the constant, we can apply this technique to figure z to find  $q_A = -8$ ,  $q_B = 2 - 2 = 0$ , and  $q_C = 5 - 5 = 0$ .

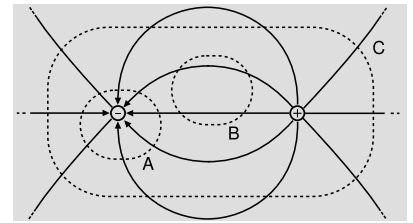
Figures x and z are drawn in two dimensions, but we should really imagine them in three dimensions, so that the regions A, B, and C in figure z are volumes bounded by surfaces. It is only because our universe happens to have three dimensions that the field line concept is as useful as it is. This is because the number of field lines per unit area perpendicularly piercing a surface can be interpreted as the strength of the electric field. To see this, consider an imaginary spherical surface of radius  $r$ , with a positive charge  $q$  at its center. The field at the surface equals  $kq/r^2$ . The number of field lines piercing the surface is independent of  $r$ , but the surface's area is proportional to  $r^2$ , so the number of piercings per unit area is proportional to  $1/r^2$ , just like the electric field. With this interpretation, we arrive at Gauss's law, which states that the field strength on a surface multiplied by the surface area equals the total charge enclosed by the surface. (This particular formulation only works in the case where the field pierces the surface perpendicularly and is constant in magnitude over the whole surface. It can be reformulated so as to eliminate these restrictions, but we won't require that reformulation for our present purposes.)

#### The field of a line of charge

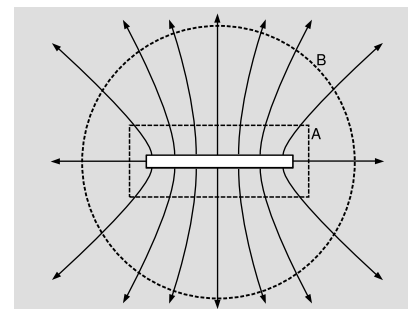
#### example 14

In example 4 on p. 607, we found by simple scaling arguments that the electric field of a line of charge is proportional to  $1/r$ , where  $r$  is the distance from the line. Since electric fields are always proportional to the Coulomb constant  $k$  and to the amount of charge, clearly we must have something of the form  $E = (\dots)k(q/L)/r$ , where  $q/L$  is the charge per unit length and  $(\dots)$  represents an unknown numerical constant (which you can easily verify is unitless). Using Gauss's law we can fill in the final piece of the puzzle, which is the value of this constant.

Figure aa shows two surfaces, a cylindrical one A and a spherical one B. Every field line that passes through A also passes through B. By making A sufficiently small and B sufficiently large, we can make the field on each surface nearly constant and perpendicular to the surface, as required by our restricted form of Gauss's law. If radius  $r_B$  is made large, the field at B made by the line of charge becomes indistinguishable from that of a point charge,  $kq/r_B^2$ . By Gauss's law, the electric field at each surface is proportional to the number of field lines divided by its area. But the number of



z / The number of field lines coming in and out of each region depends on the total charge it encloses.



aa / Example 14.

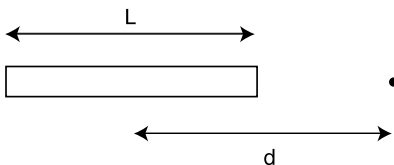
field lines is the same in both cases, so each field is inversely proportional to the corresponding surface area. We therefore have

$$\begin{aligned} E_A &= E_B \left( \frac{A_B}{A_A} \right) \\ &= \left( \frac{kq}{r_B^2} \right) \left( \frac{4\pi r_B^2}{2\pi r_A L} \right) \\ &= \frac{2kq}{Lr_B} \end{aligned}$$

The unknown numerical constant equals 2.

## 22.8 $\int$ ★ Electric field of a continuous charge distribution

Charge really comes in discrete chunks, but often it is mathematically convenient to treat a set of charges as if they were like a continuous fluid spread throughout a region of space. For example, a charged metal ball will have charge spread nearly uniformly all over its surface, and in for most purposes it will make sense to ignore the fact that this uniformity is broken at the atomic level. The electric field made by such a continuous charge distribution is the sum of the fields created by every part of it. If we let the “parts” become infinitesimally small, we have a sum of an infinite number of infinitesimal numbers, which is an integral. If it was a discrete sum, we would have a total electric field in the  $x$  direction that was the sum of all the  $x$  components of the individual fields, and similarly we’d have sums for the  $y$  and  $z$  components. In the continuous case, we have three integrals.



ab / Example 15.

### *Field of a uniformly charged rod* example 15

▷ A rod of length  $L$  has charge  $Q$  spread uniformly along it. Find the electric field at a point a distance  $d$  from the center of the rod, along the rod's axis. (This is different from examples 4 on p. 607 and 14 on p. 619, both because the point is on the axis of the rod and because the rod is of finite length.)

▷ This is a one-dimensional situation, so we really only need to do a single integral representing the total field along the axis. We imagine breaking the rod down into short pieces of length  $dz$ , each with charge  $dq$ . Since charge is uniformly spread along the rod, we have  $dq = \lambda dz$ , where  $\lambda = Q/L$  (Greek lambda) is the charge per unit length, in units of coulombs per meter. Since the pieces are infinitesimally short, we can treat them as point charges and use the expression  $k dq / r^2$  for their contributions to the field, where  $r = d - z$  is the distance from the charge at  $z$  to

the point in which we are interested.

$$\begin{aligned}
 E_z &= \int \frac{k dq}{r^2} \\
 &= \int_{-L/2}^{+L/2} \frac{k \lambda dz}{r^2} \\
 &= k \lambda \int_{-L/2}^{+L/2} \frac{dz}{(d-z)^2}
 \end{aligned}$$

The integral can be looked up in a table, or reduced to an elementary form by substituting a new variable for  $d - z$ . The result is

$$\begin{aligned}
 E_z &= k \lambda \left( \frac{1}{d-z} \right)_{-L/2}^{+L/2} \\
 &= \frac{kQ}{L} \left( \frac{1}{d-L/2} - \frac{1}{d+L/2} \right) .
 \end{aligned}$$

For large values of  $d$ , this expression gets smaller for two reasons: (1) the denominators of the fractions become large, and (2) the two fractions become nearly the same, and tend to cancel out. This makes sense, since the field should get weaker as we get farther away from the charge. In fact, the field at large distances must approach  $kQ/d^2$ , since from a great distance, the rod looks like a point.

It's also interesting to note that the field becomes infinite at the ends of the rod, but is not infinite on the interior of the rod. Can you explain physically why this happens?



## Summary

### Selected vocabulary

field . . . . .	a property of a point in space describing the forces that would be exerted on a particle if it was there
sink . . . . .	a point at which field vectors converge
source . . . . .	a point from which field vectors diverge; often used more inclusively to refer to points of either convergence or divergence
electric field . . .	the force per unit charge exerted on a test charge at a given point in space
gravitational field	the force per unit mass exerted on a test mass at a given point in space
electric dipole . .	an object that has an imbalance between positive charge on one side and negative charge on the other; an object that will experience a torque in an electric field

### Notation

$\mathbf{g}$ . . . . .	the gravitational field
$\mathbf{E}$ . . . . .	the electric field
$D$ . . . . .	an electric dipole moment

### Other terminology and notation

$d, p, m$ . . . . .	other notations for the electric dipole moment
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## Summary

Experiments show that time is not absolute: it flows at different rates depending on an observer's state of motion. This is an example of the strange effects predicted by Einstein's theory of relativity. All of these effects, however, are very small when the relative velocities are small compared to  $c$ . This makes sense, because Newton's laws have already been thoroughly tested by experiments at such speeds, so a new theory like relativity must agree with the old one in their realm of common applicability. This requirement of backwards-compatibility is known as the correspondence principle.

Since time is not absolute, simultaneity is not a well-defined concept, and therefore the universe cannot operate as Newton imagined, through instantaneous action at a distance. There is a delay in time before a change in the configuration of mass and charge in one corner of the universe will make itself felt as a change in the forces experienced far away. We imagine the outward spread of such a change as a ripple in an invisible universe-filling *field of force*.

We define the *gravitational field* at a given point as the force per unit mass exerted on objects inserted at that point, and likewise the *electric field* is defined as the force per unit charge. These fields are vectors, and the fields generated by multiple sources add according to the rules of vector addition.

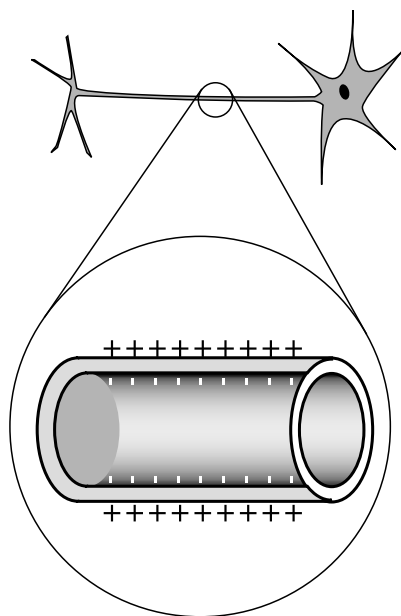
When the electric field is constant, the voltage difference between two points lying on a line parallel to the field is related to the field by the equation  $\Delta V = -Ed$ , where  $d$  is the distance between the two points.

Fields of force contain energy. The density of energy is proportional to the square of the magnitude of the field. In the case of static fields, we can calculate potential energy either using the previous definition in terms of mechanical work or by calculating the energy stored in the fields. If the fields are not static, the old method gives incorrect results and the new one must be used.

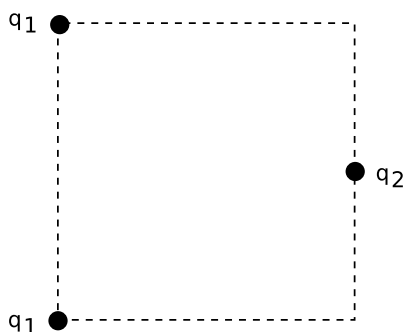
## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.



Problem 1.



Problem 4.

**1** In our by-now-familiar neuron, the voltage difference between the inner and outer surfaces of the cell membrane is about  $V_{out} - V_{in} = -70$  mV in the resting state, and the thickness of the membrane is about 6.0 nm (i.e., only about a hundred atoms thick). What is the electric field inside the membrane? ✓

**2** The gap between the electrodes in an automobile engine's spark plug is 0.060 cm. To produce an electric spark in a gasoline-air mixture, an electric field of  $3.0 \times 10^6$  V/m must be achieved. On starting a car, what minimum voltage must be supplied by the ignition circuit? Assume the field is uniform. ✓

(b) The small size of the gap between the electrodes is inconvenient because it can get blocked easily, and special tools are needed to measure it. Why don't they design spark plugs with a wider gap?

**3** (a) At time  $t = 0$ , a positively charged particle is placed, at rest, in a vacuum, in which there is a uniform electric field of magnitude  $E$ . Write an equation giving the particle's speed,  $v$ , in terms of  $t$ ,  $E$ , and its mass and charge  $m$  and  $q$ . ✓

(b) If this is done with two different objects and they are observed to have the same motion, what can you conclude about their masses and charges? (For instance, when radioactivity was discovered, it was found that one form of it had the same motion as an electron in this type of experiment.)

**4** Three charges are arranged on a square as shown. All three charges are positive. What value of  $q_2/q_1$  will produce zero electric field at the center of the square? ▷ Solution, p. 973

**5** Show that the magnitude of the electric field produced by a simple two-charge dipole, at a distant point along the dipole's axis, is to a good approximation proportional to  $D/r^3$ , where  $r$  is the distance from the dipole. [Hint: Use the approximation  $(1 + \epsilon)^p \approx 1 + p\epsilon$ , which is valid for small  $\epsilon$ .] ★

**6** Consider the electric field created by a uniform ring of total charge  $q$  and radius  $b$ . (a) Show that the field at a point on the ring's axis at a distance  $a$  from the plane of the ring is  $kqa(a^2 + b^2)^{-3/2}$ . (b) Show that this expression has the right behavior for  $a = 0$  and for  $a$  much greater than  $b$ . ★

**7** Example 4 on p. 607 showed that the electric field of a long, uniform line of charge falls off with distance as  $1/r$ . By a similar technique, show that the electric field of a uniformly charged plane has *no* dependence on the distance from the plane. ★

**8** Given that the field of a dipole is proportional to  $D/r^3$  (problem 5), show that its voltage varies as  $D/r^2$ . (Ignore positive and negative signs and numerical constants of proportionality.) ∫

**9** A carbon dioxide molecule is structured like O-C-O, with all three atoms along a line. The oxygen atoms grab a little bit of extra negative charge, leaving the carbon positive. The molecule's symmetry, however, means that it has no overall dipole moment, unlike a V-shaped water molecule, for instance. Whereas the voltage of a dipole of magnitude  $D$  is proportional to  $D/r^2$  (problem 8), it turns out that the voltage of a carbon dioxide molecule along its axis equals  $k/r^3$ , where  $r$  is the distance from the molecule and  $k$  is a constant. What would be the electric field of a carbon dioxide molecule at a distance  $r$ ? ✓ ∫

**10** A proton is in a region in which the electric field is given by  $E = a + bx^3$ . If the proton starts at rest at  $x_1 = 0$ , find its speed,  $v$ , when it reaches position  $x_2$ . Give your answer in terms of  $a$ ,  $b$ ,  $x_2$ , and  $e$  and  $m$ , the charge and mass of the proton. ✓ ∫

**11** Consider the electric field created by a uniformly charged cylinder that extends to infinity in one direction. (a) Starting from the result of problem 8, show that the field at the center of the cylinder's mouth is  $2\pi k\sigma$ , where  $\sigma$  is the density of charge on the cylinder, in units of coulombs per square meter. [Hint: You can use a method similar to the one in problem 9.] (b) This expression is independent of the radius of the cylinder. Explain why this should be so. For example, what would happen if you doubled the cylinder's radius? ∫

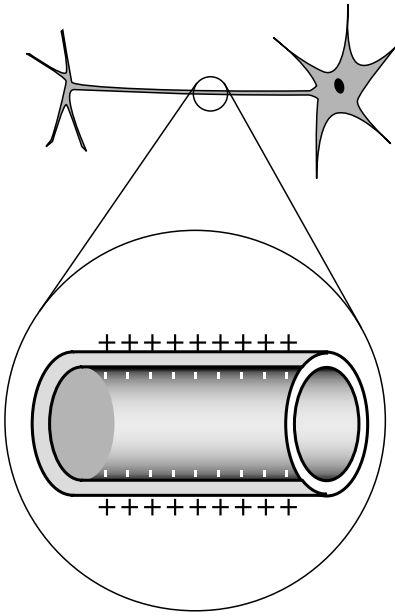
**12** In an electrical storm, the cloud and the ground act like a parallel-plate capacitor, which typically charges up due to frictional electricity in collisions of ice particles in the cold upper atmosphere. Lightning occurs when the magnitude of the electric field builds up to a critical value,  $E_c$ , at which air is ionized.

(a) Treat the cloud as a flat square with sides of length  $L$ . If it is at a height  $h$  above the ground, find the amount of energy released in the lightning strike. ✓

(b) Based on your answer from part a, which is more dangerous, a lightning strike from a high-altitude cloud or a low-altitude one?

(c) Make an order-of-magnitude estimate of the energy released by a typical lightning bolt, assuming reasonable values for its size and altitude.  $E_c$  is about  $10^6$  V/m.

See problem 16 for a note on how recent research affects this estimate.



Problem 13.

**13** The neuron in the figure has been drawn fairly short, but some neurons in your spinal cord have tails (axons) up to a meter long. The inner and outer surfaces of the membrane act as the “plates” of a capacitor. (The fact that it has been rolled up into a cylinder has very little effect.) In order to function, the neuron must create a voltage difference  $V$  between the inner and outer surfaces of the membrane. Let the membrane’s thickness, radius, and length be  $t$ ,  $r$ , and  $L$ . (a) Calculate the energy that must be stored in the electric field for the neuron to do its job. (In real life, the membrane is made out of a substance called a dielectric, whose electrical properties increase the amount of energy that must be stored. For the sake of this analysis, ignore this fact.) [Hint: The volume of the membrane is essentially the same as if it was unrolled and flattened out.]  $\checkmark$

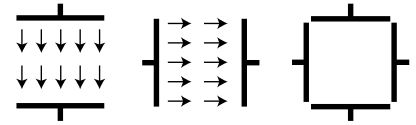
(b) An organism’s evolutionary fitness should be better if it needs less energy to operate its nervous system. Based on your answer to part a, what would you expect evolution to do to the dimensions  $t$  and  $r$ ? What other constraints would keep these evolutionary trends from going too far?

**14** To do this problem, you need to understand how to do volume integrals in cylindrical and spherical coordinates. (a) Show that if you try to integrate the energy stored in the field of a long, straight wire, the resulting energy per unit length diverges both at  $r \rightarrow 0$  and  $r \rightarrow \infty$ . Taken at face value, this would imply that a certain real-life process, the initiation of a current in a wire, would be impossible, because it would require changing from a state of zero magnetic energy to a state of infinite magnetic energy. (b) Explain why the infinities at  $r \rightarrow 0$  and  $r \rightarrow \infty$  don’t really happen in a realistic situation. (c) Show that the electric energy of a point charge diverges at  $r \rightarrow 0$ , but not at  $r \rightarrow \infty$ .

A remark regarding part (c): Nature does seem to supply us with particles that are charged and pointlike, e.g., the electron, but one could argue that the infinite energy is not really a problem, because an electron traveling around and doing things neither gains nor loses infinite energy; only an infinite *change* in potential energy would be physically troublesome. However, there are real-life processes that create and destroy pointlike charged particles, e.g., the annihilation of an electron and antielectron with the emission of two gamma rays. Physicists have, in fact, been struggling with infinities like this since about 1950, and the issue is far from resolved. Some theorists propose that apparently pointlike particles are actually not pointlike: close up, an electron might be like a little circular loop of string.

$\int \star$

**15** The figure shows cross-sectional views of two cubical capacitors, and a cross-sectional view of the same two capacitors put together so that their interiors coincide. A capacitor with the plates close together has a nearly uniform electric field between the plates, and almost zero field outside; these capacitors don't have their plates very close together compared to the dimensions of the plates, but for the purposes of this problem, assume that they still have approximately the kind of idealized field pattern shown in the figure. Each capacitor has an interior volume of  $1.00 \text{ m}^3$ , and is charged up to the point where its internal field is  $1.00 \text{ V/m}$ . (a) Calculate the energy stored in the electric field of each capacitor when they are separate. (b) Calculate the magnitude of the interior field when the two capacitors are put together in the manner shown. Ignore effects arising from the redistribution of each capacitor's charge under the influence of the other capacitor. (c) Calculate the energy of the put-together configuration. Does assembling them like this release energy, consume energy, or neither? ✓



Problem 15.

**16** In problem 12 on p. 625, you estimated the energy released in a bolt of lightning, based on the energy stored in the electric field immediately before the lightning occurs. The assumption was that the field would build up to a certain value, which is what is necessary to ionize air. However, real-life measurements always seemed to show electric field strengths roughly 10 times smaller than those required in that model. For a long time, it wasn't clear whether the field measurements were wrong, or the model was wrong. Research carried out in 2003 seems to show that the model was wrong. It is now believed that the final triggering of the bolt of lightning comes from cosmic rays that enter the atmosphere and ionize some of the air. If the field is 10 times smaller than the value assumed in problem 12, what effect does this have on the final result of problem 12? ✓

## Exercise 22: Field vectors

Apparatus:

3 solenoids

DC power supply

compass

ruler

cut-off plastic cup

At this point you've studied the gravitational field,  $\mathbf{g}$ , and the electric field,  $\mathbf{E}$ , but not the magnetic field,  $\mathbf{B}$ . However, they all have some of the same mathematical behavior: they act like vectors. Furthermore, magnetic fields are the easiest to manipulate in the lab. Manipulating gravitational fields directly would require futuristic technology capable of moving planet-sized masses around! Playing with electric fields is not as ridiculously difficult, but static electric charges tend to leak off through your body to ground, and static electricity effects are hard to measure numerically. Magnetic fields, on the other hand, are easy to make and control. Any moving charge, i.e. any current, makes a magnetic field.

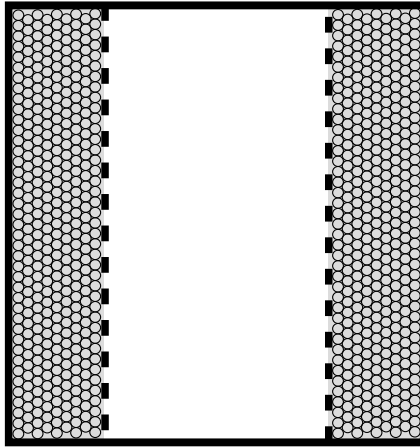
A practical device for making a strong magnetic field is simply a coil of wire, formally known as a solenoid. The field pattern surrounding the solenoid gets stronger or weaker in proportion to the amount of current passing through the wire.

1. With a single solenoid connected to the power supply and laid with its axis horizontal, use a magnetic compass to explore the field pattern inside and outside it. The compass shows you the field vector's direction, but not its magnitude, at any point you choose. Note that the field the compass experiences is a combination (vector sum) of the solenoid's field and the earth's field.
2. What happens when you bring the compass extremely far away from the solenoid?

What does this tell you about the way the solenoid's field varies with distance?

Thus although the compass doesn't tell you the field vector's magnitude numerically, you can get at least some general feel for how it depends on distance.

3. The figure below is a cross-section of the solenoid in the plane containing its axis. Make a sea-of-arrows sketch of the magnetic field in this plane. The length of each arrow should at least approximately reflect the strength of the magnetic field at that point.



Does the field seem to have sources or sinks?

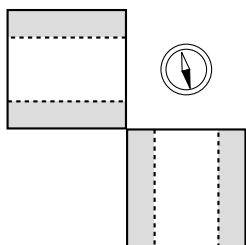
4. What do you think would happen to your sketch if you reversed the wires?

Try it.



5. Now hook up the two solenoids in parallel. You are going to measure what happens when their two fields combine in the at a certain point in space. As you've seen already, the solenoids' nearby fields are much stronger than the earth's field; so although we now theoretically have three fields involved (the earth's plus the two solenoids'), it will be safe to ignore the earth's field. The basic idea here is to place the solenoids with their axes at some angle to each other, and put the compass at the intersection of their axes, so that it is the same distance from each solenoid. Since the geometry doesn't favor either solenoid, the only factor that would make one solenoid influence the compass more than the other is current. You can use the cut-off plastic cup as a little platform to bring the compass up to the same level as the solenoids' axes.

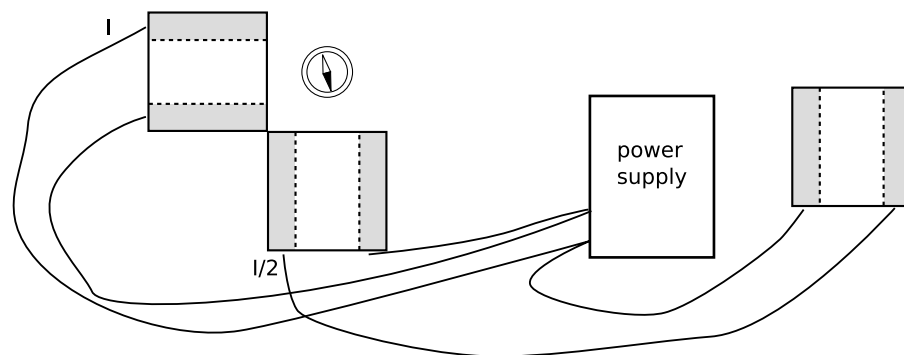
a) What do you think will happen with the solenoids' axes at 90 degrees to each other, and equal currents? Try it. Now represent the vector addition of the two magnetic fields with a diagram. Check your diagram with your instructor to make sure you're on the right track.



b) Now try to make a similar diagram of what would happen if you switched the wires on one of the solenoids.

After predicting what the compass will do, try it and see if you were right.

c) Now suppose you were to go back to the arrangement you had in part a, but you changed one of the currents to half its former value. Make a vector addition diagram, and use trig to predict the angle.



Try it. To cut the current to one of the solenoids in half, an easy and accurate method is simply to put the third solenoid in series with it, and put that third solenoid so far away that its magnetic field doesn't have any significant effect on the compass.

## Chapter 23

# Relativity and magnetism

Many people imagine Einstein's theory of relativity as something exotic and speculative. It's certainly not speculative at this point in history, since you use it every time you use a GPS receiver. But it's even less exotic than that. Every time you stick a magnet to your refrigerator, you're making use of relativity. The title of this chapter is "Electromagnetism," but we'll start out by digging a little deeper into relativity as preparation for understanding what magnetism is and where it comes from.

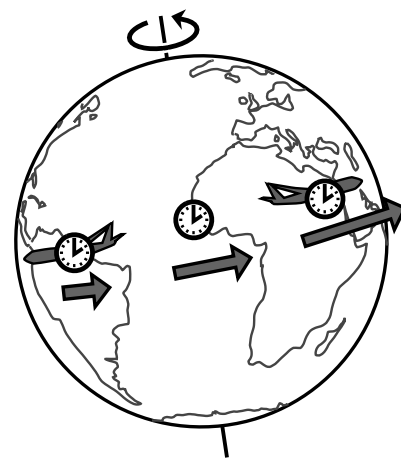
### 23.1 Relativistic distortion of space and time

#### Time distortion arising from motion and gravity

Let's refer back to the results of the Hafele-Keating experiment described on p. 598. Hafele and Keating were testing specific quantitative predictions of relativity, and they verified them to within their experiment's error bars. Let's work backward instead, and inspect the empirical results for clues as to how time works.

The east-going clock lost time, ending up off by  $-59 \pm 10$  nanoseconds, while the west-going one gained  $273 \pm 7$  ns. Since two traveling clocks experienced effects in opposite directions, we can tell that the rate at which time flows depends on the motion of the observer. The east-going clock was moving in the same direction as the earth's rotation, so its velocity relative to the earth's center was greater than that of the clock that remained in Washington, while the west-going clock's velocity was correspondingly reduced. The fact that the east-going clock fell behind, and the west-going one got ahead, shows that the effect of motion is to make time go more slowly. This effect of motion on time was predicted by Einstein in his original 1905 paper on relativity, written when he was 26.

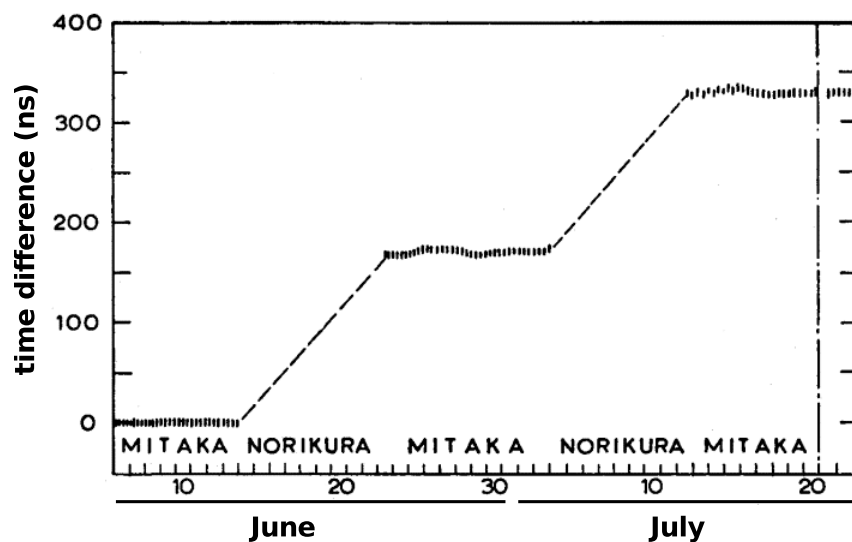
If this had been the only effect in the Hafele-Keating experiment, then we would have expected to see effects on the two flying clocks that were equal in size. Making up some simple numbers to keep the arithmetic transparent, suppose that the earth rotates from west to east at 1000 km/hr, and that the planes fly at 300 km/hr. Then the speed of the clock on the ground is 1000 km/hr, the speed of the clock on the east-going plane is 1300 km/hr, and that of the west-going clock 700 km/hr. Since the speeds of 700, 1000, and 1300 km/hr have equal spacing on either side of 1000, we would expect



a / All three clocks are moving to the east. Even though the west-going plane is moving to the west relative to the air, the air is moving to the east due to the earth's rotation.

the discrepancies of the moving clocks relative to the one in the lab to be equal in size but opposite in sign.

In fact, the two effects are unequal in size:  $-59$  ns and  $273$  ns. This implies that there is a second effect involved, simply due to the planes' being up in the air. This was verified more directly in a 1978 experiment by Iijima and Fujiwara, figure b, in which identical atomic clocks were kept at rest at the top and bottom of a mountain near Tokyo. This experiment, unlike the Hafele-Keating one, isolates one effect on time, the gravitational one: time's rate of flow increases with height in a gravitational field. Einstein didn't figure out how to incorporate gravity into relativity until 1915, after much frustration and many false starts. The simpler version of the theory without gravity is known as special relativity, the full version as general relativity. We'll restrict ourselves to special relativity until chapter 27, and that means that what we want to focus on right now is the distortion of time due to motion, not gravity.



b / A graph showing the time difference between two atomic clocks. One clock was kept at Mitaka Observatory, at 58 m above sea level. The other was moved back and forth to a second observatory, Norikura Corona Station, at the peak of the Norikura volcano, 2876 m above sea level. The plateaus on the graph are data from the periods when the clocks were compared side by side at Mitaka. The difference between one plateau and the next is the gravitational effect on the rate of flow of time, accumulated during the period when the mobile clock was at the top of Norikura. (This is the original graph showing the actual data, except for some clarification of the labels on the axes.)

We can now see in more detail how to apply the correspondence principle. The behavior of the three clocks in the Hafele-Keating experiment shows that the amount of time distortion increases as

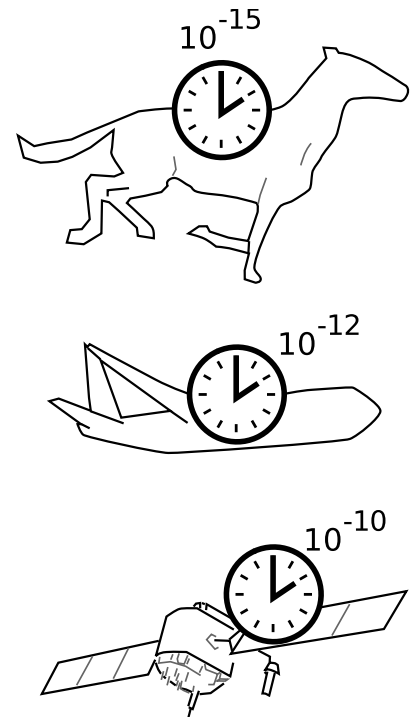
the speed of the clock's motion increases. Newton lived in an era when the fastest mode of transportation was a galloping horse, and the best pendulum clocks would accumulate errors of perhaps a minute over the course of several days. A horse is much slower than a jet plane, so the distortion of time would have had a relative size of only  $\sim 10^{-15}$  — much smaller than the clocks were capable of detecting. At the speed of a passenger jet, the effect is about  $10^{-12}$ , and state-of-the-art atomic clocks in 1971 were capable of measuring that. A GPS satellite travels much faster than a jet airplane, and the effect on the satellite turns out to be  $\sim 10^{-10}$ . The general idea here is that all physical laws are approximations, and approximations aren't simply right or wrong in different situations. Approximations are better or worse in different situations, and the question is whether a particular approximation is good enough in a given situation to serve a particular purpose. The faster the motion, the worse the Newtonian approximation of absolute time. Whether the approximation is good enough depends on what you're trying to accomplish. The correspondence principle says that the approximation must have been good enough to explain all the experiments done in the centuries before Einstein came up with relativity.

By the way, don't get an inflated idea of the importance of these atomic clock experiments. Special relativity had already been confirmed by a vast and varied body of experiments decades before the 1970's. The only reason I'm giving such a prominent role to these experiments, which were actually more important as tests of general relativity, is that they were conceptually very direct. It would be nice to have an equally simple and transparent atomic clock experiment in which only the effect of motion was singled out, with no gravitational effect. Example 2 on page 640 describes how something along these lines was eventually carried out, forty years after the Hafele-Keating experiment.

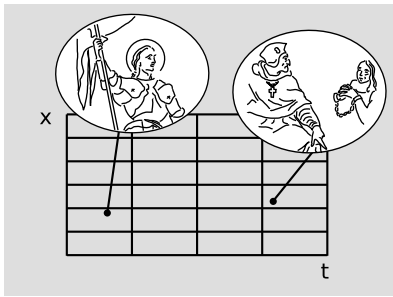
### The Lorentz transformation

Relativity says that when two observers are in different frames of reference, each observer considers the other one's perception of time to be distorted. We'll also see that something similar happens to their observations of distances, so both space and time are distorted. What exactly is this distortion? How do we even conceptualize it?

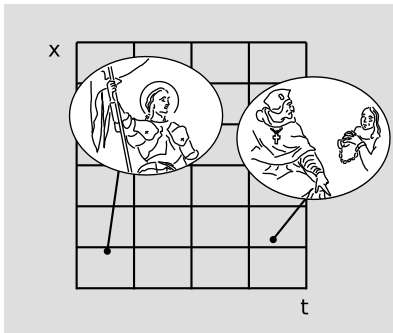
The idea isn't really as radical as it might seem at first. We can visualize the structure of space and time using a graph with position and time on its axes. These graphs are familiar by now, but we're going to look at them in a slightly different way. Before, we used them to describe the motion of objects. The grid underlying the graph was merely the stage on which the actors played their parts. Now the background comes to the foreground: it's time and space themselves that we're studying. We don't necessarily need to have a line or a curve drawn on top of the grid to represent a particu-



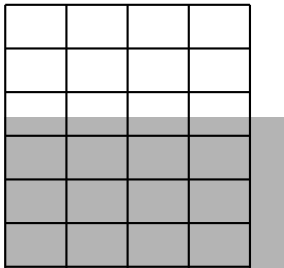
c / The correspondence principle requires that the relativistic distortion of time become small for small velocities.



d / Two events are given as points on a graph of position versus time. Joan of Arc helps to restore Charles VII to the throne. At a later time and a different position, Joan of Arc is sentenced to death.



e / A change of units distorts an  $x$ - $t$  graph. This graph depicts exactly the same events as figure d. The only change is that the  $x$  and  $t$  coordinates are measured using different units, so the grid is compressed in  $t$  and expanded in  $x$ .



f / A convention we'll use to represent a distortion of time and space.

lar object. We may, for example, just want to talk about events, depicted as points on the graph as in figure d. A distortion of the Cartesian grid underlying the graph can arise for perfectly ordinary reasons that Newton would have readily accepted. For example, we can simply change the units used to measure time and position, as in figure e.

We're going to have quite a few examples of this type, so I'll adopt the convention shown in figure f for depicting them. Figure f summarizes the relationship between figures d and e in a more compact form. The gray rectangle represents the original coordinate grid of figure d, while the grid of black lines represents the new version from figure e. Omitting the grid from the gray rectangle makes the diagram easier to decode visually.

Our goal of unraveling the mysteries of special relativity amounts to nothing more than finding out how to draw a diagram like f in the case where the two different sets of coordinates represent measurements of time and space made by two different observers, each in motion relative to the other. Galileo and Newton thought they knew the answer to this question, but their answer turned out to be only approximately right. To avoid repeating the same mistakes, we need to clearly spell out what we think are the basic properties of time and space that will be a reliable foundation for our reasoning. I want to emphasize that there is no purely logical way of deciding on this list of properties. The ones I'll list are simply a summary of the patterns observed in the results from a large body of experiments. Furthermore, some of them are only approximate. For example, property 1 below is only a good approximation when the gravitational field is weak, so it is a property that applies to special relativity, not to general relativity.

Experiments show that:

1. No point in time or space has properties that make it different from any other point.
2. Likewise, all directions in space have the same properties.
3. Motion is relative, i.e., all inertial frames of reference are equally valid.
4. Causality holds, in the sense described on page 599.
5. Time depends on the state of motion of the observer.

Most of these are not very subversive. Properties 1 and 2 date back to the time when Galileo and Newton started applying the

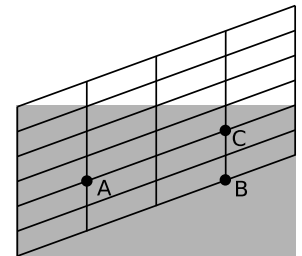
same universal laws of motion to the solar system and to the earth; this contradicted Aristotle, who believed that, for example, a rock would naturally want to move in a certain special direction (down) in order to reach a certain special location (the earth's surface). Property 3 is the reason that Einstein called his theory "relativity," but Galileo and Newton believed exactly the same thing to be true, as dramatized by Galileo's run-in with the Church over the question of whether the earth could really be in motion around the sun. Property 4 would probably surprise most people only because it asserts in such a weak and specialized way something that they feel deeply must be true. The only really strange item on the list is 5, but the Hafele-Keating experiment forces it upon us.

If it were not for property 5, we could imagine that figure g would give the correct transformation between frames of reference in motion relative to one another. Let's say that observer 1, whose grid coincides with the gray rectangle, is a hitch-hiker standing by the side of a road. Event A is a raindrop hitting his head, and event B is another raindrop hitting his head. He says that A and B occur at the same location in space. Observer 2 is a motorist who drives by without stopping; to him, the passenger compartment of his car is at rest, while the asphalt slides by underneath. He says that A and B occur at different points in space, because during the time between the first raindrop and the second, the hitch-hiker has moved backward. On the other hand, observer 2 says that events A and C occur in the same place, while the hitch-hiker disagrees. The slope of the grid-lines is simply the velocity of the relative motion of each observer relative to the other.

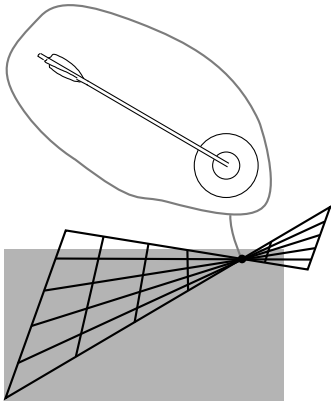
Figure g has familiar, comforting, and eminently sensible behavior, but it also happens to be wrong, because it violates property 5. The distortion of the coordinate grid has only moved the vertical lines up and down, so both observers agree that events like B and C are simultaneous. If this was really the way things worked, then all observers could synchronize all their clocks with one another for once and for all, and the clocks would never get out of sync. This contradicts the results of the Hafele-Keating experiment, in which all three clocks were initially synchronized in Washington, but later went out of sync because of their different states of motion.

It might seem as though we still had a huge amount of wiggle room available for the correct form of the distortion. It turns out, however, that properties 1-5 are sufficient to prove that there is only one answer, which is the one found by Einstein in 1905. To see why this is, let's work by a process of elimination.

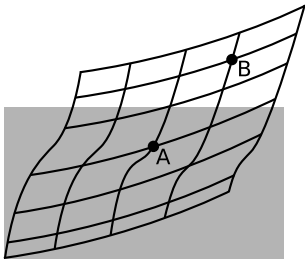
Figure h shows a transformation that might seem at first glance to be as good a candidate as any other, but it violates property 3, that



g / A Galilean version of the relationship between two frames of reference. As in all such graphs in this chapter, the original coordinates, represented by the gray rectangle, have a time axis that goes to the right, and a position axis that goes straight up.



h / A transformation that leads to disagreements about whether two events occur at the same time and place. This is not just a matter of opinion. Either the arrow hit the bull's-eye or it didn't.



i / A nonlinear transformation.

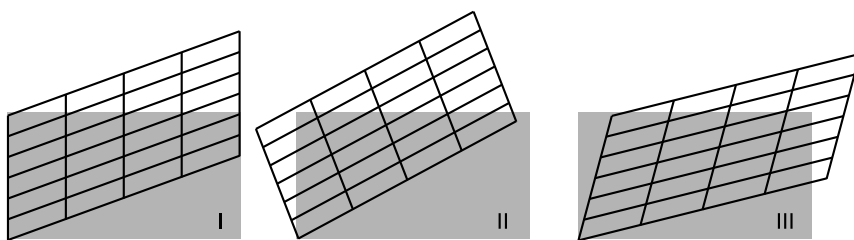
motion is relative, for the following reason. In observer 2's frame of reference, some of the grid lines cross one another. This means that observers 1 and 2 disagree on whether or not certain events are the same. For instance, suppose that event A marks the arrival of an arrow at the bull's-eye of a target, and event B is the location and time when the bull's-eye is punctured. Events A and B occur at the same location and at the same time. If one observer says that A and B coincide, but another says that they don't, we have a direct contradiction. Since the two frames of reference in figure h give contradictory results, one of them is right and one is wrong. This violates property 3, because all inertial frames of reference are supposed to be equally valid. To avoid problems like this, we clearly need to make sure that none of the grid lines ever cross one another.

The next type of transformation we want to kill off is shown in figure i, in which the grid lines curve, but never cross one another. The trouble with this one is that it violates property 1, the uniformity of time and space. The transformation is unusually "twisty" at A, whereas at B it's much more smooth. This can't be correct, because the transformation is only supposed to depend on the relative state of motion of the two frames of reference, and that given information doesn't single out a special role for any particular point in spacetime. If, for example, we had one frame of reference *rotating* relative to the other, then there would be something special about the axis of rotation. But we're only talking about *inertial* frames of reference here, as specified in property 3, so we can't have rotation; each frame of reference has to be moving in a straight line at constant speed. For frames related in this way, there is nothing that could single out an event like A for special treatment compared to B, so transformation i violates property 1.

The examples in figures h and i show that the transformation we're looking for must be linear, meaning that it must transform lines into lines, and furthermore that it has to take parallel lines to parallel lines. Einstein wrote in his 1905 paper that "...on account of the property of homogeneity [property 1] which we ascribe to time and space, the [transformation] must be linear."<sup>1</sup> Applying this to our diagrams, the original gray rectangle, which is a special type of parallelogram containing right angles, must be transformed into another parallelogram. There are three types of transformations, figure j, that have this property. Case I is the Galilean transformation of figure g on page 635, which we've already ruled out.

Case II can also be discarded. Here every point on the grid rotates

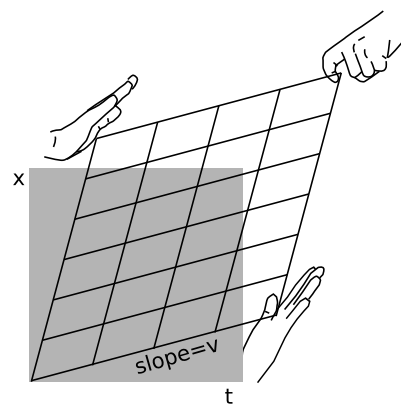
<sup>1</sup>A. Einstein, "On the Electrodynamics of Moving Bodies," *Annalen der Physik* 17 (1905), p. 891, tr. Saha and Bose, 1920



j / Three types of transformations that preserve parallelism. Their distinguishing feature is what they do to simultaneity, as shown by what happens to the left edge of the original rectangle. In I, the left edge remains vertical, so simultaneous events remain simultaneous. In II, the left edge turns counterclockwise. In III, it turns clockwise.

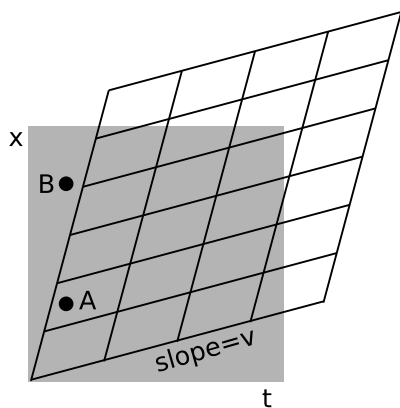
counterclockwise. What physical parameter would determine the amount of rotation? The only thing that could be relevant would be  $v$ , the relative velocity of the motion of the two frames of reference with respect to one another. But if the angle of rotation was proportional to  $v$ , then for large enough velocities the grid would have left and right reversed, and this would violate property 4, causality: one observer would say that event A caused a later event B, but another observer would say that B came first and caused A.

The only remaining possibility is case III, which I've redrawn in figure k with a couple of changes. This is the one that Einstein predicted in 1905. The transformation is known as the Lorentz transformation, after Hendrik Lorentz (1853-1928), who partially anticipated Einstein's work, without arriving at the correct interpretation. The distortion is a kind of smooshing and stretching, as suggested by the hands. Also, we've already seen in figures d-f on page 634 that we're free to stretch or compress everything as much as we like in the horizontal and vertical directions, because this simply corresponds to changing the units of measurement for time and distance. In figure k I've chosen units that give the whole drawing a convenient symmetry about a 45-degree diagonal line. Ordinarily it wouldn't make sense to talk about a 45-degree angle on a graph whose axes had different units. But in relativity, the symmetric appearance of the transformation tells us that space and time ought to be treated on the same footing, and measured in the same units.



k / In the units that are most convenient for relativity, the transformation has symmetry about a 45-degree diagonal line.

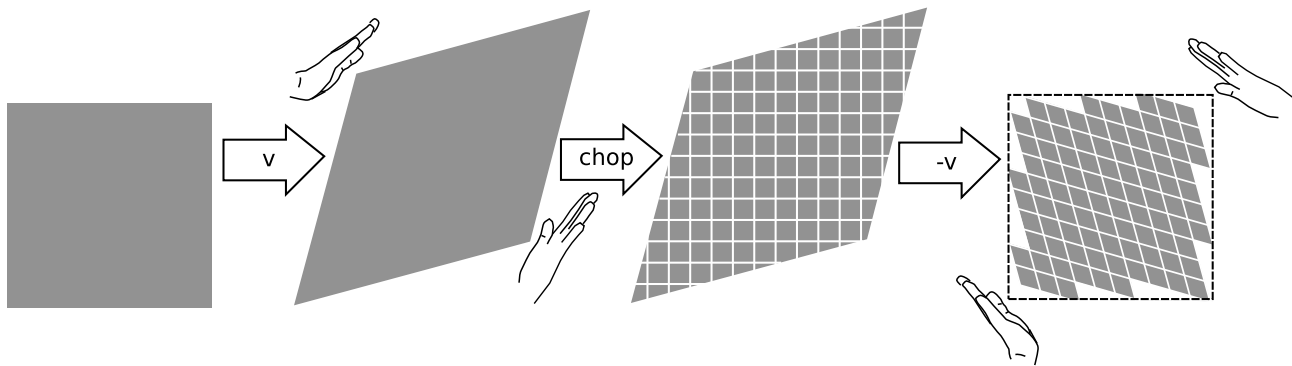




l / Interpretation of the Lorentz transformation. The slope indicated in the figure gives the relative velocity of the two frames of reference. Events A and B that were simultaneous in frame 1 are not simultaneous in frame 2, where event A occurs to the right of the  $t = 0$  line represented by the left edge of the grid, but event B occurs to its left.

As in our discussion of the Galilean transformation, slopes are interpreted as velocities, and the slope of the near-horizontal lines in figure l is interpreted as the relative velocity of the two observers. The difference between the Galilean version and the relativistic one is that now there is smooching happening from the other side as well. Lines that were vertical in the original grid, representing simultaneous events, now slant over to the right. This tells us that, as required by property 5, different observers do not agree on whether events that occur in different places are simultaneous. The Hafele-Keating experiment tells us that this non-simultaneity effect is fairly small, even when the velocity is as big as that of a passenger jet, and this is what we would have anticipated by the correspondence principle. The way that this is expressed in the graph is that if we pick the time unit to be the second, then the distance unit turns out to be hundreds of thousands of miles. In these units, the velocity of a passenger jet is an extremely small number, so the slope  $v$  in a figure like l is extremely small, and the amount of distortion is tiny — it would be much too small to see on this scale.

The only thing left to determine about the Lorentz transformation is the size of the transformed parallelogram relative to the size of the original one. Although the drawing of the hands in figure k may suggest that the grid deforms like a framework made of rigid coat-hanger wire, that is not the case. If you look carefully at the figure, you'll see that the edges of the smooshed parallelogram are actually a little longer than the edges of the original rectangle. In fact what stays the same is not lengths but *areas*, as proved in the caption to figure m.



m / Proof that Lorentz transformations don't change area: We first subject a square to a transformation with velocity  $v$ , and this increases its area by a factor  $R(v)$ , which we want to prove equals 1. We chop the resulting parallelogram up into little squares and finally apply a  $-v$  transformation; this changes each little square's area by a factor  $R(-v)$ , so the whole figure's area is also scaled by  $R(-v)$ . The final result is to restore the square to its original shape and area, so  $R(v)R(-v) = 1$ . But  $R(v) = R(-v)$  by property 2 of spacetime on page 634, which states that all directions in space have the same properties, so  $R(v) = 1$ .

## The $\gamma$ factor

Figure 1 showed us that observers in different frames disagree on whether different events are simultaneous. This is an indication that time is not absolute, so we shouldn't be surprised that time's rate of flow is also different for different observers. We use the symbol  $\gamma$  (Greek letter gamma) defined in the figure n to measure this unequal rate of flow. With a little algebra and geometry (homework problem 2, page 651), one can use the equal-area property to show that this ratio is given by

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \quad .$$

If you've had good training in physics, the first thing you probably think when you look at this equation is that it must be nonsense, because its units don't make sense. How can we take something with units of velocity squared, and subtract it from a unitless 1? But remember that this is expressed in our new relativistic units, in which the same units are used for distance and time. In this system, velocities are always unitless. This sort of thing happens frequently in physics. For instance, before James Joule discovered conservation of energy, nobody knew that heat and mechanical energy were different forms of the same thing, so instead of measuring them both in units of joules as we would do now, they measured heat in one unit (such as calories) and mechanical energy in another (such as foot-pounds). In ordinary metric units, we just need an extra conversion factor, called  $c$ , and the equation becomes

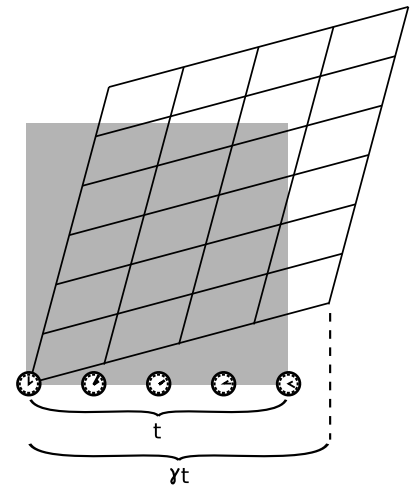
$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad .$$

When we say, "It's five hours from LA to Vegas," we're using a unit of time as a unit of distance. This works because there is a standard speed implied: the speed of a car on the freeway. Similarly, the conversion factor  $c$  can be interpreted as a speed, so that  $v/c$  is the unitless ratio of two speeds. As argued on p. 599, cause and effect can never be propagated instantaneously;  $c$  turns out to be the specific numerical speed limit on cause and effect. In particular, we'll see in section 24.3 that light travels at  $c$ , which has a numerical value of  $3.0 \times 10^8$  m/s.

Because  $\gamma$  is always greater than 1, we have the following interpretation:

### Time dilation

A clock runs fastest in the frame of reference of an observer who is at rest relative to the clock. An observer in motion relative to the clock at speed  $v$  perceives the clock as running more slowly by a factor of  $\gamma$ .



n / The clock is at rest in the original frame of reference, and it measures a time interval  $t$ . In the new frame of reference, the time interval is greater by a factor that we notate as  $\gamma$ .

Since the Lorentz transformation treats time and distance entirely symmetrically, we could just as well have defined  $\gamma$  using the upright  $x$  axis in figure n, and we therefore have a similar interpretation in terms of space:

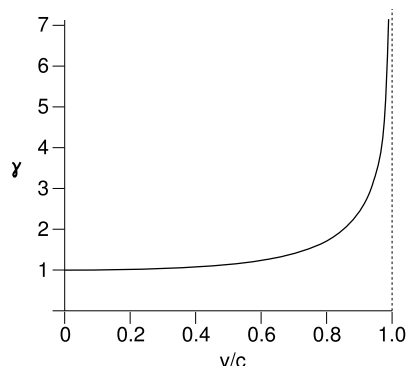
### Length contraction

A meter-stick appears longest to an observer who is at rest relative to it. An observer moving relative to the meter-stick at  $v$  observes the stick to be shortened by a factor of  $\gamma$ .

#### self-check A

What is  $\gamma$  when  $v = 0$ ? What does this mean?

▷ Answer, p. 978

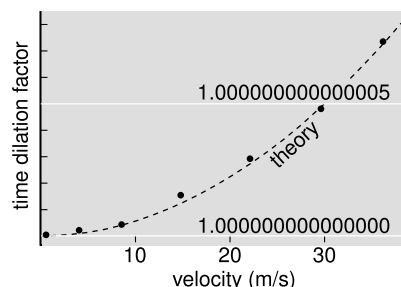


o / A graph of  $\gamma$  as a function of  $v$ .

#### The correspondence principle

#### example 1

The correspondence principle requires that  $\gamma$  be close to 1 for the velocities much less than  $c$  encountered in everyday life. Let's explicitly find the amount  $\epsilon$  by which  $\gamma$  differs from 1, when  $v$  is small. Let  $\gamma = 1 + \epsilon$ . The definition of  $\gamma$  gives  $1 = \gamma^2(1 - v^2/c^2)$ , so  $1 = (1 + 2\epsilon + \epsilon^2)(1 - v^2/c^2) \approx 1 + 2\epsilon - v^2/c^2$ , where the approximation comes from discarding very small terms such as  $\epsilon^2$  and  $\epsilon v^2/c^2$ . We find  $\epsilon = v^2/2c^2$ . As expected, this will be small when  $v$  is small compared to  $c$ .



p / Time dilation measured with an atomic clock at low speeds. The theoretical curve, shown with a dashed line, is calculated from  $\gamma = 1/\sqrt{1 - (v/c)^2}$ ; at these small velocities, the approximation of example 1 is an excellent one, so  $\gamma \approx 1 + v^2/2c^2$ , and the graph is indistinguishable from a parabola. This graph corresponds to an extreme close-up view of the lower left corner of figure o. The error bars on the experimental points are about the same size as the dots.

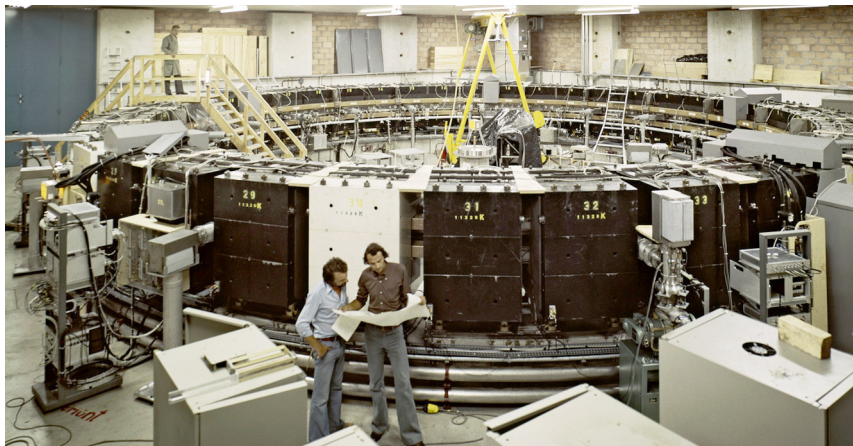
Figure o shows that the approximation found in example 1 *not* valid for large values of  $v/c$ . In fact,  $\gamma$  blows up to infinity as  $v$  gets closer and closer to  $c$ .

#### A moving atomic clock

#### example 2

Example 1 shows that when  $v$  is small, relativistic effects are approximately proportional to  $v^2$ , so it is very difficult to observe them at low speeds. For example, a car on the freeway travels at about 1/10 the speed of a passenger jet, so the resulting time dilation is only 1/100 as much. For this reason, it was not until four decades after Hafele and Keating that anyone did a conceptually simple atomic clock experiment in which the only effect was motion, not gravity; it is difficult to move a clock at a high enough velocity without putting it in some kind of aircraft, which then has to fly at some altitude. In 2010, however, Chou *et al.*<sup>2</sup> succeeded in building an atomic clock accurate enough to detect time dilation at speeds as low as 10 m/s. Figure p shows their results. Since it was not practical to move the entire clock, the experimenters only moved the aluminum atoms inside the clock that actually made it “tick.”

<sup>2</sup>Science 329 (2010) 1630



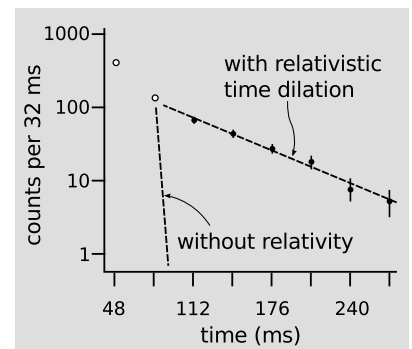
q / Apparatus used for the test of relativistic time dilation described in example 3. The prominent black and white blocks are large magnets surrounding a circular pipe with a vacuum inside. (c) 1974 by CERN.

### Large time dilation

### example 3

The time dilation effect in the Hafele-Keating experiment was very small. If we want to see a large time dilation effect, we can't do it with something the size of the atomic clocks they used; the kinetic energy would be greater than the total megatonnage of all the world's nuclear arsenals. We can, however, accelerate subatomic particles to speeds at which  $\gamma$  is large. For experimental particle physicists, relativity is something you do all day before heading home and stopping off at the store for milk. An early, low-precision experiment of this kind was performed by Rossi and Hall in 1941, using naturally occurring cosmic rays. Figure q shows a 1974 experiment<sup>3</sup> of a similar type which verified the time dilation predicted by relativity to a precision of about one part per thousand.

Particles called muons (named after the Greek letter  $\mu$ , "myoo") were produced by an accelerator at CERN, near Geneva. A muon is essentially a heavier version of the electron. Muons undergo radioactive decay, lasting an average of only  $2.197 \mu\text{s}$  before they evaporate into an electron and two neutrinos. The 1974 experiment was actually built in order to measure the magnetic properties of muons, but it produced a high-precision test of time dilation as a byproduct. Because muons have the same electric charge as electrons, they can be trapped using magnetic fields. Muons were injected into the ring shown in figure q, circling around it until they underwent radioactive decay. At the speed at which these muons were traveling, they had  $\gamma = 29.33$ , so on the average they lasted 29.33 times longer than the normal lifetime. In other words, they were like tiny alarm clocks that self-destructed at a randomly selected time. Figure r shows the number of radioactive decays counted, as a function of the time elapsed after a given stream of muons was injected into the storage ring. The two dashed lines



r / Example 3: Muons accelerated to nearly  $c$  undergo radioactive decay much more slowly than they would according to an observer at rest with respect to the muons. The first two data-points (unfilled circles) were subject to large systematic errors.

<sup>3</sup>Bailey et al., Nucl. Phys. B150(1979) 1

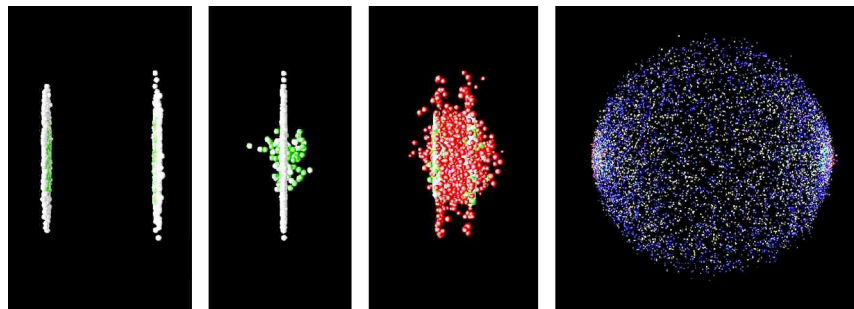
show the rates of decay predicted with and without relativity. The relativistic line is the one that agrees with experiment.

#### *An example of length contraction*

*example 4*

Figure s shows an artist's rendering of the length contraction for the collision of two gold nuclei at relativistic speeds in the RHIC accelerator in Long Island, New York, which went on line in 2000. The gold nuclei would appear nearly spherical (or just slightly lengthened like an American football) in frames moving along with them, but in the laboratory's frame, they both appear drastically foreshortened as they approach the point of collision. The later pictures show the nuclei merging to form a hot soup, in which experimenters hope to observe a new form of matter.

Example 4: Colliding nuclei show relativistic length contraction.

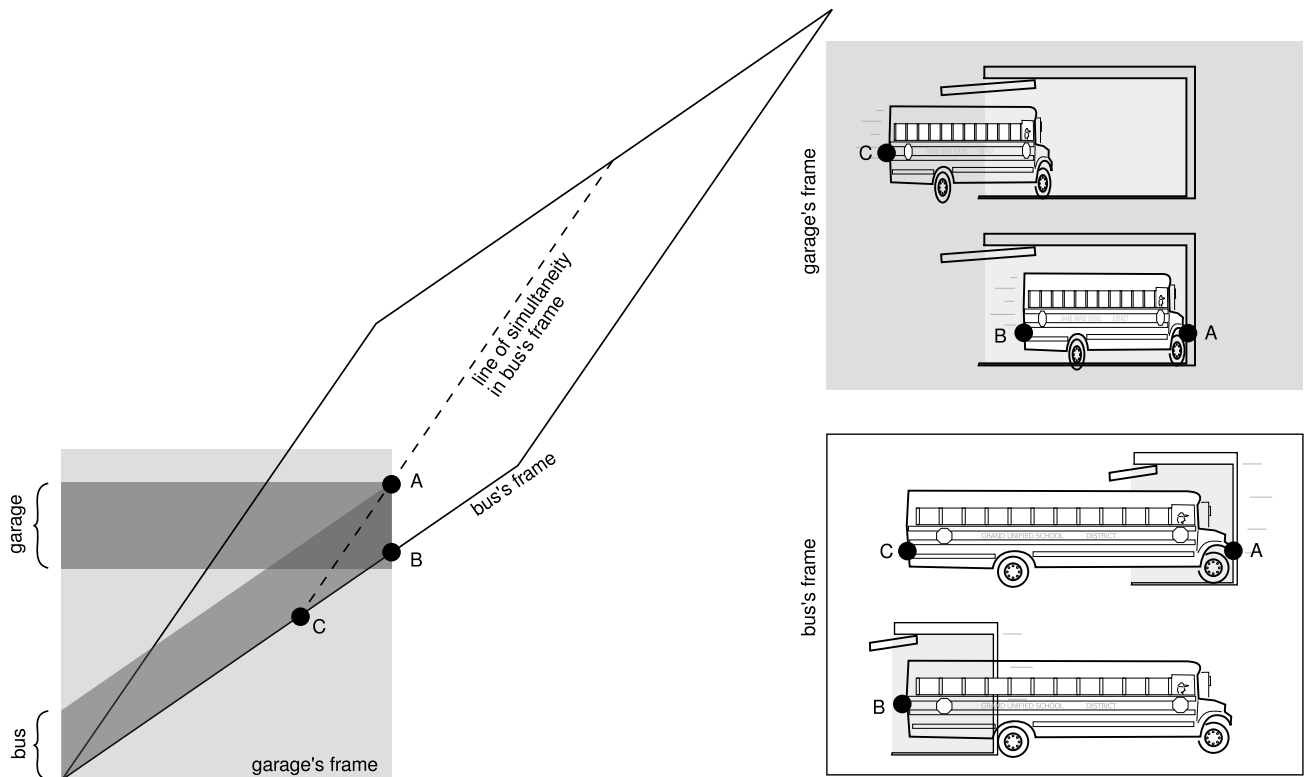


#### *The garage paradox*

*example 5*

One of the most famous of all the so-called relativity paradoxes has to do with our incorrect feeling that simultaneity is well defined. The idea is that one could take a schoolbus and drive it at relativistic speeds into a garage of ordinary size, in which it normally would not fit. Because of the length contraction, the bus would supposedly fit in the garage. The driver, however, will perceive the *garage* as being contracted and thus even less able to contain the bus.

The paradox is resolved when we recognize that the concept of fitting the bus in the garage “all at once” contains a hidden assumption, the assumption that it makes sense to ask whether the front and back of the bus can *simultaneously* be in the garage. Observers in different frames of reference moving at high relative speeds do not necessarily agree on whether things happen simultaneously. As shown in figure t, the person in the garage's frame can shut the door at an instant B he perceives to be simultaneous with the front bumper's arrival A at the back wall of the garage, but the driver would not agree about the simultaneity of these two events, and would perceive the door as having shut long after she plowed through the back wall.

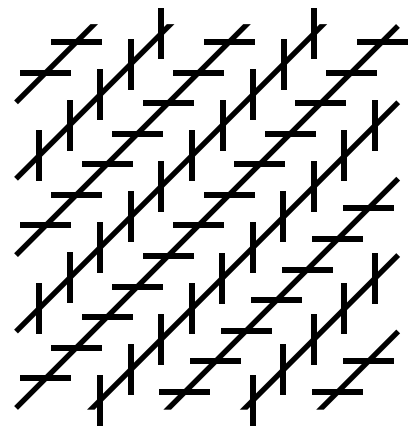


t / Example 5: In the garage's frame of reference, the bus is moving, and can fit in the garage due to its length contraction. In the bus's frame of reference, the garage is moving, and can't hold the bus due to *its* length contraction.

### Discussion questions

**A** A person in a spaceship moving at 99.99999999% of the speed of light relative to Earth shines a flashlight forward through dusty air, so the beam is visible. What does she see? What would it look like to an observer on Earth?

**B** A question that students often struggle with is whether time and space can really be distorted, or whether it just seems that way. Compare with optical illusions or magic tricks. How could you verify, for instance, that the lines in the figure are actually parallel? Are relativistic effects the same, or not?

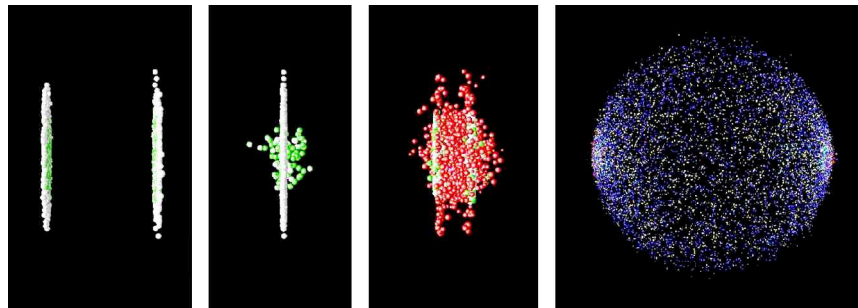


Discussion question B

**C** On a spaceship moving at relativistic speeds, would a lecture seem even longer and more boring than normal?

**D** Mechanical clocks can be affected by motion. For example, it was a significant technological achievement to build a clock that could sail aboard a ship and still keep accurate time, allowing longitude to be determined. How is this similar to or different from relativistic time dilation?

**E** Figure 23.10 from page 642, depicting the collision of two nuclei at the RHIC accelerator, is reproduced below. What would the shapes of the two nuclei look like to a microscopic observer riding on the left-hand nucleus? To an observer riding on the right-hand one? Can they agree on what is happening? If not, why not — after all, shouldn't they see the same thing if they both compare the two nuclei side-by-side at the same instant in time?



Discussion question E: colliding nuclei show relativistic length contraction.

**F** If you stick a piece of foam rubber out the window of your car while driving down the freeway, the wind may compress it a little. Does it make sense to interpret the relativistic length contraction as a type of strain that pushes an object's atoms together like this? How does this relate to discussion question E?

## 23.2 Magnetic interactions

Think not that I am come to destroy the law, or the prophets: I am not come to destroy, but to fulfill. *Matthew 5:17*

At this stage, you understand roughly as much about the classification of interactions as physicists understood around the year 1800. There appear to be three fundamentally different types of interactions: gravitational, electrical, and magnetic. Many types of interactions that appear superficially to be distinct — stickiness, chemical interactions, the energy an archer stores in a bow — are really the same: they're manifestations of electrical interactions between atoms. Is there any way to shorten the list any further? The prospects seem dim at first. For instance, we find that if we rub a piece of fur on a rubber rod, the fur does not attract or repel a magnet. The fur has an electric field, and the magnet has a magnetic field. The two are completely separate, and don't seem to affect one another. Likewise we can test whether magnetizing a piece of iron changes its weight. The weight doesn't seem to change by any measurable amount, so magnetism and gravity seem to be unrelated.

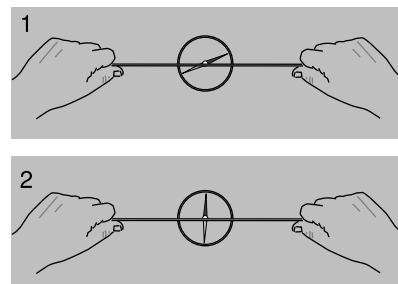
That was where things stood until 1820, when the Danish physicist Hans Christian Oersted was delivering a lecture at the University of Copenhagen, and he wanted to give his students a demonstration that would illustrate the cutting edge of research. He generated a current in a wire by making a short circuit across a battery, and held the wire near a magnetic compass. The idea was to give an example of how one could search for a previously undiscovered link between electricity (the electric current in the wire) and magnetism. One never knows how much to believe from these dramatic legends, but the story is<sup>4</sup> that the experiment he'd expected to turn out negative instead turned out positive: when he held the wire near the compass, the current in the wire caused the compass to twist!

People had tried similar experiments before, but only with static electricity, not with a moving electric current. For instance, they had hung batteries so that they were free to rotate in the earth's magnetic field, and found no effect; since the battery was not connected to a complete circuit, there was no current flowing. With Oersted's own setup, u, the effect was only produced when the "circuit was closed, but not when open, as certain very celebrated physicists in vain attempted several years ago."<sup>5</sup>

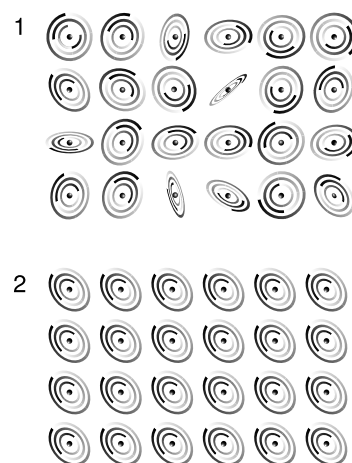
Oersted was eventually led to the conclusion that magnetism was an interaction between moving charges and other moving charges, i.e., between one current and another. A permanent magnet, he in-

<sup>4</sup>Oersted's paper describing the phenomenon says that "The first experiments on the subject ... were set on foot in the classes for electricity, galvanism, and magnetism, which were held by me in the winter just past," but that doesn't tell us whether the result was really a surprise that occurred in front of his students.

<sup>5</sup>All quotes are from the 1876 translation are by J.E. Kempe.

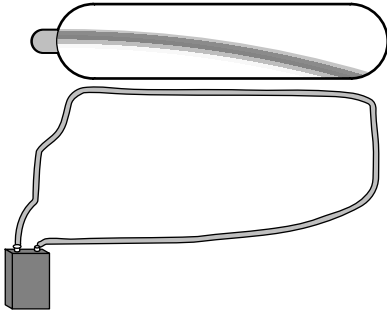


u / 1. When the circuit is incomplete, no current flows through the wire, and the magnet is unaffected. It points in the direction of the Earth's magnetic field. 2. The circuit is completed, and current flows through the wire. The wire has a strong effect on the magnet, which turns almost perpendicular to it. If the earth's field could be removed entirely, the compass would point exactly perpendicular to the wire; this is the direction of the wire's field.



v / A schematic representation of an unmagnetized material, 1, and a magnetized one, 2.





w / Magnetism is an interaction between moving charges and moving charges. The moving charges in the wire attract the moving charges in the beam of charged particles in the vacuum tube.

ferred, contained currents on a microscopic scale that simply weren't practical to measure with an ammeter. Today this seems natural to us, since we're accustomed to picturing an atom as a tiny solar system, with the electrons whizzing around the nucleus in circles. As shown in figure v, a magnetized piece of iron is different from an unmagnetized piece because the atoms in the unmagnetized piece are jumbled in random orientations, whereas the atoms in the magnetized piece are at least partially organized to face in a certain direction.

Figure w shows an example that is conceptually simple, but not very practical. If you try this with a typical vacuum tube, like a TV or computer monitor, the current in the wire probably won't be enough to produce a visible effect. A more practical method is to hold a magnet near the screen. We still have an interaction between moving charges and moving charges, but the swirling electrons in the atoms in the magnet are now playing the role played by the moving charges in the wire in figure w. Warning: if you do this, make sure your monitor has a demagnetizing button! If not, then your monitor may be permanently ruined.

### Relativity requires magnetism

So magnetism is an interaction between moving charges and moving charges. But how can that be? Relativity tells us that motion is a matter of opinion. Consider figure x. In this figure and in figure y, the dark and light coloring of the particles represents the fact that one particle has positive charge and the other negative. Observer  $x/2$  sees the two particles as flying through space side by side, so they would interact both electrically (simply because they're charged) and magnetically (because they're charges in motion). But an observer moving along with them,  $x/1$ , would say they were both at rest, and would expect only an electrical interaction. This seems like a paradox. Magnetism, however, comes not to destroy relativity but to fulfill it. Magnetic interactions *must* exist according to the theory of relativity. To understand how this can be, consider how time and space behave in relativity. Observers in different frames of reference disagree about the lengths of measuring sticks and the speeds of clocks, but the laws of physics are valid and self-consistent in either frame of reference. Similarly, observers in different frames of reference disagree about what electric and magnetic fields there are, but they agree about concrete physical events. An observer in frame of reference  $x/1$  says there are electric fields around the particles, and predicts that as time goes on, the particles will begin to accelerate towards one another, eventually colliding. She explains the collision as being due to the electrical attraction between the particles. A different observer,  $x/2$ , says the particles are moving. This observer also predicts that the particles will collide, but explains their motion in terms of both an electric field and a magnetic field. As we'll see shortly, the magnetic field is *required* in order

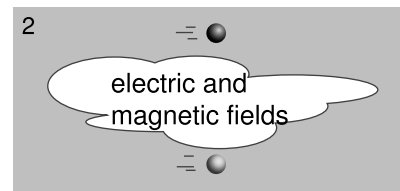
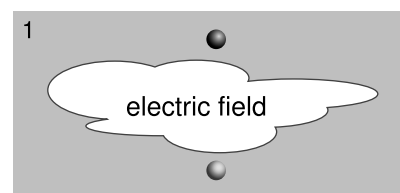
to maintain consistency between the predictions made in the two frames of reference.

To see how this really works out, we need to find a nice simple example. An example like figure x is *not* easy to handle, because in the second frame of reference, the moving charges create fields that change over time at any given location, like when the V-shaped wake of a speedboat washes over a buoy. Examples like figure w are easier, because there is a steady flow of charges, and all the fields stay the same over time. Figure y/1 shows a simplified and idealized model of figure w. The charge by itself is like one of the charged particles in the vacuum tube beam of figure w, and instead of the wire, we have two long lines of charges moving in opposite directions. Note that, as discussed in discussion question C on page 555, the currents of the two lines of charges do not cancel out. The dark and light balls represent particles with opposite charges. Because of this, the total current in the “wire” is double what it would be if we took away one line.

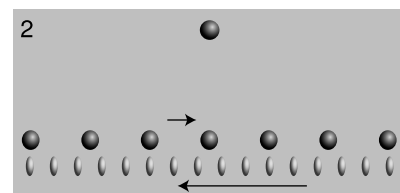
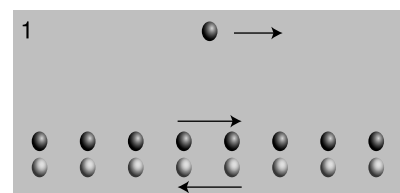
As a model of figure w, figure y/1 is partly realistic and partly unrealistic. In a real piece of copper wire, there are indeed charged particles of both types, but it turns out that the particles of one type (the protons) are locked in place, while only some of the other type (the electrons) are free to move. The model also shows the particles moving in a simple and orderly way, like cars on a two-lane road, whereas in reality most of the particles are organized into copper atoms, and there is also a great deal of random thermal motion. The model’s unrealistic features aren’t a problem, because the point of this exercise is only to find one particular situation that shows magnetic effects must exist based on relativity.

What electrical force does the lone particle in figure y/1 feel? Since the density of “traffic” on the two sides of the “road” is equal, there is zero overall electrical force on the lone particle. Each “car” that attracts the lone particle is paired with a partner on the other side of the road that repels it. If we didn’t know about magnetism, we’d think this was the whole story: the lone particle feels no force at all from the wire.

Figure y/2 shows what we’d see if we were observing all this from a frame of reference moving along with the lone charge. Here’s where the relativity comes in. Relativity tells us that moving objects appear contracted to an observer who is not moving along with them. Both lines of charge are in motion in both frames of reference, but in frame 1 they were moving at equal speeds, so their contractions were equal. In frame 2, however, their speeds are unequal. The dark charges are moving more slowly than in frame 1, so in frame 2 they are less contracted. The light-colored charges are moving more quickly, so their contraction is greater now. The “cars” on the two sides of the “road” are no longer paired off, so the electrical forces



x / One observer sees an electric field, while the other sees both an electric field and a magnetic one.



y / A model of a charged particle and a current-carrying wire, seen in two different frames of reference. The relativistic length contraction is highly exaggerated. The force on the lone particle is purely magnetic in 1, and purely electric in 2.

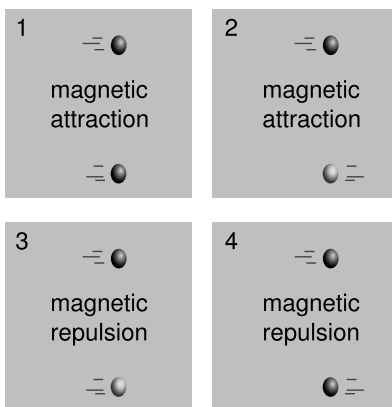
on the lone particle no longer cancel out as they did in  $y/1$ . The lone particle is attracted to the wire, because the particles attracting it are more dense than the ones repelling it. Furthermore, the attraction felt by the lone charge must be purely electrical, since the lone charge is at rest in this frame of reference, and magnetic effects occur only between moving charges and other moving charges.

Now observers in frames 1 and 2 disagree about many things, but they do agree on concrete events. Observer 2 is going to see the lone particle drift toward the wire due to the wire's electrical attraction, gradually speeding up, and eventually hit the wire. If 2 sees this collision, then 1 must as well. But 1 knows that the total electrical force on the lone particle is exactly zero. There must be some new type of force. She invents a name for this new type of force: magnetism. This was a particularly simple example, because the force was purely magnetic in one frame of reference, and purely electrical in another. In general, an observer in a certain frame of reference will measure a mixture of electric and magnetic fields, while an observer in another frame, in motion with respect to the first, says that the same volume of space contains a different mixture.

We therefore arrive at the conclusion that electric and magnetic phenomena aren't separate. They're different sides of the same coin. We refer to electric and magnetic interactions collectively as electromagnetic interactions. Our list of the fundamental interactions of nature now has two items on it instead of three: gravity and electromagnetism.

Oersted found that magnetism was an interaction between moving charges and other moving charges. We can see this in the situation described in figure  $y/1$ , in which the result of the argument depended on the fact that both the lone charge and the charges in the wire were moving. To see this in a different way, we can apply the result of example 1 on p. 640, that for small velocities the  $\gamma$  factor differs from 1 by about  $v^2/2c^2$ . Let the lone charge in figure  $y/1$  have velocity  $u$ , the ones in the wire  $\pm v$ . As we'll see on p. 660, velocities in relative motion don't exactly add and subtract relativistically, but as long as we assume that  $u$  and  $v$  are small, the correspondence principle guarantees that they will approximately add and subtract. Then the velocities in the lone charge's rest frame,  $y/2$ , are approximately 0,  $v - u$ , and  $-v - u$ . The nonzero charge density of the wire in frame  $y/2$  is then proportional to the difference in the length contractions  $\gamma_{-v-u} - \gamma_{v-u} \approx 2uv/c^2$ . This depends on the product of the velocities  $u$  and  $v$ , which is as expected if magnetism is an interaction of moving charges with moving charges.

The basic rules for magnetic attractions and repulsions, shown in figure  $z$ , aren't quite as simple as the ones for gravity and electricity. Rules  $z/1$  and  $z/2$  follow directly from our previous analysis of figure  $y$ . Rules 3 and 4 are obtained by flipping the type of charge that the



$z$  / Magnetic interactions involving only two particles at a time. In these figures, unlike figure  $y/1$ , there are electrical forces as well as magnetic ones. The electrical forces are not shown here. Don't memorize these rules!

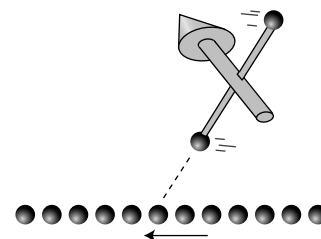
bottom particle has. For instance, rule 3 is like rule 1, except that the bottom charge is now the opposite type. This turns the attraction into a repulsion. (We know that flipping the charge reverses the interaction, because that's the way it works for electric forces, and magnetic forces are just electric forces viewed in a different frame of reference.)

*A magnetic weathervane placed near a current.* example 6

Figure 23.2.1 shows a magnetic weathervane, consisting of two charges that spin in circles around the axis of the arrow. (The magnetic field doesn't cause them to spin; a motor is needed to get them to spin in the first place.) Just like the magnetic compass in figure u, the weathervane's arrow tends to align itself in the direction perpendicular to the wire. This is its preferred orientation because the charge close to the wire is attracted to the wire, while the charge far from the wire is repelled by it.

### Discussion question

**A** Resolve the following paradox concerning the argument given in this section. We would expect that at any given time, electrons in a solid would be associated with protons in a definite way. For simplicity, let's imagine that the solid is made out of hydrogen (which actually does become a metal under conditions of very high pressure). A hydrogen atom consists of a single proton and a single electron. Even if the electrons are moving and forming an electric current, we would imagine that this would be like a game of musical chairs, with the protons as chairs and the electrons as people. Each electron has a proton that is its "friend," at least for the moment. This is the situation shown in figure y/1. How, then, can an observer in a different frame see the electrons and protons as not being paired up, as in y/2?



Example 6

# Summary

## Notation

$\gamma$  . . . . . an abbreviation for  $1/\sqrt{1 - v^2/c^2}$

## Summary

Experiments show that space and time do not have the properties claimed by Galileo and Newton. Time and space as seen by one observer are distorted compared to another observer's perceptions if they are moving relative to each other. This distortion is quantified by the factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ,$$

where  $v$  is the relative velocity of the two observers, and  $c$  is a universal velocity that is the same in all frames of reference. Light travels at  $c$ . A clock appears to run fastest to an observer who is not in motion relative to it, and appears to run too slowly by a factor of  $\gamma$  to an observer who has a velocity  $v$  relative to the clock. Similarly, a meter-stick appears longest to an observer who sees it at rest, and appears shorter to other observers. Time and space are relative, not absolute.

As a consequence of relativity, we must have not just electrical interactions of charges with charges, but also an additional magnetic interaction of moving charges with other moving charges.

## Exploring further

**Relativity Simply Explained**, Martin Gardner. A beautifully clear, nonmathematical introduction to the subject, with entertaining illustrations. A postscript, written in 1996, follows up on recent developments in some of the more speculative ideas from the 1967 edition.

**Was Einstein Right? — Putting General Relativity to the Test**, Clifford M. Will. This book makes it clear that general relativity is neither a fantasy nor holy scripture, but a scientific theory like any other.

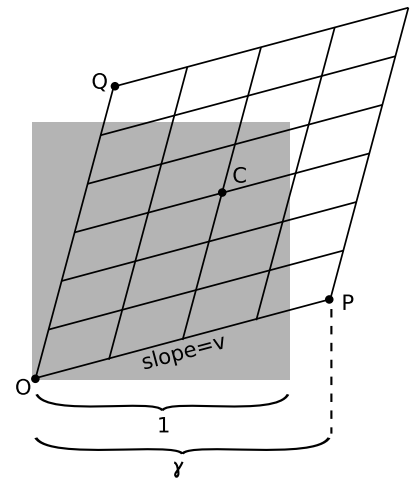
## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** What happens in the equation for  $\gamma$  when you put in a negative number for  $v$ ? Explain what this means physically, and why it makes sense.

**2** In this homework problem, you'll fill in the steps of the algebra required in order to find the equation for  $\gamma$  on page 639. To keep the algebra simple, let the time  $t$  in figure n equal 1, as suggested in the figure accompanying this homework problem. The original square then has an area of 1, and the transformed parallelogram must also have an area of 1. (a) Prove that point P is at  $x = v\gamma$ , so that its  $(t, x)$  coordinates are  $(\gamma, v\gamma)$ . (b) Find the  $(t, x)$  coordinates of point Q. (c) Find the length of the short diagonal connecting P and Q. (d) Average the coordinates of P and Q to find the coordinates of the midpoint C of the parallelogram, and then find distance OC. (e) Find the area of the parallelogram by computing twice the area of triangle PQO. [Hint: You can take PQ to be the base of the triangle.] (f) Set this area equal to 1 and solve for  $\gamma$  to prove  $\gamma = 1/\sqrt{1-v^2}$ .



Problem 2.

**3** Astronauts in three different spaceships are communicating with each other. Those aboard ships A and B agree on the rate at which time is passing, but they disagree with the ones on ship C.

- (a) Alice is aboard ship A. How does she describe the motion of her own ship, in its frame of reference?
- (b) Describe the motion of the other two ships according to Alice.
- (c) Give the description according to Betty, whose frame of reference is ship B.
- (d) Do the same for Cathy, aboard ship C.

**4** The Voyager 1 space probe, launched in 1977, is moving faster relative to the earth than any other human-made object, at 17,000 meters per second.

- (a) Calculate the probe's  $\gamma$ .
- (b) Over the course of one year on earth, slightly less than one year passes on the probe. How much less? (There are 31 million seconds in a year.)

**5** The earth is orbiting the sun, and therefore is contracted relativistically in the direction of its motion. Compute the amount by which its diameter shrinks in this direction.

**6** Page 634 lists five observed properties of space and time that sufficed to derive relativity. There are other sets of postulates that could also have been used, and in fact this was not the set that was first used by Einstein. Suppose that our first inkling that spacetime isn't Galilean consists of an experiment like the one in figure w on p. 646, and that we interpret it, correctly, as being evidence of a relativistic length contraction as in figure y on p. 647. Besides this fact, what else is needed in order to get the full structure of relativity?

# Chapter 24

## Electromagnetism

### 24.1 The magnetic field

#### No magnetic monopoles

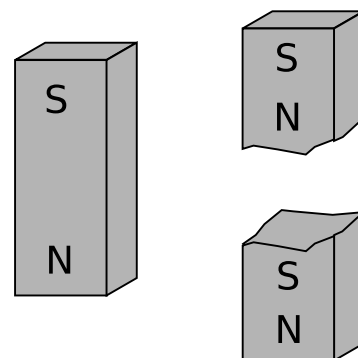
If you could play with a handful of electric dipoles and a handful of bar magnets, they would appear very similar. For instance, a pair of bar magnets wants to align themselves head-to-tail, and a pair of electric dipoles does the same thing. (It is unfortunately not that easy to make a permanent electric dipole that can be handled like this, since the charge tends to leak.)

You would eventually notice an important difference between the two types of objects, however. The electric dipoles can be broken apart to form isolated positive charges and negative charges. The two-ended device can be broken into parts that are not two-ended. But if you break a bar magnet in half, a, you will find that you have simply made two smaller two-ended objects.

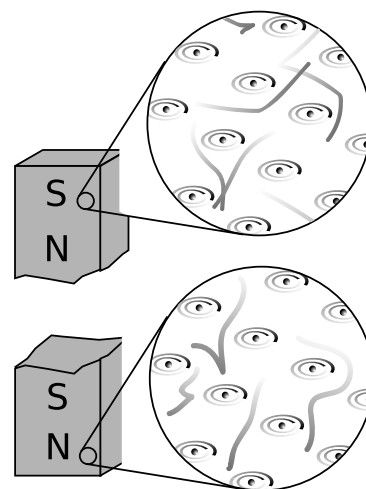
The reason for this behavior is not hard to divine from our microscopic picture of permanent iron magnets. An electric dipole has extra positive “stuff” concentrated in one end and extra negative in the other. The bar magnet, on the other hand, gets its magnetic properties not from an imbalance of magnetic “stuff” at the two ends but from the orientation of the rotation of its electrons. One end is the one from which we could look down the axis and see the electrons rotating clockwise, and the other is the one from which they would appear to go counterclockwise. There is no difference between the “stuff” in one end of the magnet and the other, b.

Nobody has ever succeeded in isolating a single magnetic pole. In technical language, we say that magnetic monopoles do not seem to exist. Electric monopoles *do* exist — that’s what charges are.

Electric and magnetic forces seem similar in many ways. Both act at a distance, both can be either attractive or repulsive, and both are intimately related to the property of matter called charge. (Recall that magnetism is an interaction between moving charges.) Physicists’s aesthetic senses have been offended for a long time because this seeming symmetry is broken by the existence of electric monopoles and the absence of magnetic ones. Perhaps some exotic form of matter exists, composed of particles that are magnetic monopoles. If such particles could be found in cosmic rays

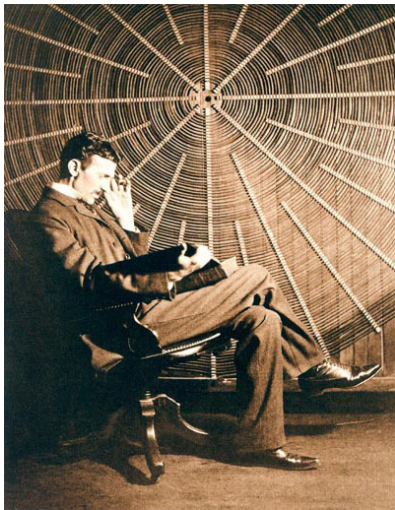


a / Breaking a bar magnet in half doesn’t create two monopoles, it creates two smaller dipoles.

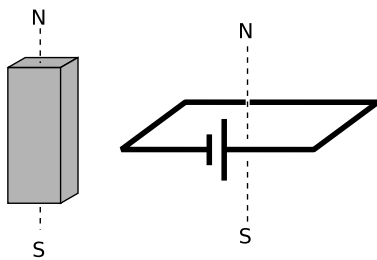


b / An explanation at the atomic level.

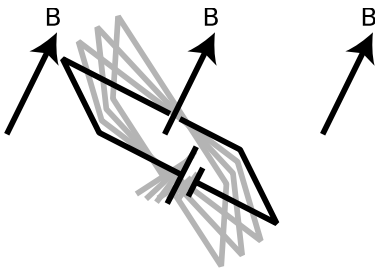




The unit of magnetic field, the tesla, is named after Serbian-American inventor Nikola Tesla.



A standard dipole made from a square loop of wire shorting across a battery. It acts very much like a bar magnet, but its strength is more easily quantified.



A dipole tends to align itself to the surrounding magnetic field.

or moon rocks, it would be evidence that the apparent asymmetry was only an asymmetry in the composition of the universe, not in the laws of physics. For these admittedly subjective reasons, there have been several searches for magnetic monopoles. Experiments have been performed, with negative results, to look for magnetic monopoles embedded in ordinary matter. Soviet physicists in the 1960s made exciting claims that they had created and detected magnetic monopoles in particle accelerators, but there was no success in attempts to reproduce the results there or at other accelerators. The most recent search for magnetic monopoles, done by reanalyzing data from the search for the top quark at Fermilab, turned up no candidates, which shows that either monopoles don't exist in nature or they are extremely massive and thus hard to create in accelerators.

### Definition of the magnetic field

Since magnetic monopoles don't seem to exist, it would not make much sense to define a magnetic field in terms of the force on a test monopole. Instead, we follow the philosophy of the alternative definition of the electric field, and define the field in terms of the torque on a magnetic test dipole. This is exactly what a magnetic compass does: the needle is a little iron magnet which acts like a magnetic dipole and shows us the direction of the earth's magnetic field.

To define the strength of a magnetic field, however, we need some way of defining the strength of a test dipole, i.e., we need a definition of the magnetic dipole moment. We could use an iron permanent magnet constructed according to certain specifications, but such an object is really an extremely complex system consisting of many iron atoms, only some of which are aligned. A more fundamental standard dipole is a square current loop. This could be little resistive circuit consisting of a square of wire shorting across a battery.

We will find that such a loop, when placed in a magnetic field, experiences a torque that tends to align plane so that its face points in a certain direction. (Since the loop is symmetric, it doesn't care if we rotate it like a wheel without changing the plane in which it lies.) It is this preferred facing direction that we will end up defining as the direction of the magnetic field.

Experiments show if the loop is out of alignment with the field, the torque on it is proportional to the amount of current, and also to the interior area of the loop. The proportionality to current makes sense, since magnetic forces are interactions between moving charges, and current is a measure of the motion of charge. The proportionality to the loop's area is also not hard to understand, because increasing the length of the sides of the square increases both the amount of charge contained in this circular "river" and

the amount of leverage supplied for making torque. Two separate physical reasons for a proportionality to length result in an overall proportionality to length squared, which is the same as the area of the loop. For these reasons, we define the magnetic dipole moment of a square current loop as

$$D_m = IA \quad , \quad \begin{array}{l} \text{[definition of the magnetic} \\ \text{dipole moment of a square current loop]} \end{array}$$

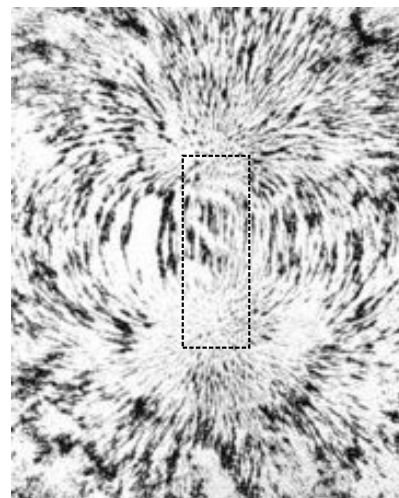
We now define the magnetic field in a manner entirely analogous to the second definition of the electric field:

#### definition of the magnetic field

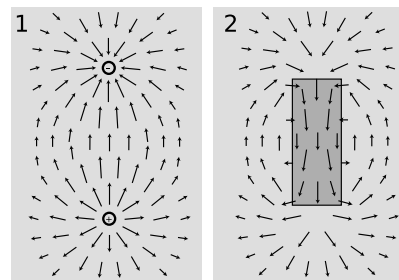
The magnetic field vector,  $\mathbf{B}$ , at any location in space is defined by observing the torque exerted on a magnetic test dipole  $D_{mt}$  consisting of a square current loop. The field's magnitude is  $|\mathbf{B}| = \tau/D_{mt} \sin \theta$ , where  $\theta$  is the angle by which the loop is misaligned. The direction of the field is perpendicular to the loop; of the two perpendiculars, we choose the one such that if we look along it, the loop's current is counterclockwise.

We find from this definition that the magnetic field has units of  $\text{N}\cdot\text{m}/\text{A}\cdot\text{m}^2 = \text{N}/\text{A}\cdot\text{m}$ . This unwieldy combination of units is abbreviated as the tesla,  $1 \text{ T} = 1 \text{ N}/\text{A}\cdot\text{m}$ . Refrain from memorizing the part about the counterclockwise direction at the end; in section 24.5 we'll see how to understand this in terms of more basic principles.

The nonexistence of magnetic monopoles means that unlike an electric field, d/1, a magnetic one, d/2, can never have sources or sinks. The magnetic field vectors lead in paths that loop back on themselves, without ever converging or diverging at a point.



c / The magnetic field pattern of a bar magnet. This picture was made by putting iron filings on a piece of paper, and bringing a bar magnet up underneath it. Note how the field pattern passes across the body of the magnet, forming closed loops, as in figure d/2. There are no sources or sinks.



d / Electric fields, 1, have sources and sinks, but magnetic fields, 2, don't.

## 24.2 Calculating magnetic fields and forces

### Magnetostatics

Our study of the electric field built on our previous understanding of electric forces, which was ultimately based on Coulomb's law for the electric force between two point charges. Since magnetism is ultimately an interaction between currents, i.e., between moving charges, it is reasonable to wish for a magnetic analog of Coulomb's law, an equation that would tell us the magnetic force between any two moving point charges.

Such a law, unfortunately, does not exist. Coulomb's law describes the special case of electrostatics: if a set of charges is sitting around and not moving, it tells us the interactions among them. Coulomb's law fails if the charges are in motion, since it does not incorporate any allowance for the time delay in the outward propagation of a change in the locations of the charges.

A pair of moving point charges will certainly exert magnetic forces on one another, but their magnetic fields are like the v-shaped bow waves left by boats. Each point charge experiences a magnetic field that originated from the other charge when it was at some previous position. There is no way to construct a force law that tells us the force between them based only on their current positions in space.

There is, however, a science of magnetostatics that covers a great many important cases. Magnetostatics describes magnetic forces among currents in the special case where the currents are steady and continuous, leading to magnetic fields throughout space that do not change over time.

The magnetic field of a long, straight wire is one example that we can say something about without resorting to fancy mathematics. We saw in examples 4 on p. 607 and 14 on p. 619 that the *electric* field of a uniform line of charge is  $E = 2kq/Lr$ , where  $r$  is the distance from the line and  $q/L$  is the charge per unit length. In a frame of reference moving at velocity  $v$  parallel to the line, this electric field will be observed as a combination of electric and magnetic fields. It therefore follows that the magnetic field of a long, straight, current-carrying wire must be proportional to  $1/r$ . We also expect that it will be proportional to the Coulomb constant, which sets the strength of electric and magnetic interactions, and to the current  $I$  in the wire. The complete expression turns out to be  $B = (k/c^2)(2I/r)$ . This is identical to the expression for  $E$  except for replacement of  $q/L$  with  $I$  and an additional factor of  $1/c^2$ . The latter occurs because magnetism is a purely relativistic effect, and the relativistic length contraction depends on  $v^2/c^2$ .

Figure e shows the equations for some of the more commonly encountered configurations, with illustrations of their field patterns. They all have a factor of  $k/c^2$  in front, which shows that magnetism is just electricity ( $k$ ) seen through the lens of relativity ( $1/c^2$ ). A convenient feature of SI units is that  $k/c^2$  has a numerical value of exactly  $10^{-7}$ , with units of  $\text{N/A}^2$ .

*Field created by a long, straight wire carrying current  $I$ :*

$$B = \frac{k}{c^2} \cdot \frac{2I}{r}$$

Here  $r$  is the distance from the center of the wire. The field vectors trace circles in planes perpendicular to the wire, going clockwise when viewed from along the direction of the current.

*Field created by a single circular loop of current:*

The field vectors form a dipole-like pattern, coming through the loop and back around on the outside. Each oval path traced out by the field vectors appears clockwise if viewed from along the direction the current is going when it punches through it. There is no simple equation for a field at an arbitrary point in space, but for a point lying *along the central axis* perpendicular to the loop, the field is

$$B = \frac{k}{c^2} \cdot 2\pi I b^2 (b^2 + z^2)^{-3/2} ,$$

where  $b$  is the radius of the loop and  $z$  is the distance of the point from the plane of the loop.

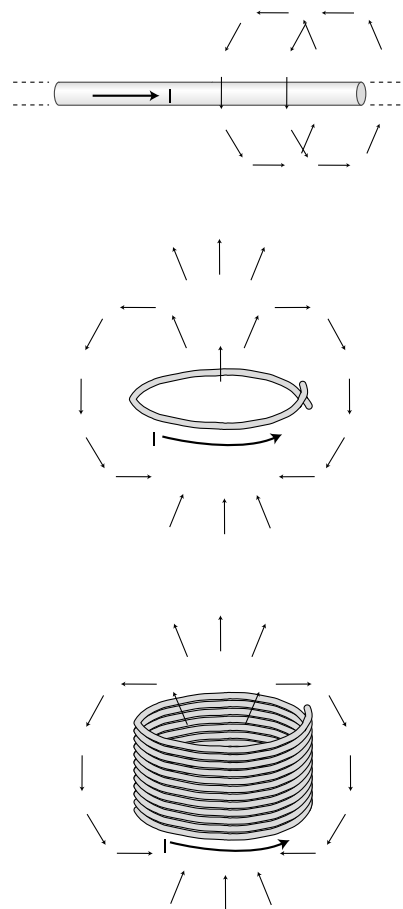
*Field created by a solenoid (cylindrical coil):*

The field pattern is similar to that of a single loop, but for a long solenoid the paths of the field vectors become very straight on the inside of the coil and on the outside immediately next to the coil. For a sufficiently long solenoid, the interior field also becomes very nearly uniform, with a magnitude of

$$B = \frac{k}{c^2} \cdot 4\pi I N / \ell ,$$

where  $N$  is the number of turns of wire and  $\ell$  is the length of the solenoid. The field near the mouths or outside the coil is not constant, and is more difficult to calculate. For a long solenoid, the exterior field is much smaller than the interior field.

Don't memorize the equations!



e / Some magnetic fields.

## Force on a charge moving through a magnetic field

We now know how to calculate magnetic fields in some typical situations, but one might also like to be able to calculate magnetic forces, such as the force of a solenoid on a moving charged particle, or the force between two parallel current-carrying wires.

We will restrict ourselves to the case of the force on a charged particle moving through a magnetic field, which allows us to calculate the force between two objects when one is a moving charged particle and the other is one whose magnetic field we know how to find. An example is the use of solenoids inside a TV tube to guide the electron beam as it paints a picture.

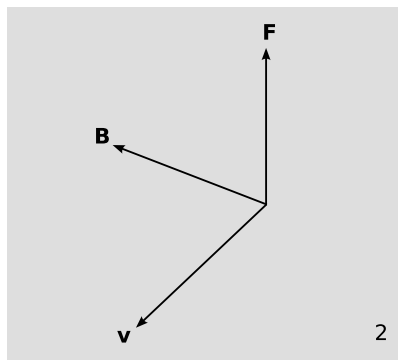
Experiments show that the magnetic force on a moving charged particle has a magnitude given by

$$|\mathbf{F}| = q|\mathbf{v}||\mathbf{B}| \sin \theta \quad ,$$

where  $\mathbf{v}$  is the velocity vector of the particle, and  $\theta$  is the angle between the  $\mathbf{v}$  and  $\mathbf{B}$  vectors. Unlike electric and gravitational forces, magnetic forces do not lie along the same line as the field vector. The force is always *perpendicular* to both  $\mathbf{v}$  and  $\mathbf{B}$ . Given two vectors, there is only one line perpendicular to both of them, so the force vector points in one of the two possible directions along this line. For a positively charged particle, the direction of the force vector can be found as follows. First, position the  $\mathbf{v}$  and  $\mathbf{B}$  vectors with their tails together. The direction of  $\mathbf{F}$  is such that if you sight along it, the  $\mathbf{B}$  vector is clockwise from the  $\mathbf{v}$  vector; for a negatively charged particle the direction of the force is reversed. Note that since the force is perpendicular to the particle's motion, the magnetic field never does work on it.



1



2

Example 1.

### Magnetic levitation

### example 1

In figure 24.2.2, a small, disk-shaped permanent magnet is stuck on the side of a battery, and a wire is clasped loosely around the battery, shorting it. A large current flows through the wire. The electrons moving through the wire feel a force from the magnetic field made by the permanent magnet, and this force levitates the wire.

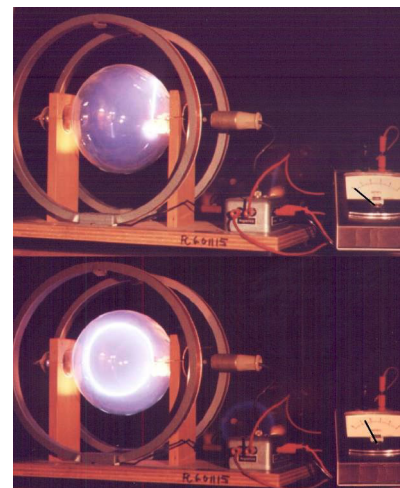
From the photo, it's possible to find the direction of the magnetic field made by the permanent magnet. The electrons in the copper wire are negatively charged, so they flow from the negative (flat) terminal of the battery to the positive terminal (the one with the bump, in front). As the electrons pass by the permanent magnet, we can imagine that they would experience a field either toward the magnet, or away from it, depending on which way the magnet was flipped when it was stuck onto the battery. Imagine sighting along the upward force vector, which you could do if you were

a tiny bug lying on your back underneath the wire. Since the electrons are negatively charged, the **B** vector must be counter-clockwise from the **v** vector, which means toward the magnet.

#### A circular orbit

#### example 2

Magnetic forces cause a beam of electrons to move in a circle. The beam is created in a vacuum tube, in which a small amount of hydrogen gas has been left. A few of the electrons strike hydrogen molecules, creating light and letting us see the beam. A magnetic field is produced by passing a current (meter) through the circular coils of wire in front of and behind the tube. In the bottom figure, with the magnetic field turned on, the force perpendicular to the electrons' direction of motion causes them to move in a circle.



Example 2.

#### Hallucinations during an MRI scan

#### example 3

During an MRI scan of the head, the patient's nervous system is exposed to intense magnetic fields. The average velocities of the charge-carrying ions in the nerve cells is fairly low, but if the patient moves her head suddenly, the velocity can be high enough that the magnetic field makes significant forces on the ions. This can result in visual and auditory hallucinations, e.g., frying bacon sounds.

### Energy in the magnetic field

On p. 611 I gave equations for the energy stored in the gravitational and electric fields. Since a magnetic field is essentially an electric field seen in a different frame of reference, we expect the magnetic-field equation to be closely analogous to the electric version, and it is:

$$\begin{aligned}\text{energy stored in the gravitational field per m}^3 &= -\frac{1}{8\pi G}|\mathbf{g}|^2 \\ \text{energy stored in the electric field per m}^3 &= \frac{1}{8\pi k}|\mathbf{E}|^2 \\ \text{energy stored in the magnetic field per m}^3 &= \frac{c^2}{8\pi k}|\mathbf{B}|^2\end{aligned}$$

The idea here is that  $k/c^2$  is the magnetic version of the electric quantity  $k$ , the  $1/c^2$  representing the fact that magnetism is a relativistic effect.

Solenoids are very common electrical devices, but they can be a hazard to someone who is working on them. Imagine a solenoid that initially has a DC current passing through it. The current creates a magnetic field inside and around it, which contains energy. Now suppose that we break the circuit. Since there is no longer a complete circuit, current will quickly stop flowing, and the magnetic field will collapse very quickly. The field had energy stored in it, and even a small amount of energy can create a dangerous power surge if released over a short enough time interval. It is prudent not to fiddle with a solenoid that has current flowing through it, since breaking the circuit could be hazardous to your health.

As a typical numerical estimate, let's assume a  $40\text{ cm} \times 40\text{ cm} \times 40\text{ cm}$  solenoid with an interior magnetic field of  $1.0\text{ T}$  (quite a strong field). For the sake of this rough estimate, we ignore the exterior field, which is weak, and assume that the solenoid is cubical in shape. The energy stored in the field is

$$\begin{aligned} (\text{energy per unit volume})(\text{volume}) &= \frac{c^2}{8\pi k} |\mathbf{B}|^2 V \\ &= 3 \times 10^4 \text{ J} \end{aligned}$$

That's a lot of energy!

## 24.3 The universal speed $c$

Let's think a little more about the role of the 45-degree diagonal in the Lorentz transformation. Slopes on these graphs are interpreted as velocities. This line has a slope of 1 in relativistic units, but that slope corresponds to  $c$  in ordinary metric units. We already know that the relativistic distance unit must be extremely large compared to the relativistic time unit, so  $c$  must be extremely large. Now note what happens when we perform a Lorentz transformation: this particular line gets stretched, but the new version of the line lies right on top of the old one, and its slope stays the same. In other words, if one observer says that something has a velocity equal to  $c$ , every other observer will agree on that velocity as well. (The same thing happens with  $-c$ .)

### Velocities don't simply add and subtract.

This is counterintuitive, since we expect velocities to add and subtract in relative motion. If a dog is running away from me at  $5\text{ m/s}$  relative to the sidewalk, and I run after it at  $3\text{ m/s}$ , the dog's velocity in my frame of reference is  $2\text{ m/s}$ . According to everything we have learned about motion (p. 81), the dog must have different speeds in the two frames:  $5\text{ m/s}$  in the sidewalk's frame and  $2\text{ m/s}$

in mine. But velocities are measured by dividing a distance by a time, and both distance and time are distorted by relativistic effects, so we actually shouldn't expect the ordinary arithmetic addition of velocities to hold in relativity; it's an approximation that's valid at velocities that are small compared to  $c$ .

### A universal speed limit

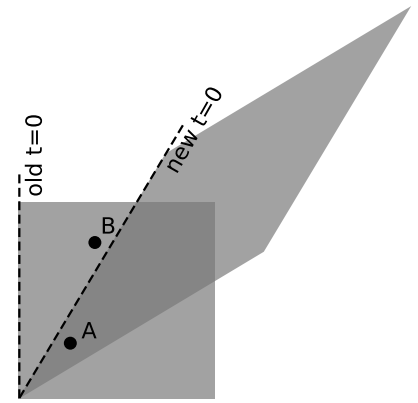
For example, suppose Janet takes a trip in a spaceship, and accelerates until she is moving at  $0.6c$  relative to the earth. She then launches a space probe in the forward direction at a speed relative to her ship of  $0.6c$ . We might think that the probe was then moving at a velocity of  $1.2c$ , but in fact the answer is still less than  $c$  (problem 1, page 675). This is an example of a more general fact about relativity, which is that  $c$  represents a universal speed limit. This is required by causality, as shown in figure f (but see section 26.7, p. 754).

### Light travels at $c$ .

Now consider a beam of light. We're used to talking casually about the "speed of light," but what does that really mean? Motion is relative, so normally if we want to talk about a velocity, we have to specify what it's measured relative to. A sound wave has a certain speed relative to the air, and a water wave has its own speed relative to the water. If we want to measure the speed of an ocean wave, for example, we should make sure to measure it in a frame of reference at rest relative to the water. But light isn't a vibration of a physical medium; it can propagate through the near-perfect vacuum of outer space, as when rays of sunlight travel to earth. This seems like a paradox: light is supposed to have a specific speed, but there is no way to decide what frame of reference to measure it in. The way out of the paradox is that light must travel at a velocity equal to  $c$ . Since all observers agree on a velocity of  $c$ , regardless of their frame of reference, everything is consistent.

### The Michelson-Morley experiment

The constancy of the speed of light had in fact already been observed when Einstein was an 8-year-old boy, but because nobody could figure out how to interpret it, the result was largely ignored. In 1887 Michelson and Morley set up a clever apparatus to measure any difference in the speed of light beams traveling east-west and north-south. The motion of the earth around the sun at 110,000 km/hour (about 0.01% of the speed of light) is to our west during the day. Michelson and Morley believed that light was a vibration of a mysterious medium called the ether, so they expected that the speed of light would be a fixed value relative to the ether. As the earth moved through the ether, they thought they would observe an effect on the velocity of light along an east-west line. For instance, if they released a beam of light in a westward direction during the day, they

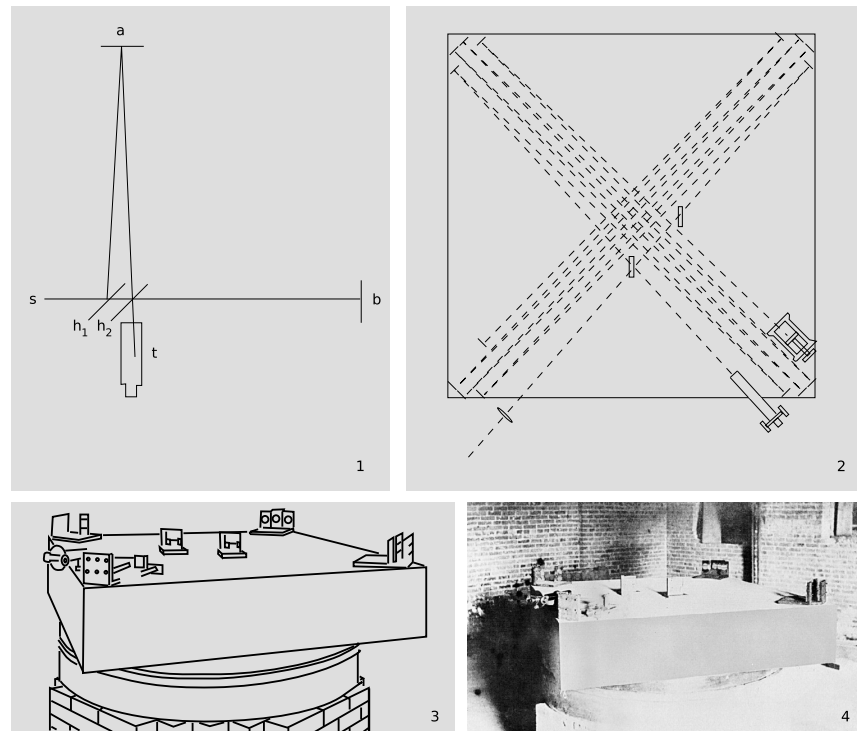


f / A proof that causality imposes a universal speed limit. In the original frame of reference, represented by the square, event A happens a little before event B. In the new frame, shown by the parallelogram, A happens after  $t = 0$ , but B happens before  $t = 0$ ; that is, B happens before A. The time ordering of the two events has been reversed. This can only happen because events A and B are very close together in time and fairly far apart in space. The line segment connecting A and B has a slope greater than 1, meaning that if we wanted to be present at both events, we would have to travel at a speed greater than  $c$  (which equals 1 in the units used on this graph). You will find that if you pick any two points for which the slope of the line segment connecting them is less than 1, you can never get them to straddle the new  $t = 0$  line in this funny, time-reversed way. Since different observers disagree on the time order of events like A and B, causality requires that information never travel from A to B or from B to A; if it did, then we would have time-travel paradoxes. The conclusion is that  $c$  is the maximum speed of cause and effect in relativity.



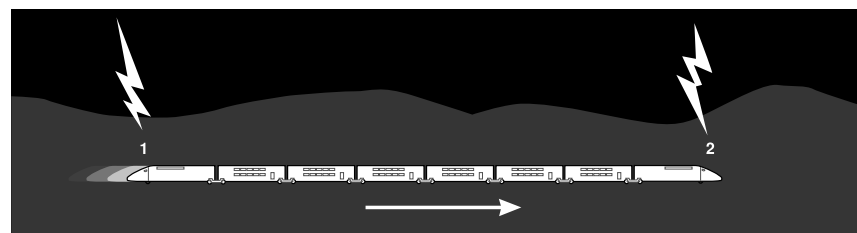
expected that it would move away from them at less than the normal speed because the earth was chasing it through the ether. They were surprised when they found that the expected 0.01% change in the speed of light did not occur.

The Michelson-Morley experiment, shown in photographs, and drawings from the original 1887 paper. 1. A simplified drawing of the apparatus. A beam of light from the source,  $s$ , is partially reflected and partially transmitted by the half-silvered mirror  $h_1$ . The two half-intensity parts of the beam are reflected by the mirrors at  $a$  and  $b$ , reunited, and observed in the telescope,  $t$ . If the earth's surface was supposed to be moving through the ether, then the times taken by the two light waves to pass through the moving ether would be unequal, and the resulting time lag would be detectable by observing the interference between the waves when they were reunited. 2. In the real apparatus, the light beams were reflected multiple times. The effective length of each arm was increased to 11 meters, which greatly improved its sensitivity to the small expected difference in the speed of light. 3. In an earlier version of the experiment, they had run into problems with its "extreme sensitiveness to vibration," which was "so great that it was impossible to see the interference fringes except at brief intervals ... even at two o'clock in the morning." They therefore mounted the whole thing on a massive stone floating in a pool of mercury, which also made it possible to rotate it easily. 4. A photo of the apparatus.



### Discussion question

**A** The figure shows a famous thought experiment devised by Einstein. A train is moving at constant velocity to the right when bolts of lightning strike the ground near its front and back. Alice, standing on the dirt at the midpoint of the flashes, observes that the light from the two flashes arrives simultaneously, so she says the two strikes must have occurred simultaneously. Bob, meanwhile, is sitting aboard the train, at its middle. At the moment when Alice sees the two flashes arrive, Bob is right alongside her, inside the train, so he also sees them. But Bob reasons that when the flashes were emitted, he was closer to flash 1 than to flash 2. Therefore if light from both flashes is reaching him simultaneously, and both traveled at  $c$ , flash 2 must have had more time to travel. How can this be reconciled with Alice's belief that the flashes were simultaneous?



## 24.4 Induction

### The principle of induction

Physicists of Michelson and Morley's generation thought that light was a mechanical vibration of the ether, but we now know that it is a ripple in the electric and magnetic fields. With hindsight, relativity essentially requires this:

1. Relativity requires that changes in any field propagate as waves at a finite speed (p. 599).
2. Relativity says that if a wave has a fixed speed but is not a mechanical disturbance in a physical medium, then it must travel at the universal velocity  $c$  (p. 661).

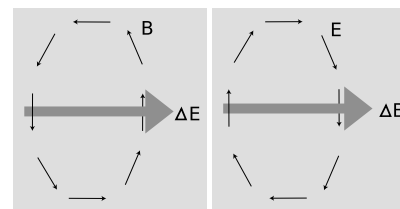
What is less obvious is that there are not two separate kinds of waves, electric and magnetic. In fact an electric wave can't exist without a magnetic one, or a magnetic one without an electric one. This new fact follows from the principle of induction, which was discovered experimentally by Faraday in 1831, seventy-five years before Einstein. Let's state Faraday's idea first, and then see how something like it must follow inevitably from relativity:

#### the principle of induction

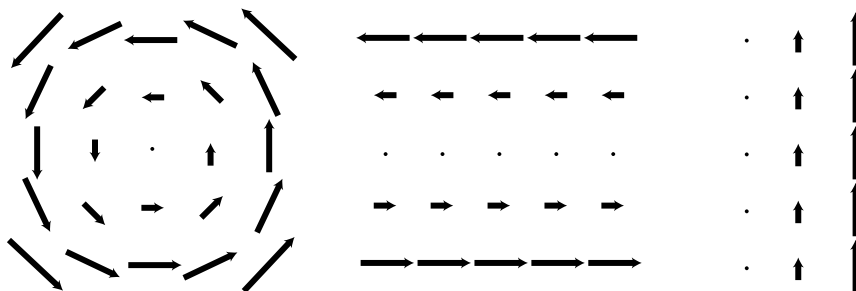
Any electric field that changes over time will produce a magnetic field in the space around it.

Any magnetic field that changes over time will produce an electric field in the space around it.

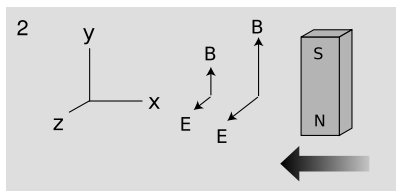
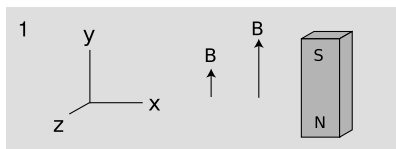
The induced field tends to have a whirlpool pattern, as shown in figure i, but the whirlpool image is not to be taken too literally; the principle of induction really just requires a field pattern such that, if one inserted a paddlewheel in it, the paddlewheel would spin. All of the field patterns shown in figure j are ones that could be created by induction; all have a counterclockwise "curl" to them.



i / The geometry of induced fields. The induced field tends to form a whirlpool pattern around the change in the vector producing it. Note how they circulate in opposite directions.



j / Three fields with counterclockwise "curls."

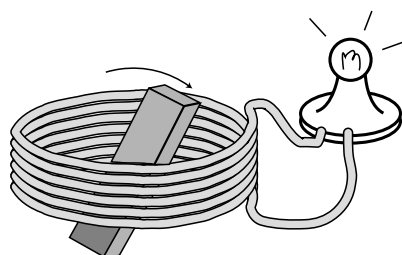


k / Observer 1 is at rest with respect to the bar magnet, and observes magnetic fields that have different strengths at different distances from the magnet. Observer 2, hanging out in the region to the left of the magnet, sees the magnet moving toward her, and detects that the magnetic field in that region is getting stronger as time passes.

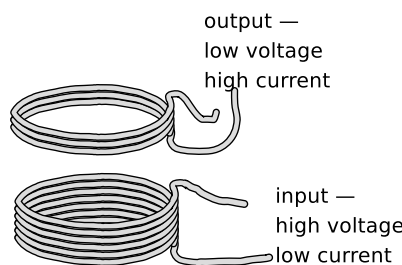
Figure k shows an example of the fundamental reason why a changing  $\mathbf{B}$  field must create an  $\mathbf{E}$  field. In section 23.2 we established that according to relativity, what one observer describes as a purely magnetic field, an observer in a different state of motion describes as a mixture of magnetic and electric fields. This is why there must be both an  $\mathbf{E}$  and a  $\mathbf{B}$  in observer 2's frame. Observer 2 cannot explain the electric field as coming from any charges. In frame 2, the  $\mathbf{E}$  can only be explained as an effect caused by the changing  $\mathbf{B}$ .

Observer 1 says, "2 feels a changing  $\mathbf{B}$  field because he's moving through a static field." Observer 2 says, "I feel a changing  $\mathbf{B}$  because the magnet is getting closer."

Although this argument doesn't prove the "whirlpool" geometry, we can verify that the fields I've drawn in figure k are consistent with it. The  $\Delta\mathbf{B}$  vector is upward, and the electric field has a curliness to it: a paddlewheel inserted in the electric field would spin clockwise as seen from above, since the clockwise torque made by the strong electric field on the right is greater than the counterclockwise torque made by the weaker electric field on the left.



l / A generator.



m / A transformer.

#### The generator

#### example 5

A generator, l, consists of a permanent magnet that rotates within a coil of wire. The magnet is turned by a motor or crank, (not shown). As it spins, the nearby magnetic field changes. According to the principle of induction, this changing magnetic field results in an electric field, which has a whirlpool pattern. This electric field pattern creates a current that whips around the coils of wire, and we can tap this current to light the lightbulb.

#### self-check A

When you're driving a car, the engine recharges the battery continuously using a device called an alternator, which is really just a generator like the one shown on the previous page, except that the coil rotates while the permanent magnet is fixed in place. Why can't you use the alternator to start the engine if your car's battery is dead?  $\triangleright$  Answer, p. 978

#### The transformer

#### example 6

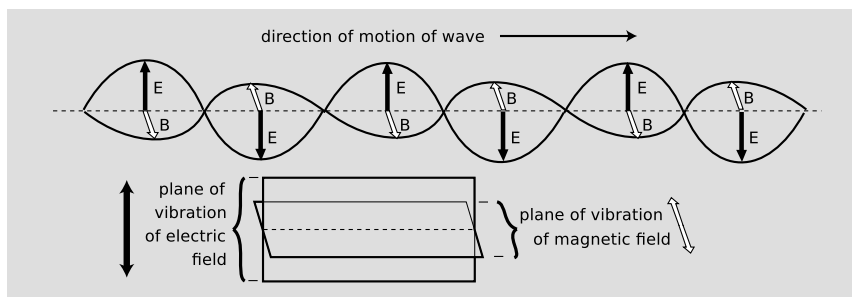
In example 18 on p. 572 we discussed the advantages of transmitting power over electrical lines using high voltages and low currents. However, we don't want our wall sockets to operate at 10000 volts! For this reason, the electric company uses a device called a transformer, m, to convert to lower voltages and higher currents inside your house. The coil on the input side creates a magnetic field. Transformers work with alternating current, so the

magnetic field surrounding the input coil is always changing. This induces an electric field, which drives a current around the output coil.

If both coils were the same, the arrangement would be symmetric, and the output would be the same as the input, but an output coil with a smaller number of coils gives the electric forces a smaller distance through which to push the electrons. Less mechanical work per unit charge means a lower voltage. Conservation of energy, however, guarantees that the amount of power on the output side must equal the amount put in originally,  $I_{in} V_{in} = I_{out} V_{out}$ , so this reduced voltage must be accompanied by an increased current.

## 24.5 Electromagnetic waves

The most important consequence of induction is the existence of electromagnetic waves. Whereas a gravitational wave would consist of nothing more than a rippling of gravitational fields, the principle of induction tells us that there can be no purely electrical or purely magnetic waves. Instead, we have waves in which there are both electric and magnetic fields, such as the sinusoidal one shown in the figure. Maxwell proved that such waves were a direct consequence of his equations, and derived their properties mathematically. The derivation would be beyond the mathematical level of this book, so we will just state the results.



An electromagnetic wave.

A sinusoidal electromagnetic wave has the geometry shown above. The  $\mathbf{E}$  and  $\mathbf{B}$  fields are perpendicular to the direction of motion, and are also perpendicular to each other. If you look along the direction of motion of the wave, the  $\mathbf{B}$  vector is always 90 degrees clockwise from the  $\mathbf{E}$  vector. The magnitudes of the two fields are related by the equation  $|\mathbf{E}| = c|\mathbf{B}|$ .

How is an electromagnetic wave created? It could be emitted, for example, by an electron orbiting an atom or currents going back and forth in a transmitting antenna. In general any accelerating charge will create an electromagnetic wave, although only a current that varies sinusoidally with time will create a sinusoidal wave. Once

created, the wave spreads out through space without any need for charges or currents along the way to keep it going. As the electric field oscillates back and forth, it induces the magnetic field, and the oscillating magnetic field in turn creates the electric field. The whole wave pattern propagates through empty space at the velocity  $c$ .

#### *Einstein's motorcycle*

#### *example 7*

As a teenage physics student, Einstein imagined the following paradox. (See p. 483.) What if he could get on a motorcycle and ride at speed  $c$ , alongside a beam of light? In his frame of reference, he observes constant electric and magnetic fields. But only a *changing* electric field can induce a magnetic field, and only a *changing* magnetic field can induce an electric field. The laws of physics are violated in his frame, and this seems to violate the principle that all frames of reference are equally valid.

The resolution of the paradox is that  $c$  is a universal speed limit, so the motorcycle can't be accelerated to  $c$ . Observers can never be at rest relative to a light wave, so no observer can have a frame of reference in which a light wave is observed to be at rest.

### **Polarization**

Two electromagnetic waves traveling in the same direction through space can differ by having their electric and magnetic fields in different directions, a property of the wave called its polarization.

### **Light is an electromagnetic wave**

Once Maxwell had derived the existence of electromagnetic waves, he became certain that they were the same phenomenon as light. Both are transverse waves (i.e., the vibration is perpendicular to the direction the wave is moving), and the velocity is the same.

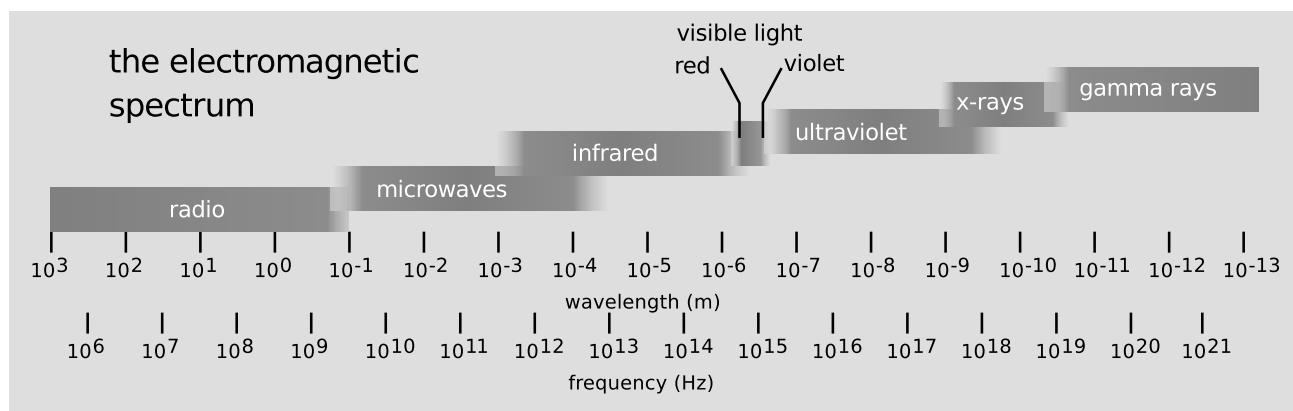
Heinrich Hertz (for whom the unit of frequency is named) verified Maxwell's ideas experimentally. Hertz was the first to succeed in producing, detecting, and studying electromagnetic waves in detail using antennas and electric circuits. To produce the waves, he had to make electric currents oscillate very rapidly in a circuit. In fact, there was really no hope of making the current reverse directions at the frequencies of  $10^{15}$  Hz possessed by visible light. The fastest electrical oscillations he could produce were  $10^9$  Hz, which would give a wavelength of about 30 cm. He succeeded in showing that, just like light, the waves he produced were polarizable, and could be reflected and refracted (i.e., bent, as by a lens), and he built devices such as parabolic mirrors that worked according to the same optical principles as those employing light. Hertz's results were convincing evidence that light and electromagnetic waves were one and the same.



Heinrich Hertz (1857-1894).

### **The electromagnetic spectrum**

Today, electromagnetic waves with frequencies in the range employed by Hertz are known as radio waves. Any remaining doubts that the “Hertzian waves,” as they were then called, were the same type of wave as light waves were soon dispelled by experiments in the whole range of frequencies in between, as well as the frequencies outside that range. In analogy to the spectrum of visible light, we speak of the entire electromagnetic spectrum, of which the visible spectrum is one segment.



The terminology for the various parts of the spectrum is worth memorizing, and is most easily learned by recognizing the logical relationships between the wavelengths and the properties of the waves with which you are already familiar. Radio waves have wavelengths that are comparable to the size of a radio antenna, i.e., meters to tens of meters. Microwaves were named that because they have much shorter wavelengths than radio waves; when food heats unevenly in a microwave oven, the small distances between neighboring hot and cold spots is half of one wavelength of the standing wave the oven creates. The infrared, visible, and ultraviolet obviously have much shorter wavelengths, because otherwise the wave nature of light would have been as obvious to humans as the wave nature of ocean waves. To remember that ultraviolet, x-rays, and gamma rays all lie on the short-wavelength side of visible, recall that all three of these can cause cancer. (As we’ll discuss later in the course, there is a basic physical reason why the cancer-causing disruption of DNA can only be caused by very short-wavelength electromagnetic waves. Contrary to popular belief, microwaves cannot cause cancer, which is why we have microwave ovens and not x-ray ovens!)

*Why the sky is blue*

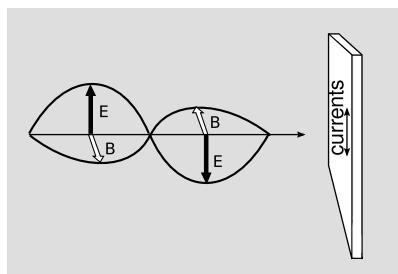
*example 8*

When sunlight enters the upper atmosphere, a particular air molecule finds itself being washed over by an electromagnetic wave of frequency  $f$ . The molecule’s charged particles (nuclei and electrons) act like oscillators being driven by an oscillating force, and

respond by vibrating at the same frequency  $f$ . Energy is sucked out of the incoming beam of sunlight and converted into the kinetic energy of the oscillating particles. However, these particles are accelerating, so they act like little radio antennas that put the energy back out as spherical waves of light that spread out in all directions. An object oscillating at a frequency  $f$  has an acceleration proportional to  $f^2$ , and an accelerating charged particle creates an electromagnetic wave whose fields are proportional to its acceleration, so the field of the reradiated spherical wave is proportional to  $f^2$ . The energy of a field is proportional to the square of the field, so the energy of the reradiated wave is proportional to  $f^4$ . Since blue light has about twice the frequency of red light, this process is about  $2^4 = 16$  times as strong for blue as for red, and that's why the sky is blue.

### Momentum of light

Newton defined momentum as  $mv$ , and that would lead us to believe that light, which has no mass, should have no momentum. However, Newton's laws only work at velocities that are small compared to the speed of light, and light travels *at* the speed of light, so there is no reason to trust Newton here. In fact, it's straightforward to show that electromagnetic waves have momentum. If a light wave strikes an ohmic surface, as in figure 24.6, the wave's electric field causes charges to vibrate back and forth in the surface. These currents then experience a magnetic force from the wave's magnetic field, and application of the geometrical rule on p. 658 shows that the resulting force is in the direction of propagation of the wave. Thus the light wave acts as if it has momentum and inertia. This is explored further in problem 12 on p. 759.



o / An electromagnetic wave strikes an ohmic surface. The wave's electric field causes currents to flow up and down. The wave's magnetic field then acts on these currents, producing a force in the direction of the wave's propagation. This is a pre-relativistic argument that light must possess inertia.

## 24.6 ★ Symmetry and handedness

The physicist Richard Feynman helped to get me hooked on physics with an educational film containing the following puzzle. Imagine that you establish radio contact with an alien on another planet. Neither of you even knows where the other one's planet is, and you aren't able to establish any landmarks that you both recognize. You manage to learn quite a bit of each other's languages, but you're stumped when you try to establish the definitions of left and right (or, equivalently, clockwise and counterclockwise). Is there any way to do it?

If there was any way to do it without reference to external landmarks, then it would imply that the laws of physics themselves were asymmetric, which would be strange. Why should they distinguish left from right? The gravitational field pattern surrounding a star or planet looks the same in a mirror, and the same goes for electric fields. However, the field patterns shown in section 24.2 seem to vio-

late this principle, but do they really? Could you use these patterns to explain left and right to the alien? In fact, the answer is no. If you look back at the definition of the magnetic field in section 24.1, it also contains a reference to handedness: the counterclockwise direction of the loop's current as viewed along the magnetic field. The aliens might have reversed their definition of the magnetic field, in which case their drawings of field patterns would look like mirror images of ours.

Until the middle of the twentieth century, physicists assumed that any reasonable set of physical laws would have to have this kind of symmetry between left and right. An asymmetry would be grotesque. Whatever their aesthetic feelings, they had to change their opinions about reality when experiments showed that the weak nuclear force (section 22.4) violates right-left symmetry! It is still a mystery why right-left symmetry is observed so scrupulously in general, but is violated by one particular type of physical process.

## 24.7 ★ Doppler shifts and clock time

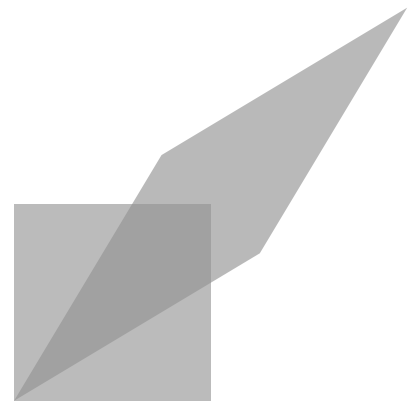
Figure p shows our now-familiar method of visualizing a Lorentz transformation, in a case where the numbers come out to be particularly simple. This diagram has two geometrical features that we have referred to before without digging into their physical significance: the *stretch factor* of the diagonals, and the *area*. In this section we'll see that the former can be related to the Doppler effect, and the latter to clock time.

### Doppler shifts of light

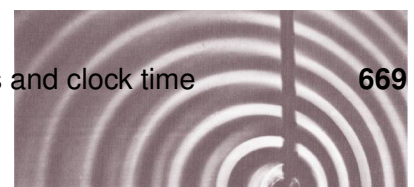
When Doppler shifts happen to ripples on a pond or the sound waves from an airplane, they can depend on the relative motion of three different objects: the source, the receiver, and the medium. But light waves don't have a medium. Therefore Doppler shifts of light can only depend on the relative motion of the source and observer.

One simple case is the one in which the relative motion of the source and the receiver is perpendicular to the line connecting them. That is, the motion is transverse. Nonrelativistic Doppler shifts happen because the distance between the source and receiver is changing, so in nonrelativistic physics we don't expect any Doppler shift at all when the motion is transverse, and this is what is in fact observed to high precision. For example, the photo shows shortened and lengthened wavelengths to the right and left, along the source's line of motion, but an observer above or below the source measures just the normal, unshifted wavelength and frequency. But relativistically, we have a time dilation effect, so for light waves emitted transversely, there is a Doppler shift of  $1/\gamma$  in frequency (or  $\gamma$  in wavelength).

The other simple case is the one in which the relative motion of the



p / A graphical representation of the Lorentz transformation for a velocity of  $(3/5)c$ . The long diagonal is stretched by a factor of two, the short one is half its former length, and the area is the same as before.





source and receiver is longitudinal, i.e., they are either approaching or receding from one another. For example, distant galaxies are receding from our galaxy due to the expansion of the universe, and this expansion was originally detected because Doppler shifts toward the red (low-frequency) end of the spectrum were observed.

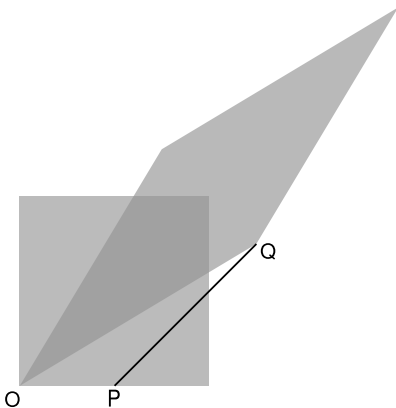
Nonrelativistically, we would expect the light from such a galaxy to be Doppler shifted down in frequency by some factor, which would depend on the relative velocities of three different objects: the source, the wave's medium, and the receiver. Relativistically, things get simpler, because light isn't a vibration of a physical medium, so the Doppler shift can only depend on a single velocity  $v$ , which is the rate at which the separation between the source and the receiver is increasing.

The square in figure r is the “graph paper” used by someone who considers the source to be at rest, while the parallelogram plays a similar role for the receiver. The figure is drawn for the case where  $v = 3/5$  (in units where  $c = 1$ ), and in this case the stretch factor of the long diagonal is 2. To keep the area the same, the short diagonal has to be squished to half its original size. But now it's a matter of simple geometry to show that  $OP$  equals half the width of the square, and this tells us that the Doppler shift is a factor of  $1/2$  in frequency. That is, the squish factor of the short diagonal is interpreted as the Doppler shift. To get this as a general equation for velocities other than  $3/5$ , one can show by straightforward fiddling with the result of part c of problem 2 on p. 651 that the Doppler shift is

$$D(v) = \sqrt{\frac{1-v}{1+v}} \quad .$$

Here  $v > 0$  is the case where the source and receiver are getting farther apart,  $v < 0$  the case where they are approaching. (This is the opposite of the sign convention used in section 19.5. It is convenient to change conventions here so that we can use positive values of  $v$  in the case of cosmological red-shifts, which are the most important application.)

Suppose that Alice stays at home on earth while her twin Betty takes off in her rocket ship at  $3/5$  of the speed of light. When I first learned relativity, the thing that caused me the most pain was understanding how each observer could say that the other was the one whose time was slow. It seemed to me that if I could take a pill that would speed up my mind and my body, then naturally I would see everybody *else* as being slow. Shouldn't the same apply to relativity? But suppose Alice and Betty get on the radio and try to settle who is the fast one and who is the slow one. Each twin's voice sounds sloooooow doooooowwwwn to the other. If Alice claps her hands twice, at a time interval of one second by her clock, Betty hears the hand-claps coming over the radio two seconds apart, but the situation is exactly symmetric, and Alice hears the same thing if Betty claps.



r / At event O, the source and the receiver are on top of each other, so as the source emits a wave crest, it is received without any time delay. At P, the source emits another wave crest, and at Q the receiver receives it.

Each twin analyzes the situation using a diagram identical to r, and attributes her sister's observations to a complicated combination of time distortion, the time taken by the radio signals to propagate, and the motion of her twin relative to her.

*self-check B*

Turn your book upside-down and reinterpret figure r. ▷ Answer, p. 978

*A symmetry property of the Doppler effect* *example 9*

Suppose that A and B are at rest relative to one another, but C is moving along the line between A and B. A transmits a signal to C, who then retransmits it to B. The signal accumulates two Doppler shifts, and the result is their product  $D(v)D(-v)$ . But this product must equal 1, so we must have  $D(-v)D(v) = 1$ , which can be verified directly from the equation.

*The Ives-Stilwell experiment* *example 10*

The result of example 9 was the basis of one of the earliest laboratory tests of special relativity, by Ives and Stilwell in 1938. They observed the light emitted by excited by a beam of  $H_2^+$  and  $H_3^+$  ions with speeds of a few tenths of a percent of  $c$ . Measuring the light from both ahead of and behind the beams, they found that the product of the Doppler shifts  $D(v)D(-v)$  was equal to 1, as predicted by relativity. If relativity had been false, then one would have expected the product to differ from 1 by an amount that would have been detectable in their experiment. In 2003, Saathoff et al. carried out an extremely precise version of the Ives-Stilwell technique with  $Li^+$  ions moving at 6.4% of  $c$ . The frequencies observed, in units of MHz, were:

$$\begin{aligned} f_0 &= 546466918.8 \pm 0.4 \\ &\quad \text{(unshifted frequency)} \\ f_0 D(-v) &= 582490203.44 \pm .09 \\ &\quad \text{(shifted frequency, forward)} \\ f_0 D(v) &= 512671442.9 \pm 0.5 \\ &\quad \text{(shifted frequency, backward)} \\ \sqrt{f_0 D(-v) \cdot f_0 D(v)} &= 546466918.6 \pm 0.3 \end{aligned}$$

The results show incredibly precise agreement between  $f_0$  and  $\sqrt{f_0 D(-v) \cdot f_0 D(v)}$ , as expected relativistically because  $D(v)D(-v)$  is supposed to equal 1. The agreement extends to 9 significant figures, whereas if relativity had been false there should have been a relative disagreement of about  $v^2 = .004$ , i.e., a discrepancy in the third significant figure. The spectacular agreement with theory has made this experiment a lightning rod for anti-relativity kooks.

We saw on p. 660 that relativistic velocities should not be expected to be exactly additive, and problem 1 on p. 675 verifies this in the special case where A moves relative to B at  $0.6c$  and B relative to

$C$  at  $0.6c$  — the result *not* being  $1.2c$ . The relativistic Doppler shift provides a simple way of deriving a general equation for the relativistic combination of velocities; problem 21 on p. 680 guides you through the steps of this derivation.

### Clock time

On p. 638 we proved that the Lorentz transformation doesn't change the area of a shape in the  $x$ - $t$  plane. We used this only as a stepping stone toward the Lorentz transformation, but it is natural to wonder whether this kind of area has any physical interest of its own.

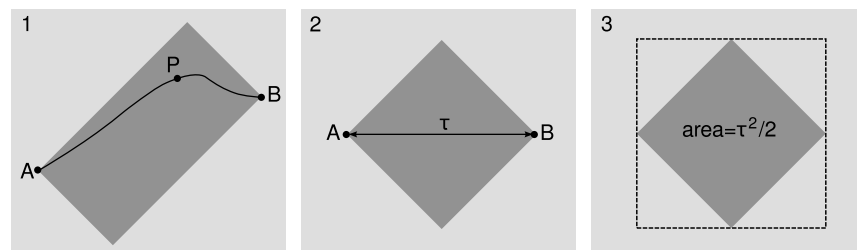
The equal-area result is not relativistic, since the proof never appeals to property 5 on page 634. Cases I and II on page 637 also have the equal-area property. We can see this clearly in a Galilean transformation like figure g on p. 635, where the distortion of the rectangle could be accomplished by cutting it into vertical slices and then displacing the slices upward without changing their areas.

But the area does have a nice interpretation in the relativistic case. Suppose that we have events A (Charles VII is restored to the throne) and B (Joan of Arc is executed). Now imagine that technologically advanced aliens want to be present at both A and B, but in the interim they wish to fly away in their spaceship, be present at some other event P (perhaps a news conference at which they give an update on the events taking place on earth), but get back in time for B. Since nothing can go faster than  $c$  (which we take to equal 1 in appropriate units), P cannot be too far away. The set of all possible events P forms a rectangle, figure s/1, in the  $x$  -  $t$  plane that has A and B at opposite corners and whose edges have slopes equal to  $\pm 1$ . We call this type of rectangle a light-rectangle, because its sides could represent the motion of rays of light.

s / 1. The gray light-rectangle represents the set of all events such as P that could be visited after A and before B.

2. The rectangle becomes a square in the frame in which A and B occur at the same location in space.

3. The area of the dashed square is  $\tau^2$ , so the area of the gray square is  $\tau^2/2$ .



The area of this rectangle will be the same regardless of one's frame of reference. In particular, we could choose a special frame of reference, panel 2 of the figure, such that A and B occur in the same place. (They do not occur at the same place, for example, in the sun's frame, because the earth is spinning and going around the sun.) Since the speed  $c$ , which equals 1 in our units, is the same in all frames of reference, and the sides of the rectangle had slopes  $\pm 1$  in frame 1, they must still have slopes  $\pm 1$  in frame 2. The rectangle

becomes a square with its diagonals parallel to the  $x$  and  $t$  axes, and the length of these diagonals equals the time  $\tau$  elapsed on a clock that is at rest in frame 2, i.e., a clock that glides through space at constant velocity from A to B, meeting up with the planet earth at the appointed time. As shown in panel 3 of the figure, the area of the gray regions can be interpreted as half the square of this gliding-clock time. If events A and B are separated by a distance  $x$  and a time  $t$ , then in general  $t^2 - x^2$  gives the square of the gliding-clock time.<sup>1</sup>

When  $|x|$  is greater than  $|t|$ , events A and B are so far apart in space and so close together in time that it would be impossible to have a cause and effect relationship between them, since  $c = 1$  is the maximum speed of cause and effect. In this situation  $t^2 - x^2$  is negative and cannot be interpreted as a clock time, but it can be interpreted as minus the square of the distance between A and B as measured by rulers at rest in a frame in which A and B are simultaneous.

No matter what,  $t^2 - x^2$  is the same as measured in all frames of reference. Geometrically, it plays the same role in the  $x$ - $t$  plane that ruler measurements play in the Euclidean plane. In Euclidean geometry, the ruler-distance between any two points stays the same regardless of rotation, i.e., regardless of the angle from which we view the scene; according to the Pythagorean theorem, the square of this distance is  $x^2 + y^2$ . In the  $x$ - $t$  plane,  $t^2 - x^2$  stays the same regardless of the frame of reference.

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<sup>1</sup>Proof: Based on units, the expression must have the form  $(\dots)t^2 + (\dots)tx + (\dots)x^2$ , where each  $(\dots)$  represents a unitless constant. The  $tx$  coefficient must be zero by property 2 on p. 634. For consistency with figure s/3, the  $t^2$  coefficient must equal 1. Since the area vanishes for  $x = t$ , the  $x^2$  coefficient must equal  $-1$ .

# Summary

## Selected vocabulary

magnetic field . .	a field of force, defined in terms of the torque exerted on a test dipole
magnetic dipole .	an object, such as a current loop, an atom, or a bar magnet, that experiences torques due to magnetic forces; the strength of magnetic dipoles is measured by comparison with a standard dipole consisting of a square loop of wire of a given size and carrying a given amount of current
induction . . . . .	the production of an electric field by a changing magnetic field, or vice-versa

## Notation

$\mathbf{B}$ . . . . .	the magnetic field
$D_m$ . . . . .	magnetic dipole moment

## Summary

The magnetic field is defined in terms of the torque on a magnetic test dipole. It has no sources or sinks; magnetic field patterns never converge on or diverge from a point.

Relativity dictates a maximum speed limit  $c$  for cause and effect. This speed is the same in all frames of reference.

Relativity requires that the magnetic and electric fields be intimately related. The principle of induction states that any changing electric field produces a magnetic field in the surrounding space, and vice-versa. These induced fields tend to form whirlpool patterns.

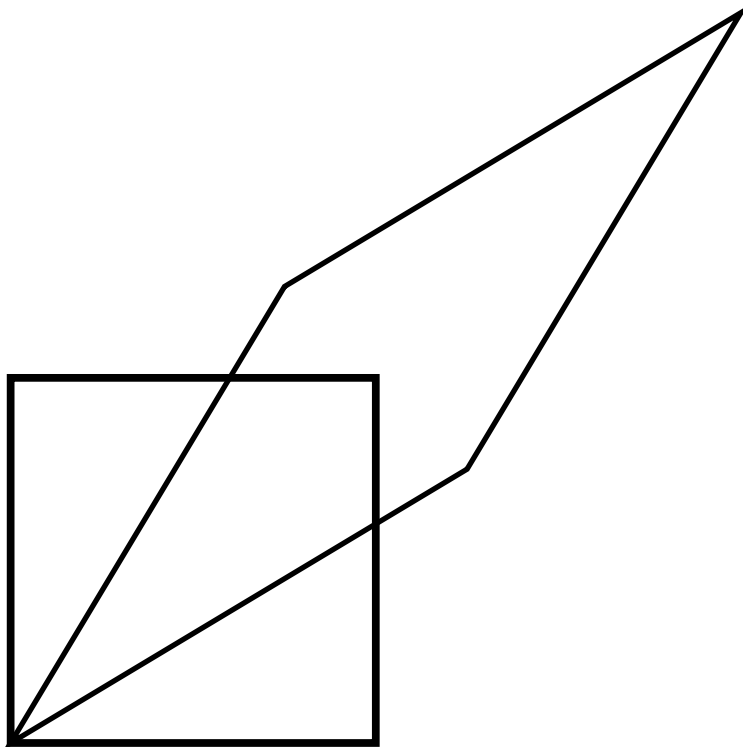
The most important consequence of the principle of induction is that there are no purely magnetic or purely electric waves. Electromagnetic disturbances propagate outward at  $c$  as combined magnetic and electric waves, with a well-defined relationship between the magnitudes and directions of the electric and magnetic fields. These electromagnetic waves are what light is made of, but other forms of electromagnetic waves exist besides visible light, including radio waves, x-rays, and gamma rays.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

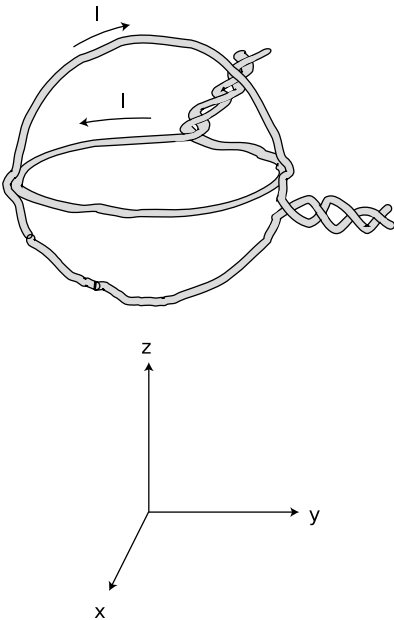
1 The figure illustrates a Lorentz transformation using the conventions employed in section 23.1. For simplicity, the transformation chosen is one that lengthens one diagonal by a factor of 2. Since Lorentz transformations preserve area, the other diagonal is shortened by a factor of 2. Let the original frame of reference, depicted with the square, be A, and the new one B. (a) By measuring with a ruler on the figure, show that the velocity of frame B relative to frame A is  $0.6c$ . (b) Print out a copy of the page. With a ruler, draw a third parallelogram that represents a second successive Lorentz transformation, one that lengthens the long diagonal by another factor of 2. Call this third frame C. Use measurements with a ruler to determine frame C's velocity relative to frame A. Does it equal double the velocity found in part a? Explain why it should be expected to turn out the way it does. ✓



**2** (a) In this chapter we've represented Lorentz transformations as distortions of a square into various parallelograms, with the degree of distortion depending on the velocity of one frame of reference relative to another. Suppose that one frame of reference was moving at  $c$  relative to another. Discuss what would happen in terms of distortion of a square, and show that this is impossible by using an argument similar to the one used to rule out transformations like the one in figure h on page 636.

(b) Resolve the following paradox. Two pen-pointer lasers are placed side by side and aimed in parallel directions. Their beams both travel at  $c$  relative to the hardware, but each beam has a velocity of zero relative to the neighboring beam. But the speed of light can't be zero; it's supposed to be the same in all frames of reference.

**3** Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is inside the big one with their currents circulating in the same direction, and a second configuration in which the currents circulate in opposite directions. Compare the energies of these configurations with the energy when the solenoids are far apart. Based on this reasoning, which configuration is stable, and in which configuration will the little solenoid tend to get twisted around or spit out? [Hint: A stable system has low energy; energy would have to be added to change its configuration.]



Problem 4.

**4** The figure shows a nested pair of circular wire loops used to create magnetic fields. (The twisting of the leads is a practical trick for reducing the magnetic fields they contribute, so the fields are very nearly what we would expect for an ideal circular current loop.) The coordinate system below is to make it easier to discuss directions in space. One loop is in the  $y - z$  plane, the other in the  $x - y$  plane. Each of the loops has a radius of 1.0 cm, and carries 1.0 A in the direction indicated by the arrow.

(a) Using the equation in optional section 24.2, calculate the magnetic field that would be produced by *one* such loop, at its center. ✓

(b) Describe the direction of the magnetic field that would be produced, at its center, by the loop in the  $x - y$  plane alone.

(c) Do the same for the other loop.

(d) Calculate the magnitude of the magnetic field produced by the two loops in combination, at their common center. Describe its direction. ✓

**5** One model of the hydrogen atom has the electron circling around the proton at a speed of  $2.2 \times 10^6$  m/s, in an orbit with a radius of 0.05 nm. (Although the electron and proton really orbit around their common center of mass, the center of mass is very close to the proton, since it is 2000 times more massive. For this problem, assume the proton is stationary.) In homework problem 19 on page 584, you calculated the electric current created.

(a) Now estimate the magnetic field created at the center of the atom by the electron. We are treating the circling electron as a current loop, even though it's only a single particle. ✓

(b) Does the proton experience a nonzero force from the electron's magnetic field? Explain.

(c) Does the electron experience a magnetic field from the proton? Explain.

(d) Does the electron experience a magnetic field created by its own current? Explain.

(e) Is there an electric force acting between the proton and electron? If so, calculate it. ✓

(f) Is there a gravitational force acting between the proton and electron? If so, calculate it.

(g) An inward force is required to keep the electron in its orbit – otherwise it would obey Newton's first law and go straight, leaving the atom. Based on your answers to the previous parts, which force or forces (electric, magnetic and gravitational) contributes significantly to this inward force?

**6** Suppose a charged particle is moving through a region of space in which there is an electric field perpendicular to its velocity vector, and also a magnetic field perpendicular to both the particle's velocity vector and the electric field. Show that there will be one particular velocity at which the particle can be moving that results in a total force of zero on it. Relate this velocity to the magnitudes of the electric and magnetic fields. (Such an arrangement, called a velocity filter, is one way of determining the speed of an unknown particle.)

**7** If you put four times more current through a solenoid, how many times more energy is stored in its magnetic field? ✓

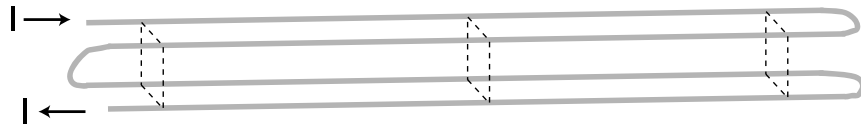
**8** Suppose we are given a permanent magnet with a complicated, asymmetric shape. Describe how a series of measurements with a magnetic compass could be used to determine the strength and direction of its magnetic field at some point of interest. Assume that you are only able to see the direction to which the compass needle settles; you cannot measure the torque acting on it. ★



**9** Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough to act like ideal solenoids, so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is partly inside and partly hanging out of the big one, with their currents circulating in the same direction. Their axes are constrained to coincide.

(a) Find the magnetic potential energy as a function of the length  $x$  of the part of the small solenoid that is inside the big one. (Your equation will include other relevant variables describing the two solenoids.)

(b) Based on your answer to part (a), find the force acting between the solenoids.



**Problem 10.**

**10** Four long wires are arranged, as shown, so that their cross-section forms a square, with connections at the ends so that current flows through all four before exiting. Note that the current is to the right in the two back wires, but to the left in the front wires. If the dimensions of the cross-sectional square (height and front-to-back) are  $b$ , find the magnetic field (magnitude and direction) along the long central axis. ✓

**11** The purpose of this problem is to find the force experienced by a straight, current-carrying wire running perpendicular to a uniform magnetic field. (a) Let  $A$  be the cross-sectional area of the wire,  $n$  the number of free charged particles per unit volume,  $q$  the charge per particle, and  $v$  the average velocity of the particles. Show that the current is  $I = Avnq$ . (b) Show that the magnetic force per unit length is  $AvnqB$ . (c) Combining these results, show that the force on the wire per unit length is equal to  $IB$ . ▷ Solution, p. 974

**12** Suppose two long, parallel wires are carrying current  $I_1$  and  $I_2$ . The currents may be either in the same direction or in opposite directions. (a) Using the information from section 24.2, determine under what conditions the force is attractive, and under what conditions it is repulsive. Note that, because of the difficulties explored in problem 14, it's possible to get yourself tied up in knots if you use the energy approach of section 22.4. (b) Starting from the result of problem 11, calculate the force per unit length.

▷ Solution, p. 974

**13** Section 24.2 states the following rule:

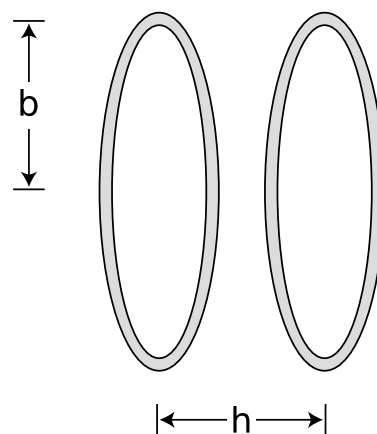
For a positively charged particle, the direction of the  $F$  vector is the one such that if you sight along it, the  $\mathbf{B}$  vector is clockwise from the  $v$  vector.

Make a three-dimensional model of the three vectors using pencils or rolled-up pieces of paper to represent the vectors assembled with their tails together. Now write down every possible way in which the rule could be rewritten by scrambling up the three symbols  $F$ ,  $\mathbf{B}$ , and  $v$ . Referring to your model, which are correct and which are incorrect?

**14** Prove that any two planar current loops with the same value of  $IA$  will experience the same torque in a magnetic field, regardless of their shapes. In other words, the dipole moment of a current loop can be defined as  $IA$ , regardless of whether its shape is a square.

★

**15** A Helmholtz coil is defined as a pair of identical circular coils separated by a distance,  $h$ , equal to their radius,  $b$ . (Each coil may have more than one turn of wire.) Current circulates in the same direction in each coil, so the fields tend to reinforce each other in the interior region. This configuration has the advantage of being fairly open, so that other apparatus can be easily placed inside and subjected to the field while remaining visible from the outside. The choice of  $h = b$  results in the most uniform possible field near the center. (a) Find the percentage drop in the field at the center of one coil, compared to the full strength at the center of the whole apparatus. (b) What value of  $h$  (not equal to  $b$ ) would make this percentage difference equal to zero?



Problem 15.

**16** (a) In the photo of the vacuum tube apparatus in section 24.2, infer the direction of the magnetic field from the motion of the electron beam. (b) Based on your answer to a, find the direction of the currents in the coils. (c) What direction are the electrons in the coils going? (d) Are the currents in the coils repelling or attracting the currents consisting of the beam inside the tube? Compare with figure z on p. 648.

▷ Solution, p. 974

**17** In the photo of the vacuum tube apparatus in section 24.2, an approximately uniform magnetic field caused circular motion. Is there any other possibility besides a circle? What can happen in general?

★

**18** This problem is now problem 16 on p. 627

**19** In section 24.2 I gave an equation for the magnetic field in the interior of a solenoid, but that equation doesn't give the right answer near the mouths or on the outside. Although in general the computation of the field in these other regions is complicated, it is possible to find a precise, simple result for the field at the center of one of the mouths, using only symmetry and vector addition. What is it? ▷ Solution, p. 975 ★

**20** Prove that in an electromagnetic wave, half the energy is in the electric field and half in the magnetic field.

**21** As promised in section 24.7, this problem will lead you through the steps of finding an equation for the combination of velocities in relativity, generalizing the numerical result found in problem 1. Suppose that A moves relative to B at velocity  $u$ , and B relative to C at  $v$ . We want to find A's velocity  $w$  relative to C, in terms of  $u$  and  $v$ . Suppose that A emits light with a certain frequency. This will be observed by B with a Doppler shift  $D(u)$ . C detects a further shift of  $D(v)$  relative to B. We therefore expect the Doppler shifts simply multiply,  $D(w) = D(u)D(v)$ , and this provides an implicit rule for determining  $w$  if  $u$  and  $v$  are known. (a) Using the expression for  $D$  given in section 24.7.1, write down an equation relating  $u$ ,  $v$ , and  $w$ . (b) Solve for  $w$  in terms of  $u$  and  $v$ . (c) Show that your answer to part b satisfies the correspondence principle. ▷ Solution, p. 975

## Exercise 24: Polarization

Apparatus:

calcite (Iceland spar) crystal

polaroid film

1. Lay the crystal on a piece of paper that has print on it. You will observe a double image. See what happens if you rotate the crystal.

Evidently the crystal does something to the light that passes through it on the way from the page to your eye. One beam of light enters the crystal from underneath, but two emerge from the top; by conservation of energy the energy of the original beam must be shared between them. Consider the following three possible interpretations of what you have observed:

(a) The two new beams differ from each other, and from the original beam, only in energy. Their other properties are the same.

(b) The crystal adds to the light some mysterious new property (not energy), which comes in two flavors, X and Y. Ordinary light doesn't have any of either. One beam that emerges from the crystal has some X added to it, and the other beam has Y.

(c) There is some mysterious new property that is possessed by all light. It comes in two flavors, X and Y, and most ordinary light sources make an equal mixture of type X and type Y light. The original beam is an even mixture of both types, and this mixture is then split up by the crystal into the two purified forms.

In parts 2 and 3 you'll make observations that will allow you to figure out which of these is correct.

2. Now place a polaroid film over the crystal and see what you observe. What happens when you rotate the film in the horizontal plane? Does this observation allow you to rule out any of the three interpretations?

3. Now put the polaroid film under the crystal and try the same thing. Putting together all your observations, which interpretation do you think is correct?

4. Look at an overhead light fixture through the polaroid, and try rotating it. What do you observe? What does this tell you about the light emitted by the lightbulb?

5. Now position yourself with your head under a light fixture and directly over a shiny surface, such as a glossy tabletop. You'll see the lamp's reflection, and the light coming from the lamp

to your eye will have undergone a reflection through roughly a 180-degree angle (i.e. it very nearly reversed its direction). Observe this reflection through the polaroid, and try rotating it. Finally, position yourself so that you are seeing glancing reflections, and try the same thing. Summarize what happens to light with properties X and Y when it is reflected. (This is the principle behind polarizing sunglasses.)

# Chapter 25

## Capacitance and inductance

The long road leading from the light bulb to the computer started with one very important step: the introduction of feedback into electronic circuits. Although the principle of feedback has been understood and applied to mechanical systems for centuries, and to electrical ones since the early twentieth century, for most of us the word evokes an image of Jimi Hendrix (or some more recent guitar hero) intentionally creating earsplitting screeches, or of the school principal doing the same inadvertently in the auditorium. In the guitar example, the musician stands in front of the amp and turns it up so high that the sound waves coming from the speaker come back to the guitar string and make it shake harder. This is an example of *positive* feedback: the harder the string vibrates, the stronger the sound waves, and the stronger the sound waves, the harder the string vibrates. The only limit is the power-handling ability of the amplifier.

Negative feedback is equally important. Your thermostat, for example, provides negative feedback by kicking the heater off when the house gets warm enough, and by firing it up again when it gets too cold. This causes the house's temperature to oscillate back and forth within a certain range. Just as out-of-control exponential freak-outs are a characteristic behavior of positive-feedback systems, oscillation is typical in cases of negative feedback. You have already studied negative feedback extensively in *Vibrations and Waves* in the case of a mechanical system, although we didn't call it that.

### 25.1 Capacitance and inductance

In a mechanical oscillation, energy is exchanged repetitively between potential and kinetic forms, and may also be siphoned off in the form of heat dissipated by friction. In an electrical circuit, resistors are the circuit elements that dissipate heat. What are the electrical analogs of storing and releasing the potential and kinetic energy of a vibrating object? When you think of energy storage in an electrical circuit, you are likely to imagine a battery, but even rechargeable batteries can only go through 10 or 100 cycles before they wear out. In addition, batteries are not able to exchange energy on a short enough time scale for most applications. The circuit in a musical

synthesizer may be called upon to oscillate thousands of times a second, and your microwave oven operates at gigahertz frequencies. Instead of batteries, we generally use capacitors and inductors to store energy in oscillating circuits. Capacitors, which you've already encountered, store energy in electric fields. An inductor does the same with magnetic fields.

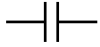
## Capacitors

A capacitor's energy exists in its surrounding electric fields. It is proportional to the square of the field strength, which is proportional to the charges on the plates. If we assume the plates carry charges that are the same in magnitude,  $+q$  and  $-q$ , then the energy stored in the capacitor must be proportional to  $q^2$ . For historical reasons, we write the constant of proportionality as  $1/2C$ ,

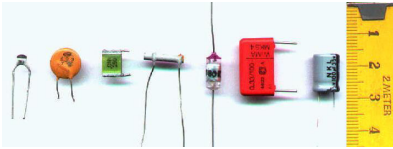
$$E_C = \frac{1}{2C}q^2 \quad .$$

The constant  $C$  is a geometrical property of the capacitor, called its capacitance.

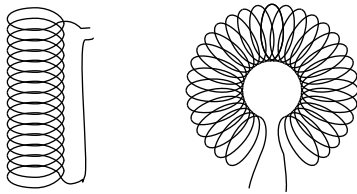
Based on this definition, the units of capacitance must be coulombs squared per joule, and this combination is more conveniently abbreviated as the farad,  $1 \text{ F} = 1 \text{ C}^2/\text{J}$ . "Condenser" is a less formal term for a capacitor. Note that the labels printed on capacitors often use MF to mean  $\mu\text{F}$ , even though MF should really be the symbol for megafarads, not microfarads. Confusion doesn't result from this nonstandard notation, since picofarad and microfarad values are the most common, and it wasn't until the 1990's that even millifarad and farad values became available in practical physical sizes. Figure a shows the symbol used in schematics to represent a capacitor.



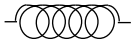
a / The symbol for a capacitor.



b / Some capacitors.



c / Two common geometries for inductors. The cylindrical shape on the left is called a solenoid.



d / The symbol for an inductor.



e / Some inductors.

## Inductors

Any current will create a magnetic field, so in fact every current-carrying wire in a circuit acts as an inductor! However, this type of "stray" inductance is typically negligible, just as we can usually ignore the stray resistance of our wires and only take into account the actual resistors. To store any appreciable amount of magnetic energy, one usually uses a coil of wire designed specifically to be an inductor. All the loops' contribution to the magnetic field add together to make a stronger field. Unlike capacitors and resistors, practical inductors are easy to make by hand. One can for instance spool some wire around a short wooden dowel, put the spool inside a plastic aspirin bottle with the leads hanging out, and fill the bottle with epoxy to make the whole thing rugged. An inductor like this, in the form cylindrical coil of wire, is called a solenoid, c, and a stylized solenoid, d, is the symbol used to represent an inductor in a circuit regardless of its actual geometry.

How much energy does an inductor store? The energy density is

proportional to the square of the magnetic field strength, which is in turn proportional to the current flowing through the coiled wire, so the energy stored in the inductor must be proportional to  $I^2$ . We write  $L/2$  for the constant of proportionality, giving

$$E_L = \frac{L}{2} I^2 \quad .$$

As in the definition of capacitance, we have a factor of  $1/2$ , which is purely a matter of definition. The quantity  $L$  is called the *inductance* of the inductor, and we see that its units must be joules per ampere squared. This clumsy combination of units is more commonly abbreviated as the henry,  $1 \text{ henry} = 1 \text{ J/A}^2$ . Rather than memorizing this definition, it makes more sense to derive it when needed from the definition of inductance. Many people know inductors simply as “coils,” or “chokes,” and will not understand you if you refer to an “inductor,” but they will still refer to  $L$  as the “inductance,” not the “coilance” or “chokeance!”

#### Identical inductances in series

#### example 1

If two inductors are placed in series, any current that passes through the combined double inductor must pass through both its parts. Thus by the definition of inductance, the inductance is doubled as well. In general, inductances in series add, just like resistances. The same kind of reasoning also shows that the inductance of a solenoid is approximately proportional to its length, assuming the number of turns per unit length is kept constant.

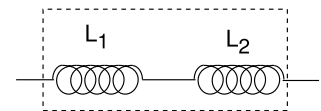
#### Identical capacitances in parallel

#### example 2

When two identical capacitances are placed in parallel, any charge deposited at the terminals of the combined double capacitor will divide itself evenly between the two parts. The electric fields surrounding each capacitor will be half the intensity, and therefore store one quarter the energy. Two capacitors, each storing one quarter the energy, give half the total energy storage. Since capacitance is inversely related to energy storage, this implies that identical capacitances in parallel give double the capacitance. In general, capacitances in parallel add. This is unlike the behavior of inductors and resistors, for which series configurations give addition.

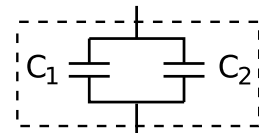
This is consistent with the fact that the capacitance of a single parallel-plate capacitor is proportional to the area of the plates. If we have two parallel-plate capacitors, and we combine them in parallel and bring them very close together side by side, we have produced a single capacitor with plates of double the area, and it has approximately double the capacitance.

Inductances in parallel and capacitances in series are explored in homework problems 4 and 6.



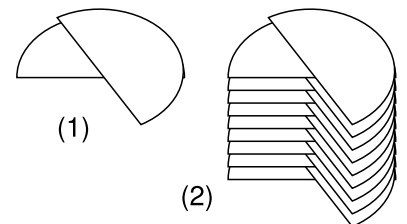
$$L = L_1 + L_2$$

f / Inductances in series add.



$$C = C_1 + C_2$$

g / Capacitances in parallel add.



h / A variable capacitor.



Figure h/1 shows the construction of a variable capacitor out of two parallel semicircles of metal. One plate is fixed, while the other can be rotated about their common axis with a knob. The opposite charges on the two plates are attracted to one another, and therefore tend to gather in the overlapping area. This overlapping area, then, is the only area that effectively contributes to the capacitance, and turning the knob changes the capacitance. The simple design can only provide very small capacitance values, so in practice one usually uses a bank of capacitors, wired in parallel, with all the moving parts on the same shaft.

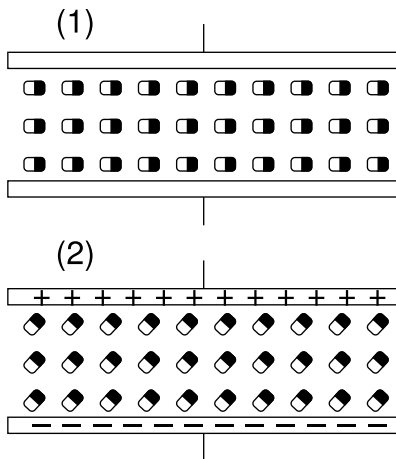
### Discussion questions

**A** Suppose that two parallel-plate capacitors are wired in parallel, and are placed very close together, side by side, so that their fields overlap. Will the resulting capacitance be too small, or too big? Could you twist the circuit into a different shape and make the effect be the other way around, or make the effect vanish? How about the case of two inductors in series?

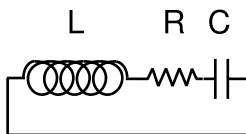
**B** Most practical capacitors do not have an air gap or vacuum gap between the plates; instead, they have an insulating substance called a dielectric. We can think of the molecules in this substance as dipoles that are free to rotate (at least a little), but that are not free to move around, since it is a solid. The figure shows a highly stylized and unrealistic way of visualizing this. We imagine that all the dipoles are initially turned sideways, (1), and that as the capacitor is charged, they all respond by turning through a certain angle, (2). (In reality, the scene might be much more random, and the alignment effect much weaker.)

For simplicity, imagine inserting just one electric dipole into the vacuum gap. For a given amount of charge on the plates, how does this affect the amount of energy stored in the electric field? How does this affect the capacitance?

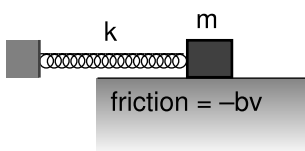
Now redo the analysis in terms of the mechanical work needed in order to charge up the plates.



i / Discussion question B.



j / A series LRC circuit.



k / A mechanical analogy for the LRC circuit.

## 25.2 Oscillations

Figure j shows the simplest possible oscillating circuit. For any useful application it would actually need to include more components. For example, if it was a radio tuner, it would need to be connected to an antenna and an amplifier. Nevertheless, all the essential physics is there.

We can analyze it without any sweat or tears whatsoever, simply by constructing an analogy with a mechanical system. In a mechanical oscillator,  $k$ , we have two forms of stored energy,

$$E_{\text{spring}} = \frac{1}{2} kx^2 \quad (1)$$

$$K = \frac{1}{2} mv^2 \quad (2)$$

In the case of a mechanical oscillator, we have usually assumed a friction force of the form that turns out to give the nicest mathematical results,  $F = -bv$ . In the circuit, the dissipation of energy into heat occurs via the resistor, with no mechanical force involved, so in order to make the analogy, we need to restate the role of the friction force in terms of energy. The power dissipated by friction equals the mechanical work it does in a time interval  $\Delta t$ , divided by  $\Delta t$ ,  $P = W/\Delta t = F\Delta x/\Delta t = Fv = -bv^2$ , so

$$\text{rate of heat dissipation} = -bv^2 \quad . \quad (3)$$

*self-check A*

Equation (1) has  $x$  squared, and equations (2) and (3) have  $v$  squared. Because they're squared, the results don't depend on whether these variables are positive or negative. Does this make physical sense?  $\triangleright$

Answer, p. 978

In the circuit, the stored forms of energy are

$$E_C = \frac{1}{2C}q^2 \quad (1')$$

$$E_L = \frac{1}{2}LI^2 \quad , \quad (2')$$

and the rate of heat dissipation in the resistor is

$$\text{rate of heat dissipation} = -RI^2 \quad . \quad (3')$$

Comparing the two sets of equations, we first form analogies between quantities that represent the state of the system at some moment in time:

$$\begin{aligned} x &\leftrightarrow q \\ v &\leftrightarrow I \end{aligned}$$

*self-check B*

How is  $v$  related mathematically to  $x$ ? How is  $I$  connected to  $q$ ? Are the two relationships analogous?  $\triangleright$  Answer, p. 978

Next we relate the ones that describe the system's permanent characteristics:

$$\begin{aligned} k &\leftrightarrow 1/C \\ m &\leftrightarrow L \\ b &\leftrightarrow R \end{aligned}$$

Since the mechanical system naturally oscillates with a period  $T = 2\pi\sqrt{m/k}$ , we can immediately solve the electrical version by analogy, giving

$$T = 2\pi\sqrt{LC} \quad .$$

Rather than period,  $T$ , and frequency,  $f$ , it turns out to be more convenient if we work with the quantity  $\omega = 2\pi f$ , which can be interpreted as the number of radians per second. Then

$$\omega = \frac{1}{\sqrt{LC}} \quad .$$

Since the resistance  $R$  is analogous to  $b$  in the mechanical case, we find that the  $Q$  (quality factor, not charge) of the resonance is inversely proportional to  $R$ , and the width of the resonance is directly proportional to  $R$ .

---

*Tuning a radio receiver* *example 4*

A radio receiver uses this kind of circuit to pick out the desired station. Since the receiver resonates at a particular frequency, stations whose frequencies are far off will not excite any response in the circuit. The value of  $R$  has to be small enough so that only one station at a time is picked up, but big enough so that the tuner isn't too touchy. The resonant frequency can be tuned by adjusting either  $L$  or  $C$ , but variable capacitors are easier to build than variable inductors.

---

*A numerical calculation* *example 5*

The phone company sends more than one conversation at a time over the same wire, which is accomplished by shifting each voice signal into different range of frequencies during transmission. The number of signals per wire can be maximized by making each range of frequencies (known as a bandwidth) as small as possible. It turns out that only a relatively narrow range of frequencies is necessary in order to make a human voice intelligible, so the phone company filters out all the extreme highs and lows. (This is why your phone voice sounds different from your normal voice.)

▷ If the filter consists of an LRC circuit with a broad resonance centered around 1.0 kHz, and the capacitor is 1  $\mu\text{F}$  (microfarad), what inductance value must be used?

▷ Solving for  $L$ , we have

$$\begin{aligned} L &= \frac{1}{C\omega^2} \\ &= \frac{1}{(10^{-6} \text{ F})(2\pi \times 10^3 \text{ s}^{-1})^2} \\ &= 2.5 \times 10^{-3} \text{ F}^{-1} \text{ s}^2 \end{aligned}$$

Checking that these really are the same units as henries is a little tedious, but it builds character:

$$\begin{aligned} \text{F}^{-1} \text{s}^2 &= (\text{C}^2/\text{J})^{-1} \text{s}^2 \\ &= \text{J} \cdot \text{C}^{-2} \text{s}^2 \\ &= \text{J}/\text{A}^2 \\ &= \text{H} \end{aligned}$$

The result is 25 mH (millihenries).

This is actually quite a large inductance value, and would require a big, heavy, expensive coil. In fact, there is a trick for making this kind of circuit small and cheap. There is a kind of silicon chip called an op-amp, which, among other things, can be used to simulate the behavior of an inductor. The main limitation of the op-amp is that it is restricted to low-power applications.

## 25.3 Voltage and current

What is physically happening in one of these oscillating circuits? Let's first look at the mechanical case, and then draw the analogy to the circuit. For simplicity, let's ignore the existence of damping, so there is no friction in the mechanical oscillator, and no resistance in the electrical one.

Suppose we take the mechanical oscillator and pull the mass away from equilibrium, then release it. Since friction tends to resist the spring's force, we might naively expect that having zero friction would allow the mass to leap instantaneously to the equilibrium position. This can't happen, however, because the mass would have to have infinite velocity in order to make such an instantaneous leap. Infinite velocity would require infinite kinetic energy, but the only kind of energy that is available for conversion to kinetic is the energy stored in the spring, and that is finite, not infinite. At each step on its way back to equilibrium, the mass's velocity is controlled exactly by the amount of the spring's energy that has so far been converted into kinetic energy. After the mass reaches equilibrium, it overshoots due to its own momentum. It performs identical oscillations on both sides of equilibrium, and it never loses amplitude because friction is not available to convert mechanical energy into heat.

Now with the electrical oscillator, the analog of position is charge. Pulling the mass away from equilibrium is like depositing charges  $+q$  and  $-q$  on the plates of the capacitor. Since resistance tends to resist the flow of charge, we might imagine that with no friction present, the charge would instantly flow through the inductor (which is, after all, just a piece of wire), and the capacitor would discharge instantly. However, such an instant discharge is impossible, because it would require infinite current for one instant. Infinite current would create infinite magnetic fields surrounding the inductor, and these fields would have infinite energy. Instead, the rate of flow of current is controlled at each instant by the relationship between the amount of energy stored in the magnetic field and the amount of current that must exist in order to have that strong a field. After the capacitor reaches  $q = 0$ , it overshoots. The circuit has its own kind of electrical "inertia," because if charge was to stop flowing, there would have to be zero current through the inductor. But the current in the inductor must be related to the amount of energy stored in

its magnetic fields. When the capacitor is at  $q = 0$ , all the circuit's energy is in the inductor, so it must therefore have strong magnetic fields surrounding it and quite a bit of current going through it.

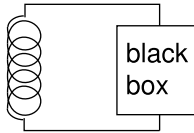
The only thing that might seem spooky here is that we used to speak as if the current in the inductor caused the magnetic field, but now it sounds as if the field causes the current. Actually this is symptomatic of the elusive nature of cause and effect in physics. It's equally valid to think of the cause and effect relationship in either way. This may seem unsatisfying, however, and for example does not really get at the question of what brings about a voltage difference across the resistor (in the case where the resistance is finite); there must be such a voltage difference, because without one, Ohm's law would predict zero current through the resistor.

Voltage, then, is what is really missing from our story so far.

Let's start by studying the voltage across a capacitor. Voltage is electrical potential energy per unit charge, so the voltage difference between the two plates of the capacitor is related to the amount by which its energy would increase if we increased the absolute values of the charges on the plates from  $q$  to  $q + \Delta q$ :

$$\begin{aligned} V_C &= (E_{q+\Delta q} - E_q) / \Delta q \\ &= \frac{\Delta E_C}{\Delta q} \\ &= \frac{\Delta}{\Delta q} \left( \frac{1}{2C} q^2 \right) \\ &= \frac{q}{C} \end{aligned}$$

Many books use this as the definition of capacitance. This equation, by the way, probably explains the historical reason why  $C$  was defined so that the energy was *inversely* proportional to  $C$  for a given value of  $C$ : the people who invented the definition were thinking of a capacitor as a device for storing charge rather than energy, and the amount of charge stored for a fixed voltage (the charge "capacity") is proportional to  $C$ .



1 / The inductor releases energy and gives it to the black box.

In the case of an inductor, we know that if there is a steady, constant current flowing through it, then the magnetic field is constant, and so is the amount of energy stored; no energy is being exchanged between the inductor and any other circuit element. But what if the current is changing? The magnetic field is proportional to the current, so a change in one implies a change in the other. For concreteness, let's imagine that the magnetic field and the current are both decreasing. The energy stored in the magnetic field is therefore decreasing, and by conservation of energy, this energy can't just go away — some other circuit element must be taking energy from the inductor. The simplest example, shown in figure 1, is a series circuit consisting of the inductor plus one other circuit element. It

doesn't matter what this other circuit element is, so we just call it a black box, but if you like, we can think of it as a resistor, in which case the energy lost by the inductor is being turned into heat by the resistor. The junction rule tells us that both circuit elements have the same current through them, so  $I$  could refer to either one, and likewise the loop rule tells us  $V_{inductor} + V_{black\ box} = 0$ , so the two voltage drops have the same absolute value, which we can refer to as  $V$ . Whatever the black box is, the rate at which it is taking energy from the inductor is given by  $|P| = |IV|$ , so

$$\begin{aligned} |IV| &= \left| \frac{\Delta E_L}{\Delta t} \right| \\ &= \left| \frac{\Delta}{\Delta t} \left( \frac{1}{2} L I^2 \right) \right| \\ &= \left| L I \frac{\Delta I}{\Delta t} \right| \quad , \end{aligned}$$

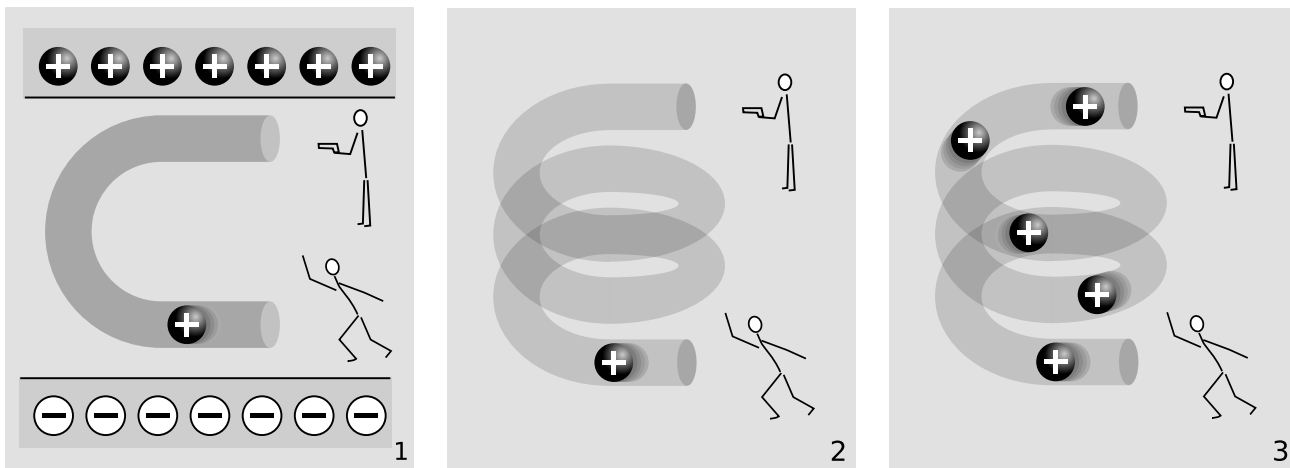
or

$$|V| = \left| L \frac{\Delta I}{\Delta t} \right| \quad ,$$

which in many books is taken to be the definition of inductance. The direction of the voltage drop (plus or minus sign) is such that the inductor resists the change in current.

There's one very intriguing thing about this result. Suppose, for concreteness, that the black box in figure 1 is a resistor, and that the inductor's energy is decreasing, and being converted into heat in the resistor. The voltage drop across the resistor indicates that it has an electric field across it, which is driving the current. But where is this electric field coming from? There are no charges anywhere that could be creating it! What we've discovered is one special case of a more general principle, the principle of induction: a changing magnetic field creates an electric field, which is in addition to any electric field created by charges. (The reverse is also true: any electric field that changes over time creates a magnetic field.) Induction forms the basis for such technologies as the generator and the transformer, and ultimately it leads to the existence of light, which is a wave pattern in the electric and magnetic fields. These are all topics for chapter 24, but it's truly remarkable that we could come to this conclusion without yet having learned any details about magnetism.

The cartoons in figure m compares electric fields made by charges, 1, to electric fields made by changing magnetic fields, 2-3. In m/1, two physicists are in a room whose ceiling is positively charged and whose floor is negatively charged. The physicist on the bottom



m / Electric fields made by charges, 1, and by changing magnetic fields, 2 and 3.

throws a positively charged bowling ball into the curved pipe. The physicist at the top uses a radar gun to measure the speed of the ball as it comes out of the pipe. They find that the ball has slowed down by the time it gets to the top. By measuring the change in the ball's kinetic energy, the two physicists are acting just like a voltmeter. They conclude that the top of the tube is at a higher voltage than the bottom of the pipe. A difference in voltage indicates an electric field, and this field is clearly being caused by the charges in the floor and ceiling.

In m/2, there are no charges anywhere in the room except for the charged bowling ball. Moving charges make magnetic fields, so there is a magnetic field surrounding the helical pipe while the ball is moving through it. A magnetic field has been created where there was none before, and that field has energy. Where could the energy have come from? It can only have come from the ball itself, so the ball must be losing kinetic energy. The two physicists working together are again acting as a voltmeter, and again they conclude that there is a voltage difference between the top and bottom of the pipe. This indicates an electric field, but this electric field can't have been created by any charges, because there aren't any in the room. This electric field was created by the change in the magnetic field.

The bottom physicist keeps on throwing balls into the pipe, until the pipe is full of balls, m/3, and finally a steady current is established. While the pipe was filling up with balls, the energy in the magnetic field was steadily increasing, and that energy was being stolen from the balls' kinetic energy. But once a steady current is established, the energy in the magnetic field is no longer changing. The balls no longer have to give up energy in order to build up the field, and the physicist at the top finds that the balls are exiting the pipe at full speed again. There is no voltage difference any more. Although

there is a current,  $\Delta I/\Delta t$  is zero.

### Discussion question

**A** What happens when the physicist at the bottom in figure m/3 starts getting tired, and decreases the current?

## 25.4 Decay

Up until now I've soft-pedaled the fact that by changing the characteristics of an oscillator, it is possible to produce non-oscillatory behavior. For example, imagine taking the mass-on-a-spring system and making the spring weaker and weaker. In the limit of small  $k$ , it's as though there was no spring whatsoever, and the behavior of the system is that if you kick the mass, it simply starts slowing down. For friction proportional to  $v$ , as we've been assuming, the result is that the velocity approaches zero, but never actually reaches zero. This is unrealistic for the mechanical oscillator, which will not have vanishing friction at low velocities, but it is quite realistic in the case of an electrical circuit, for which the voltage drop across the resistor really does approach zero as the current approaches zero.

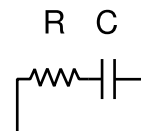
Electrical circuits can exhibit all the same behavior. For simplicity we will analyze only the cases of LRC circuits with  $L = 0$  or  $C = 0$ .

### The RC circuit

We first analyze the RC circuit, n. In reality one would have to “kick” the circuit, for example by briefly inserting a battery, in order to get any interesting behavior. We start with Ohm's law and the equation for the voltage across a capacitor:

$$V_R = IR$$

$$V_C = q/C$$



n / An RC circuit.

The loop rule tells us

$$V_R + V_C = 0 \quad ,$$

and combining the three equations results in a relationship between  $q$  and  $I$ :

$$I = -\frac{1}{RC}q$$

The negative sign tells us that the current tends to reduce the charge on the capacitor, i.e. to discharge it. It makes sense that the current is proportional to  $q$ : if  $q$  is large, then the attractive forces between the  $+q$  and  $-q$  charges on the plates of the capacitor are large, and charges will flow more quickly through the resistor in order to reunite. If there was zero charge on the capacitor plates, there would be no reason for current to flow. Since amperes, the unit of current, are the same as coulombs per second, it appears that the quantity



$RC$  must have units of seconds, and you can check for yourself that this is correct.  $RC$  is therefore referred to as the time constant of the circuit.

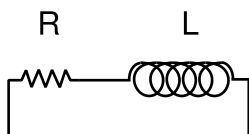
How exactly do  $I$  and  $q$  vary with time? Rewriting  $I$  as  $\Delta q/\Delta t$ , we have

$$\frac{\Delta q}{\Delta t} = -\frac{1}{RC}q \quad .$$

This equation describes a function  $q(t)$  that always gets smaller over time, and whose rate of decrease is big at first, when  $q$  is big, but gets smaller and smaller as  $q$  approaches zero. As an example of this type of mathematical behavior, we could imagine a man who has 1024 weeds in his backyard, and resolves to pull out half of them every day. On the first day, he pulls out half, and has 512 left. The next day, he pulls out half of the remaining ones, leaving 256. The sequence continues exponentially: 128, 64, 32, 16, 8, 4, 2, 1. Returning to our electrical example, the function  $q(t)$  apparently needs to be an exponential, which we can write in the form  $ae^{bt}$ , where  $e = 2.718\dots$  is the base of natural logarithms. We could have written it with base 2, as in the story of the weeds, rather than base  $e$ , but the math later on turns out simpler if we use  $e$ . It doesn't make sense to plug a number that has units into a function like an exponential, so  $bt$  must be unitless, and  $b$  must therefore have units of inverse seconds. The number  $b$  quantifies how fast the exponential decay is. The only physical parameters of the circuit on which  $b$  could possibly depend are  $R$  and  $C$ , and the only way to put units of ohms and farads together to make units of inverse seconds is by computing  $1/RC$ . Well, actually we could use  $7/RC$  or  $3\pi/RC$ , or any other unitless number divided by  $RC$ , but this is where the use of base  $e$  comes in handy: for base  $e$ , it turns out that the correct unitless constant *is* 1. Thus our solution is

$$q = q_0 \exp\left(-\frac{t}{RC}\right) \quad .$$

The number  $RC$ , with units of seconds, is called the  $RC$  time constant of the circuit, and it tells us how long we have to wait if we want the charge to fall off by a factor of  $1/e$ .



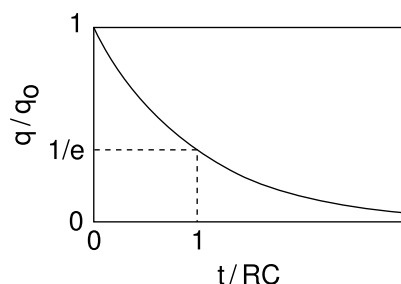
p / An RL circuit.

### The RL circuit

The RL circuit, p, can be attacked by similar methods, and it can easily be shown that it gives

$$I = I_0 \exp\left(-\frac{R}{L}t\right) \quad .$$

The RL time constant equals  $L/R$ .



o / Over a time interval  $RC$ , the charge on the capacitor is reduced by a factor of  $e$ .

When we suddenly break an RL circuit, what will happen? It might seem that we're faced with a paradox, since we only have two forms of energy, magnetic energy and heat, and if the current stops suddenly, the magnetic field must collapse suddenly. But where does the lost magnetic energy go? It can't go into resistive heating of the resistor, because the circuit has now been broken, and current can't flow!

The way out of this conundrum is to recognize that the open gap in the circuit has a resistance which is large, but not infinite. This large resistance causes the RL time constant  $L/R$  to be very small. The current thus continues to flow for a very brief time, and flows straight across the air gap where the circuit has been opened. In other words, there is a spark!

We can determine based on several different lines of reasoning that the voltage drop from one end of the spark to the other must be very large. First, the air's resistance is large, so  $V = IR$  requires a large voltage. We can also reason that all the energy in the magnetic field is being dissipated in a short time, so the power dissipated in the spark,  $P = IV$ , is large, and this requires a large value of  $V$ . ( $I$  isn't large — it is decreasing from its initial value.) Yet a third way to reach the same result is to consider the equation  $V_L = \Delta I / \Delta t$ : since the time constant is short, the time derivative  $\Delta I / \Delta t$  is large.

This is exactly how a car's spark plugs work. Another application is to electrical safety: it can be dangerous to break an inductive circuit suddenly, because so much energy is released in a short time. There is also no guarantee that the spark will discharge across the air gap; it might go through your body instead, since your body might have a lower resistance.

### Discussion question

**A** A gopher gnaws through one of the wires in the DC lighting system in your front yard, and the lights turn off. At the instant when the circuit becomes open, we can consider the bare ends of the wire to be like the plates of a capacitor, with an air gap (or gopher gap) between them. What kind of capacitance value are we talking about here? What would this tell you about the  $RC$  time constant?

## 25.5 Impedance

So far we have been thinking in terms of the free oscillations of a circuit. This is like a mechanical oscillator that has been kicked but then left to oscillate on its own without any external force to keep the vibrations from dying out. Suppose an LRC circuit is driven with a sinusoidally varying voltage, such as will occur when a radio tuner is hooked up to a receiving antenna. We know that a current will flow in the circuit, and we know that there will be resonant behavior, but it is not necessarily simple to relate current to voltage in the most general case. Let's start instead with the special cases of LRC circuits consisting of only a resistance, only a capacitance, or only an inductance. We are interested only in the steady-state response.

The purely resistive case is easy. Ohm's law gives

$$I = \frac{V}{R} \quad .$$

In the purely capacitive case, the relation  $V = q/C$  lets us calculate

$$\begin{aligned} I &= \frac{\Delta q}{\Delta t} \\ &= C \frac{\Delta V}{\Delta t} \quad . \end{aligned}$$

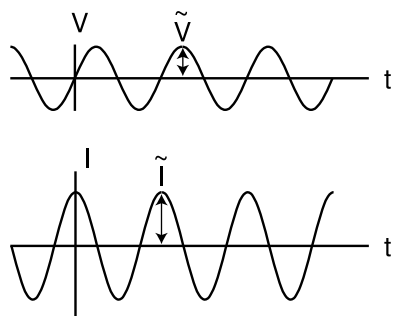
If the voltage varies as, for example,  $V(t) = \tilde{V} \sin(\omega t)$ , then it can be shown using calculus that the current will be  $I(t) = \omega C \tilde{V} \cos(\omega t)$ , so the maximum current is  $\tilde{I} = \omega C \tilde{V}$ . By analogy with Ohm's law, we can then write

$$\tilde{I} = \frac{\tilde{V}}{Z_C} \quad ,$$

where the quantity

$$Z_C = \frac{1}{\omega C} \quad , \quad [\text{impedance of a capacitor}]$$

having units of ohms, is called the *impedance* of the capacitor at this frequency. Note that it is only the *maximum* current,  $\tilde{I}$ , that is proportional to the *maximum* voltage,  $\tilde{V}$ , so the capacitor is not behaving like a resistor. The maxima of  $V$  and  $I$  occur at different times, as shown in figure q. It makes sense that the impedance becomes infinite at zero frequency. Zero frequency means that it would take an infinite time before the voltage would change by any amount. In other words, this is like a situation where the capacitor has been connected across the terminals of a battery and been allowed to settle down to a state where there is constant charge on both terminals. Since the electric fields between the plates are constant, there is no energy being added to or taken out of the field. A capacitor that can't exchange energy with any other circuit component is nothing more than a broken (open) circuit.



q / In a capacitor, the current is  $90^\circ$  ahead of the voltage in phase.

### self-check C

Why can't a capacitor have its impedance printed on it along with its capacitance?

▷ Answer, p. 978

Similar math gives

$$Z_L = \omega L \quad [\text{impedance of an inductor}]$$

for an inductor. It makes sense that the inductor has lower impedance at lower frequencies, since at zero frequency there is no change in the magnetic field over time. No energy is added to or released from the magnetic field, so there are no induction effects, and the inductor acts just like a piece of wire with negligible resistance. The term “choke” for an inductor refers to its ability to “choke out” high frequencies.

The phase relationships shown in figures q and r can be remembered using my own mnemonic, “eVIL,” which shows that the voltage (V) leads the current (I) in an inductive circuit, while the opposite is true in a capacitive one. A more traditional mnemonic is “ELI the ICE man,” which uses the notation E for emf, a concept closely related to voltage.

### Low-pass and high-pass filters

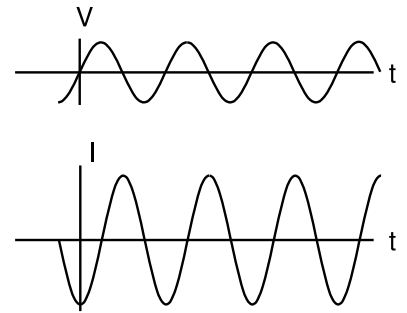
### example 7

An LRC circuit only responds to a certain range (band) of frequencies centered around its resonant frequency. As a filter, this is known as a bandpass filter. If you turn down both the bass and the treble on your stereo, you have created a bandpass filter.

To create a high-pass or low-pass filter, we only need to insert a capacitor or inductor, respectively, in series. For instance, a very basic surge protector for a computer could be constructed by inserting an inductor in series with the computer. The desired 60 Hz power from the wall is relatively low in frequency, while the surges that can damage your computer show much more rapid time variation. Even if the surges are not sinusoidal signals, we can think of a rapid “spike” qualitatively as if it was very high in frequency — like a high-frequency sine wave, it changes very rapidly.

Inductors tend to be big, heavy, expensive circuit elements, so a simple surge protector would be more likely to consist of a capacitor in *parallel* with the computer. (In fact one would normally just connect one side of the power circuit to ground via a capacitor.) The capacitor has a very high impedance at the low frequency of the desired 60 Hz signal, so it siphons off very little of the current. But for a high-frequency signal, the capacitor's impedance is very small, and it acts like a zero-impedance, easy path into which the current is diverted.

The main things to be careful about with impedance are that (1) the concept only applies to a circuit that is being driven sinusoidally, (2)



r / The current through an inductor lags behind the voltage by a phase angle of  $90^\circ$ .

the impedance of an inductor or capacitor is frequency-dependent, and (3) impedances in parallel and series don't combine according to the same rules as resistances. It is possible, however, to get around the third limitation, as discussed in subsection .

### Discussion questions

**A** Figure q on page 696 shows the voltage and current for a capacitor. Sketch the  $q$ - $t$  graph, and use it to give a physical explanation of the phase relationship between the voltage and current. For example, why is the current zero when the voltage is at a maximum or minimum?

**B** Relate the features of the graph in figure r on page 697 to the story told in cartoons in figure m/2-3 on page 692.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** If an FM radio tuner consisting of an LRC circuit contains a  $1.0\ \mu\text{H}$  inductor, what range of capacitances should the variable capacitor be able to provide? ✓

**2** (a) Show that the equation  $V_L = L\Delta I/\Delta t$  has the right units.

(b) Verify that  $RC$  has units of time.

(c) Verify that  $L/R$  has units of time.

**3** Find the energy stored in a capacitor in terms of its capacitance and the voltage difference across it. ✓

**4** Find the inductance of two identical inductors in parallel.

**5** The wires themselves in a circuit can have resistance, inductance, and capacitance. Would “stray” inductance and capacitance be most important for low-frequency or for high-frequency circuits? For simplicity, assume that the wires act like they’re in *series* with an inductor or capacitor.

**6** (a) Find the capacitance of two identical capacitors in series.

(b) Based on this, how would you expect the capacitance of a parallel-plate capacitor to depend on the distance between the plates?

**7** Find the capacitance of the surface of the earth, assuming there is an outer spherical “plate” at infinity. (In reality, this outer plate would just represent some distant part of the universe to which we carried away some of the earth’s charge in order to charge up the earth.) ✓

**8** Starting from the relation  $V = L\Delta I/\Delta t$  for the voltage difference across an inductor, show that an inductor has an impedance equal to  $L\omega$ .





a / Marie and Pierre Curie were the first to purify radium in significant quantities. Radium's intense radioactivity made possible the experiments that led to the modern planetary model of the atom, in which electrons orbit a nucleus made of protons and neutrons.

## Chapter 26

# The atom and $E=mc^2$

### 26.1 Atoms

I was brought up to look at the atom as a nice, hard fellow, red or grey in color according to taste.

*Rutherford*

#### The chemical elements

How would one find out what types of atoms there were? Today, it doesn't seem like it should have been very difficult to work out an experimental program to classify the types of atoms. For each type of atom, there should be a corresponding element, i.e., a pure substance made out of nothing but that type of atom. Atoms are supposed to be unsplitable, so a substance like milk could not possibly be elemental, since churning it vigorously causes it to split up into two separate substances: butter and whey. Similarly, rust could not be an element, because it can be made by combining two substances: iron and oxygen. Despite its apparent reasonableness, no such program was carried out until the eighteenth century.

By 1900, however, chemists had done a reasonably good job of find-



$$\frac{m_{\text{He}}}{m_{\text{H}}} = 3.97$$

$$\frac{m_{\text{Ne}}}{m_{\text{H}}} = 20.01$$

$$\frac{m_{\text{Sc}}}{m_{\text{H}}} = 44.60$$

b / Examples of masses of atoms compared to that of hydrogen. Note how some, but not all, are close to integers.

ing out what the elements were. They also had determined the ratios of the different atoms' masses fairly accurately. A typical technique would be to measure how many grams of sodium (Na) would combine with one gram of chlorine (Cl) to make salt (NaCl). (This assumes you've already decided based on other evidence that salt consisted of equal numbers of Na and Cl atoms.) The masses of individual atoms, as opposed to the mass ratios, were known only to within a few orders of magnitude based on indirect evidence, and plenty of physicists and chemists denied that individual atoms were anything more than convenient symbols.

The following table gives the atomic masses of all the elements, on a standard scale in which the mass of hydrogen is very close to 1.0. The absolute calibration of the whole scale was only very roughly known for a long time, but was eventually tied down, with the mass of a hydrogen atom being determined to be about  $1.7 \times 10^{-27}$  kg.

Ag	107.9	Eu	152.0	Mo	95.9	Sc	45.0
Al	27.0	F	19.0	N	14.0	Se	79.0
Ar	39.9	Fe	55.8	Na	23.0	Si	28.1
As	74.9	Ga	69.7	Nb	92.9	Sn	118.7
Au	197.0	Gd	157.2	Nd	144.2	Sr	87.6
B	10.8	Ge	72.6	Ne	20.2	Ta	180.9
Ba	137.3	H	1.0	Ni	58.7	Tb	158.9
Be	9.0	He	4.0	O	16.0	Te	127.6
Bi	209.0	Hf	178.5	Os	190.2	Ti	47.9
Br	79.9	Hg	200.6	P	31.0	Tl	204.4
C	12.0	Ho	164.9	Pb	207.2	Tm	168.9
Ca	40.1	In	114.8	Pd	106.4	U	238
Ce	140.1	Ir	192.2	Pt	195.1	V	50.9
Cl	35.5	K	39.1	Pr	140.9	W	183.8
Co	58.9	Kr	83.8	Rb	85.5	Xe	131.3
Cr	52.0	La	138.9	Re	186.2	Y	88.9
Cs	132.9	Li	6.9	Rh	102.9	Yb	173.0
Cu	63.5	Lu	175.0	Ru	101.1	Zn	65.4
Dy	162.5	Mg	24.3	S	32.1	Zr	91.2
Er	167.3	Mn	54.9	Sb	121.8		

### Making sense of the elements

As the information accumulated, the challenge was to find a way of systematizing it; the modern scientist's aesthetic sense rebels against complication. This hodgepodge of elements was an embarrassment. One contemporary observer, William Crookes, described the elements as extending "before us as stretched the wide Atlantic before the gaze of Columbus, mocking, taunting and murmuring strange riddles, which no man has yet been able to solve." It wasn't long before people started recognizing that many atoms' masses were nearly integer multiples of the mass of hydrogen, the lightest element. A few excitable types began speculating that hydrogen was the basic building block, and that the heavier elements were made of clusters of hydrogen. It wasn't long, however, before their parade was rained on by more accurate measurements, which showed that not all of the elements had atomic masses that were near integer

c / A modern periodic table. Elements in the same column have similar chemical properties. The modern atomic numbers, discussed on p. 719, were not known in Mendeleev's time, since the table could be flipped in various ways.

1 H																	2 He			
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne			
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar			
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr			
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe			
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn			
87 Fr	88 Ra	89 Ac	104 Rf	105 Ha	106	107	108	109	110	111	112	113	114	115	116	117	118			
		*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu				
		**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr				

Chemistry professor Dmitri Mendeleev, preparing his lectures in 1869, wanted to find some way to organize his knowledge for his students to make it more understandable. He wrote the names of all the elements on cards and began arranging them in different ways on his desk, trying to find an arrangement that would make sense of the muddle. The row-and-column scheme he came up with is essentially our modern periodic table. The columns of the modern version represent groups of elements with similar chemical properties, and each row is more massive than the one above it. Going across each row, this almost always resulted in placing the atoms in sequence by weight as well. What made the system significant was its predictive value. There were three places where Mendeleev had to leave gaps in his checkerboard to keep chemically similar elements in the same column. He predicted that elements would exist to fill these gaps, and extrapolated or interpolated from other elements in the same column to predict their numerical properties, such as masses, boiling points, and densities. Mendeleev's professional stock skyrocketed when his three elements (later named gallium, scandium and germanium) were discovered and found to have very nearly the properties he had predicted.

One thing that Mendeleev's table made clear was that mass was not the basic property that distinguished atoms of different elements. To make his table work, he had to deviate from ordering the elements strictly by mass. For instance, iodine atoms are lighter than tellurium, but Mendeleev had to put iodine after tellurium so that it would lie in a column with chemically similar elements.

The success of the kinetic theory of heat was taken as strong evidence that, in addition to the motion of any object as a whole, there is an invisible type of motion all around us: the random motion of atoms within each object. But many conservatives were not con-

vinced that atoms really existed. Nobody had ever seen one, after all. It wasn't until generations after the kinetic theory of heat was developed that it was demonstrated conclusively that atoms really existed and that they participated in continuous motion that never died out.

The smoking gun to prove atoms were more than mathematical abstractions came when some old, obscure observations were reexamined by an unknown Swiss patent clerk named Albert Einstein. A botanist named Brown, using a microscope that was state of the art in 1827, observed tiny grains of pollen in a drop of water on a microscope slide, and found that they jumped around randomly for no apparent reason. Wondering at first if the pollen he'd assumed to be dead was actually alive, he tried looking at particles of soot, and found that the soot particles also moved around. The same results would occur with any small grain or particle suspended in a liquid. The phenomenon came to be referred to as Brownian motion, and its existence was filed away as a quaint and thoroughly unimportant fact, really just a nuisance for the microscopist.

It wasn't until 1906 that Einstein found the correct interpretation for Brown's observation: the water molecules were in continuous random motion, and were colliding with the particle all the time, kicking it in random directions. After all the millennia of speculation about atoms, at last there was solid proof. Einstein's calculations dispelled all doubt, since he was able to make accurate predictions of things like the average distance traveled by the particle in a certain amount of time. (Einstein received the Nobel Prize not for his theory of relativity but for his papers on Brownian motion and the photoelectric effect.)

### Discussion questions

- A** How could knowledge of the size of an individual aluminum atom be used to infer an estimate of its mass, or vice versa?
- B** How could one test Einstein's interpretation of Brownian motion by observing it at different temperatures?

## 26.2 Quantization of charge

Proving that atoms actually existed was a big accomplishment, but demonstrating their existence was different from understanding their properties. Note that the Brown-Einstein observations had nothing at all to do with electricity, and yet we know that matter is inherently electrical, and we have been successful in interpreting certain electrical phenomena in terms of mobile positively and negatively charged particles. Are these particles atoms? Parts of atoms? Particles that are entirely separate from atoms? It is perhaps premature to attempt to answer these questions without any conclusive evidence in favor of the charged-particle model of electricity.

Strong support for the charged-particle model came from a 1911 experiment by physicist Robert Millikan at the University of Chicago. Consider a jet of droplets of perfume or some other liquid made by blowing it through a tiny pinhole. The droplets emerging from the pinhole must be smaller than the pinhole, and in fact most of them are even more microscopic than that, since the turbulent flow of air tends to break them up. Millikan reasoned that the droplets would acquire a little bit of electric charge as they rubbed against the channel through which they emerged, and if the charged-particle model of electricity was right, the charge might be split up among so many minuscule liquid drops that a single drop might have a total charge amounting to an excess of only a few charged particles — perhaps an excess of one positive particle on a certain drop, or an excess of two negative ones on another.

Millikan's ingenious apparatus, e, consisted of two metal plates, which could be electrically charged as needed. He sprayed a cloud of oil droplets into the space between the plates, and selected one drop through a microscope for study. First, with no charge on the plates, he would determine the drop's mass by letting it fall through the air and measuring its terminal velocity, i.e., the velocity at which the force of air friction canceled out the force of gravity. The force of air drag on a slowly moving sphere had already been found by experiment to be  $bvr^2$ , where  $b$  was a constant. Setting the total force equal to zero when the drop is at terminal velocity gives

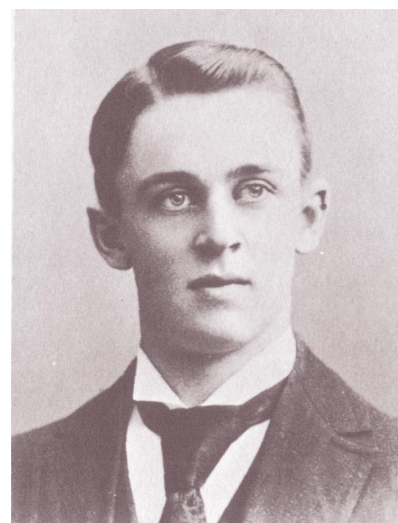
$$bvr^2 - mg = 0 \quad ,$$

and setting the known density of oil equal to the drop's mass divided by its volume gives a second equation,

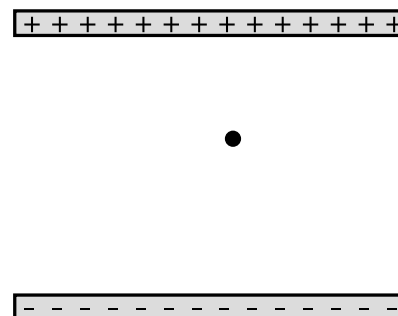
$$\rho = \frac{m}{\frac{4}{3}\pi r^3} \quad .$$

Everything in these equations can be measured directly except for  $m$  and  $r$ , so these are two equations in two unknowns, which can be solved in order to determine how big the drop is.

Next Millikan charged the metal plates, adjusting the amount of charge so as to exactly counteract gravity and levitate the drop. If, for instance, the drop being examined happened to have a total charge that was negative, then positive charge put on the top plate would attract it, pulling it up, and negative charge on the bottom plate would repel it, pushing it up. (Theoretically only one plate would be necessary, but in practice a two-plate arrangement like this gave electrical forces that were more uniform in strength throughout the space where the oil drops were.) When the drop was being levitated, the gravitational and electric forces canceled, so  $qE = mg$ . Since the mass of the drop, the gravitational field  $g$ , and the electric field  $E$  were all known, the charge  $q$  could be determined.



d / A young Robert Millikan.



e / A simplified diagram of Millikan's apparatus.

$q$ (C)	$q / (1.64 \times 10^{-19} \text{ C})$
$-1.970 \times 10^{-18}$	-12.02
$-0.987 \times 10^{-18}$	-6.02
$-2.773 \times 10^{-18}$	-16.93

f / A few samples of Millikan's data.

Table f shows a few of the results from Millikan's 1911 paper. (Millikan took data on both negatively and positively charged drops, but in his paper he gave only a sample of his data on negatively charged drops, so these numbers are all negative.) Even a quick look at the data leads to the suspicion that the charges are not simply a series of random numbers. For instance, the second charge is almost exactly equal to half the first one. Millikan explained the observed charges as all being integer multiples of a single number,  $1.64 \times 10^{-19} \text{ C}$ . In the second column, dividing by this constant gives numbers that are essentially integers, allowing for the random errors present in the experiment. Millikan states in his paper that these results were a

...direct and tangible demonstration ...of the correctness of the view advanced many years ago and supported by evidence from many sources that all electrical charges, however produced, are exact multiples of one definite, elementary electrical charge, or in other words, that an electrical charge instead of being spread uniformly over the charged surface has a definite granular structure, consisting, in fact, of ...specks, or atoms of electricity, all precisely alike, peppered over the surface of the charged body.

In other words, he had provided direct evidence for the charged-particle model of electricity and against models in which electricity was described as some sort of fluid. The basic charge is notated  $e$ , and the modern value is  $e = 1.60 \times 10^{-19} \text{ C}$ . The word "*quantized*" is used in physics to describe a quantity that can only have certain numerical values, and cannot have any of the values between those. In this language, we would say that Millikan discovered that charge is quantized. The charge  $e$  is referred to as the quantum of charge.

#### self-check A

Is money quantized? What is the quantum of money?    ▷ Answer, p. 978

#### A historical note on Millikan's fraud

Very few undergraduate physics textbooks mention the well-documented fact that although Millikan's conclusions were correct, he was guilty of scientific fraud. His technique was difficult and painstaking to perform, and his original notebooks, which have been preserved, show that the data were far less perfect than he claimed in his published scientific papers. In his publications, he stated categorically that every single oil drop observed had had a charge that was a multiple of  $e$ , with no Exceptions or omissions. But his notebooks are replete with notations such as "beautiful data, keep," and "bad run, throw out." Millikan, then, appears to have earned his Nobel Prize by advocating a correct position with dishonest descriptions of his data.

## 26.3 The electron

### Cathode rays

Nineteenth-century physicists spent a lot of time trying to come up with wild, random ways to play with electricity. The best experiments of this kind were the ones that made big sparks or pretty colors of light.

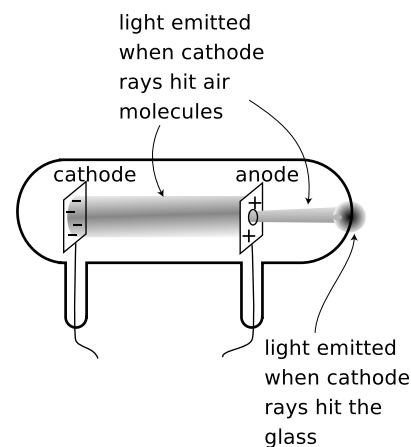
One such parlor trick was the cathode ray. To produce it, you first had to hire a good glassblower and find a good vacuum pump. The glassblower would create a hollow tube and embed two pieces of metal in it, called the electrodes, which were connected to the outside via metal wires passing through the glass. Before letting him seal up the whole tube, you would hook it up to a vacuum pump, and spend several hours huffing and puffing away at the pump's hand crank to get a good vacuum inside. Then, while you were still pumping on the tube, the glassblower would melt the glass and seal the whole thing shut. Finally, you would put a large amount of positive charge on one wire and a large amount of negative charge on the other. Metals have the property of letting charge move through them easily, so the charge deposited on one of the wires would quickly spread out because of the repulsion of each part of it for every other part. This spreading-out process would result in nearly all the charge ending up in the electrodes, where there is more room to spread out than there is in the wire. For obscure historical reasons a negative electrode is called a cathode and a positive one is an anode.

Figure g shows the light-emitting stream that was observed. If, as shown in this figure, a hole was made in the anode, the beam would extend on through the hole until it hit the glass. Drilling a hole in the cathode, however would not result in any beam coming out on the left side, and this indicated that the stuff, whatever it was, was coming from the cathode. The rays were therefore christened “cathode rays.” (The terminology is still used today in the term “cathode ray tube” or “CRT” for the picture tube of a TV or computer monitor.)

### Were cathode rays a form of light, or of matter?

Were cathode rays a form of light, or matter? At first no one really cared what they were, but as their scientific importance became more apparent, the light-versus-matter issue turned into a controversy along nationalistic lines, with the Germans advocating light and the English holding out for matter. The supporters of the material interpretation imagined the rays as consisting of a stream of atoms ripped from the substance of the cathode.

One of our defining characteristics of matter is that material objects cannot pass through each other. Experiments showed that cathode



g / Cathode rays observed in a vacuum tube.

rays could penetrate at least some small thickness of matter, such as a metal foil a tenth of a millimeter thick, implying that they were a form of light.

Other experiments, however, pointed to the contrary conclusion. Light is a wave phenomenon, and one distinguishing property of waves is demonstrated by speaking into one end of a paper towel roll. The sound waves do not emerge from the other end of the tube as a focused beam. Instead, they begin spreading out in all directions as soon as they emerge. This shows that waves do not necessarily travel in straight lines. If a piece of metal foil in the shape of a star or a cross was placed in the way of the cathode ray, then a “shadow” of the same shape would appear on the glass, showing that the rays traveled in straight lines. This straight-line motion suggested that they were a stream of small particles of matter.

These observations were inconclusive, so what was really needed was a determination of whether the rays had mass and weight. The trouble was that cathode rays could not simply be collected in a cup and put on a scale. When the cathode ray tube is in operation, one does not observe any loss of material from the cathode, or any crust being deposited on the anode.

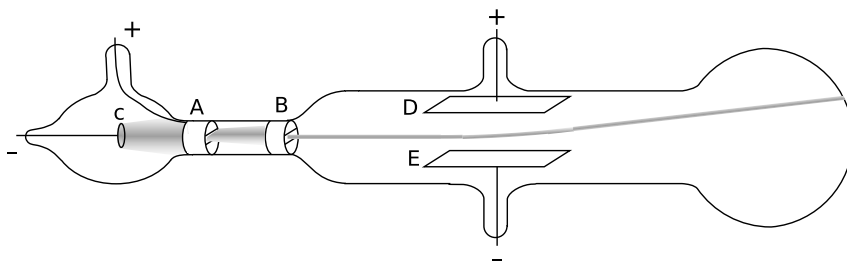
Nobody could think of a good way to weigh cathode rays, so the next most obvious way of settling the light/matter debate was to check whether the cathode rays possessed electrical charge. Light was known to be uncharged. If the cathode rays carried charge, they were definitely matter and not light, and they were presumably being made to jump the gap by the simultaneous repulsion of the negative charge in the cathode and attraction of the positive charge in the anode. The rays would overshoot the anode because of their momentum. (Although electrically charged particles do not normally leap across a gap of vacuum, very large amounts of charge were being used, so the forces were unusually intense.)

### Thomson's experiments

Physicist J.J. Thomson at Cambridge carried out a series of definitive experiments on cathode rays around the year 1897. By turning them slightly off course with electrical forces, i, he showed that they were indeed electrically charged, which was strong evidence that they were material. Not only that, but he proved that they had mass, and measured the ratio of their mass to their charge,  $m/q$ . Since their mass was not zero, he concluded that they were a form of matter, and presumably made up of a stream of microscopic, negatively charged particles. When Millikan published his results fourteen years later, it was reasonable to assume that the charge of one such particle equaled minus one fundamental charge,  $q = -e$ , and from the combination of Thomson's and Millikan's results one could therefore determine the mass of a single cathode ray particle.



h / J.J. Thomson in the lab.



i / Thomson's experiment proving cathode rays had electric charge (redrawn from his original paper). The cathode, c, and anode, A, are as in any cathode ray tube. The rays pass through a slit in the anode, and a second slit, B, is interposed in order to make the beam thinner and eliminate rays that were not going straight. Charging plates D and E shows that cathode rays have charge: they are attracted toward the positive plate D and repelled by the negative plate E.

The basic technique for determining  $m/q$  was simply to measure the angle through which the charged plates bent the beam. The electric force acting on a cathode ray particle while it was between the plates was

$$F_{elec} = qE \quad .$$

By Newton's second law,  $a = F/m$ , we can find  $m/q$ :

$$\frac{m}{q} = \frac{E}{a}$$

There was just one catch. Thomson needed to know the cathode ray particles' velocity in order to figure out their acceleration. At that point, however, nobody had even an educated guess as to the speed of the cathode rays produced in a given vacuum tube. The beam appeared to leap across the vacuum tube practically instantaneously, so it was no simple matter of timing it with a stopwatch!

Thomson's clever solution was to observe the effect of both electric and magnetic forces on the beam. The magnetic force exerted by a particular magnet would depend on both the cathode ray's charge and its speed:

$$F_{mag} = qvB$$

Thomson played with the electric and magnetic forces until either one would produce an equal effect on the beam, allowing him to solve for the speed,

$$v = \frac{E}{B} \quad .$$

Knowing the speed (which was on the order of 10% of the speed of light for his setup), he was able to find the acceleration and thus the mass-to-charge ratio  $m/q$ . Thomson's techniques were relatively crude (or perhaps more charitably we could say that they stretched the state of the art of the time), so with various methods he came up with  $m/q$  values that ranged over about a factor of two, even for cathode rays extracted from a cathode made of a single material. The best modern value is  $m/q = 5.69 \times 10^{-12} \text{ kg/C}$ , which is consistent with the low end of Thomson's range.



## The cathode ray as a subatomic particle: the electron

What was significant about Thomson's experiment was not the actual numerical value of  $m/q$ , however, so much as the fact that, combined with Millikan's value of the fundamental charge, it gave a mass for the cathode ray particles that was thousands of times smaller than the mass of even the lightest atoms. Even without Millikan's results, which were 14 years in the future, Thomson recognized that the cathode rays'  $m/q$  was thousands of times smaller than the  $m/q$  ratios that had been measured for electrically charged atoms in chemical solutions. He correctly interpreted this as evidence that the cathode rays were smaller building blocks — he called them *electrons* — out of which atoms themselves were formed. This was an extremely radical claim, coming at a time when atoms had not yet been proven to exist! Even those who used the word “atom” often considered them no more than mathematical abstractions, not literal objects. The idea of searching for structure inside of “un-splittable” atoms was seen by some as lunacy, but within ten years Thomson's ideas had been amply verified by many more detailed experiments.

### Discussion questions

**A** Thomson started to become convinced during his experiments that the “cathode rays” observed coming from the cathodes of vacuum tubes were building blocks of atoms — what we now call electrons. He then carried out observations with cathodes made of a variety of metals, and found that  $m/q$  was roughly the same in every case, considering his limited accuracy. Given his suspicion, why did it make sense to try different metals? How would the consistent values of  $m/q$  test his hypothesis?

**B** My students have frequently asked whether the  $m/q$  that Thomson measured was the value for a single electron, or for the whole beam. Can you answer this question?

**C** Thomson found that the  $m/q$  of an electron was thousands of times smaller than that of charged atoms in chemical solutions. Would this imply that the electrons had more charge? Less mass? Would there be no way to tell? Explain. Remember that Millikan's results were still many years in the future, so  $q$  was unknown.

**D** Can you guess any practical reason why Thomson couldn't just let one electron fly across the gap before disconnecting the battery and turning off the beam, and then measure the amount of charge deposited on the anode, thus allowing him to measure the charge of a single electron directly?

**E** Why is it not possible to determine  $m$  and  $q$  themselves, rather than just their ratio, by observing electrons' motion in electric and magnetic fields?

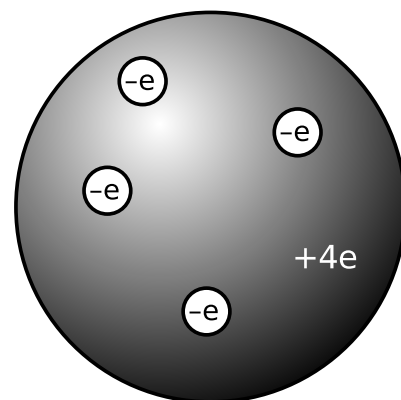
### The raisin cookie model

Based on his experiments, Thomson proposed a picture of the atom which became known as the raisin cookie model. In the neutral atom, j, there are four electrons with a total charge of  $-4e$ , sitting

in a sphere (the “cookie”) with a charge of  $+4e$  spread throughout it. It was known that chemical reactions could not change one element into another, so in Thomson’s scenario, each element’s cookie sphere had a permanently fixed radius, mass, and positive charge, different from those of other elements. The electrons, however, were not a permanent feature of the atom, and could be tacked on or pulled out to make charged ions. Although we now know, for instance, that a neutral atom with four electrons is the element beryllium, scientists at the time did not know how many electrons the various neutral atoms possessed.

This model is clearly different from the one you’ve learned in grade school or through popular culture, where the positive charge is concentrated in a tiny nucleus at the atom’s center. An equally important change in ideas about the atom has been the realization that atoms and their constituent subatomic particles behave entirely differently from objects on the human scale. For instance, we’ll see later that an electron can be in more than one place at one time. The raisin cookie model was part of a long tradition of attempts to make mechanical models of phenomena, and Thomson and his contemporaries never questioned the appropriateness of building a mental model of an atom as a machine with little parts inside. Today, mechanical models of atoms are still used (for instance the tinker-toy-style molecular modeling kits like the ones used by Watson and Crick to figure out the double helix structure of DNA), but scientists realize that the physical objects are only aids to help our brains’ symbolic and visual processes think about atoms.

Although there was no clear-cut experimental evidence for many of the details of the raisin cookie model, physicists went ahead and started working out its implications. For instance, suppose you had a four-electron atom. All four electrons would be repelling each other, but they would also all be attracted toward the center of the “cookie” sphere. The result should be some kind of stable, symmetric arrangement in which all the forces canceled out. People sufficiently clever with math soon showed that the electrons in a four-electron atom should settle down at the vertices of a pyramid with one less side than the Egyptian kind, i.e., a regular tetrahedron. This deduction turns out to be wrong because it was based on incorrect features of the model, but the model also had many successes, a few of which we will now discuss.



j / The raisin cookie model of the atom with four units of charge, which we now know to be beryllium.

#### *Flow of electrical charge in wires*

#### *example 1*

One of my former students was the son of an electrician, and had become an electrician himself. He related to me how his father had remained refused to believe all his life that electrons really flowed through wires. If they had, he reasoned, the metal would have gradually become more and more damaged, eventually crumbling to dust.

His opinion is not at all unreasonable based on the fact that electrons are material particles, and that matter cannot normally pass through matter without making a hole through it. Nineteenth-century physicists would have shared his objection to a charged-particle model of the flow of electrical charge. In the raisin-cookie model, however, the electrons are very low in mass, and therefore presumably very small in size as well. It is not surprising that they can slip between the atoms without damaging them.

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*Flow of electrical charge across cell membranes*      *example 2*

Your nervous system is based on signals carried by charge moving from nerve cell to nerve cell. Your body is essentially all liquid, and atoms in a liquid are mobile. This means that, unlike the case of charge flowing in a solid wire, entire charged atoms can flow in your nervous system

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*Emission of electrons in a cathode ray tube*      *example 3*

Why do electrons detach themselves from the cathode of a vacuum tube? Certainly they are encouraged to do so by the repulsion of the negative charge placed on the cathode and the attraction from the net positive charge of the anode, but these are not strong enough to rip electrons out of atoms by main force — if they were, then the entire apparatus would have been instantly vaporized as every atom was simultaneously ripped apart!

The raisin cookie model leads to a simple explanation. We know that heat is the energy of random motion of atoms. The atoms in any object are therefore violently jostling each other all the time, and a few of these collisions are violent enough to knock electrons out of atoms. If this occurs near the surface of a solid object, the electron may come loose. Ordinarily, however, this loss of electrons is a self-limiting process; the loss of electrons leaves the object with a net positive charge, which attracts the lost sheep home to the fold. (For objects immersed in air rather than vacuum, there will also be a balanced exchange of electrons between the air and the object.)

This interpretation explains the warm and friendly yellow glow of the vacuum tubes in an antique radio. To encourage the emission of electrons from the vacuum tubes' cathodes, the cathodes are intentionally warmed up with little heater coils.

### **Discussion questions**

**A**    Today many people would define an ion as an atom (or molecule) with missing electrons or extra electrons added on. How would people have defined the word "ion" before the discovery of the electron?

**B**    Since electrically neutral atoms were known to exist, there had to be positively charged subatomic stuff to cancel out the negatively charged electrons in an atom. Based on the state of knowledge immediately after the Millikan and Thomson experiments, was it possible that the positively charged stuff had an unquantized amount of charge? Could it be quan-

tized in units of  $+e$ ? In units of  $+2e$ ? In units of  $+5/7e$ ?

## 26.4 The nucleus

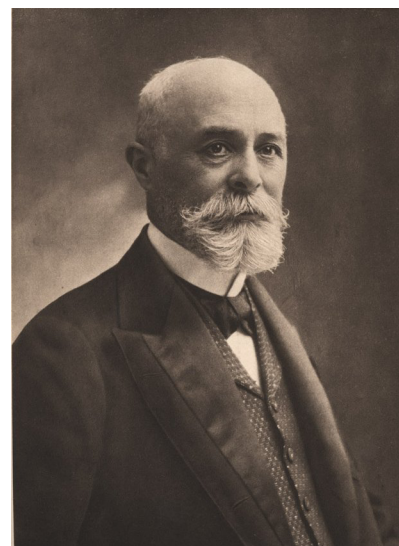
### Radioactivity

#### *Becquerel's discovery of radioactivity*

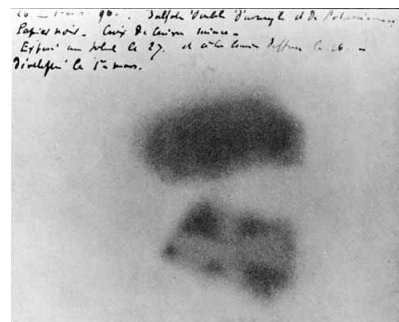
How did physicists figure out that the raisin cookie model was incorrect, and that the atom's positive charge was concentrated in a tiny, central nucleus? The story begins with the discovery of radioactivity by the French chemist Becquerel. Up until radioactivity was discovered, all the processes of nature were thought to be based on chemical reactions, which were rearrangements of combinations of atoms. Atoms exert forces on each other when they are close together, so sticking or unsticking them would either release or store electrical energy. That energy could be converted to and from other forms, as when a plant uses the energy in sunlight to make sugars and carbohydrates, or when a child eats sugar, releasing the energy in the form of kinetic energy.

Becquerel discovered a process that seemed to release energy from an unknown new source that was not chemical. Becquerel, whose father and grandfather had also been physicists, spent the first twenty years of his professional life as a successful civil engineer, teaching physics on a part-time basis. He was awarded the chair of physics at the Musée d'Histoire Naturelle in Paris after the death of his father, who had previously occupied it. Having now a significant amount of time to devote to physics, he began studying the interaction of light and matter. He became interested in the phenomenon of phosphorescence, in which a substance absorbs energy from light, then releases the energy via a glow that only gradually goes away. One of the substances he investigated was a uranium compound, the salt  $\text{UKSO}_5$ . One day in 1896, cloudy weather interfered with his plan to expose this substance to sunlight in order to observe its fluorescence. He stuck it in a drawer, coincidentally on top of a blank photographic plate — the old-fashioned glass-backed counterpart of the modern plastic roll of film. The plate had been carefully wrapped, but several days later when Becquerel checked it in the darkroom before using it, he found that it was ruined, as if it had been completely exposed to light.

History provides many examples of scientific discoveries that happened this way: an alert and inquisitive mind decides to investigate a phenomenon that most people would not have worried about explaining. Becquerel first determined by further experiments that the effect was produced by the uranium salt, despite a thick wrapping of paper around the plate that blocked out all light. He tried a variety of compounds, and found that it was the uranium that did it: the effect was produced by any uranium compound, but not



k / Henri Becquerel (1852-1908).



l / Becquerel's photographic plate. In the exposure at the bottom of the image, he has found that he could absorb the radiations, casting the shadow of a Maltese cross that was placed between the plate and the uranium salts.

by any compound that didn't include uranium atoms. The effect could be at least partially blocked by a sufficient thickness of metal, and he was able to produce silhouettes of coins by interposing them between the uranium and the plate. This indicated that the effect traveled in a straight line., so that it must have been some kind of ray rather than, e.g., the seepage of chemicals through the paper. He used the word "radiations," since the effect radiated out from the uranium salt.

At this point Becquerel still believed that the uranium atoms were absorbing energy from light and then gradually releasing the energy in the form of the mysterious rays, and this was how he presented it in his first published lecture describing his experiments. Interesting, but not earth-shattering. But he then tried to determine how long it took for the uranium to use up all the energy that had supposedly been stored in it by light, and he found that it never seemed to become inactive, no matter how long he waited. Not only that, but a sample that had been exposed to intense sunlight for a whole afternoon was no more or less effective than a sample that had always been kept inside. Was this a violation of conservation of energy? If the energy didn't come from exposure to light, where did it come from?

#### *Three kinds of "radiations"*

Unable to determine the source of the energy directly, turn-of-the-century physicists instead studied the behavior of the "radiations" once they had been emitted. Becquerel had already shown that the radioactivity could penetrate through cloth and paper, so the first obvious thing to do was to investigate in more detail what thickness of material the radioactivity could get through. They soon learned that a certain fraction of the radioactivity's intensity would be eliminated by even a few inches of air, but the remainder was not eliminated by passing through more air. Apparently, then, the radioactivity was a mixture of more than one type, of which one was blocked by air. They then found that of the part that could penetrate air, a further fraction could be eliminated by a piece of paper or a very thin metal foil. What was left after that, however, was a third, extremely penetrating type, some of whose intensity would still remain even after passing through a brick wall. They decided that this showed there were three types of radioactivity, and without having the faintest idea of what they really were, they made up names for them. The least penetrating type was arbitrarily labeled  $\alpha$  (alpha), the first letter of the Greek alphabet, and so on through  $\beta$  (beta) and finally  $\gamma$  (gamma) for the most penetrating type.

#### *Radium: a more intense source of radioactivity*

The measuring devices used to detect radioactivity were crude: photographic plates or even human eyeballs (radioactivity makes flashes

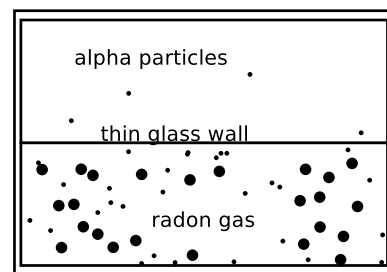
of light in the jelly-like fluid inside the eye, which can be seen by the eyeball's owner if it is otherwise very dark). Because the ways of detecting radioactivity were so crude and insensitive, further progress was hindered by the fact that the amount of radioactivity emitted by uranium was not really very great. The vital contribution of physicist/chemist Marie Curie and her husband Pierre was to discover the element radium, and to purify and isolate significant quantities of it. Radium emits about a million times more radioactivity per unit mass than uranium, making it possible to do the experiments that were needed to learn the true nature of radioactivity. The dangers of radioactivity to human health were then unknown, and Marie died of leukemia thirty years later. (Pierre was run over and killed by a horsecart.)

### *Tracking down the nature of alphas, betas, and gammas*

As radium was becoming available, an apprentice scientist named Ernest Rutherford arrived in England from his native New Zealand and began studying radioactivity at the Cavendish Laboratory. The young colonial's first success was to measure the mass-to-charge ratio of beta rays. The technique was essentially the same as the one Thomson had used to measure the mass-to-charge ratio of cathode rays by measuring their deflections in electric and magnetic fields. The only difference was that instead of the cathode of a vacuum tube, a nugget of radium was used to supply the beta rays. Not only was the technique the same, but so was the result. Beta rays had the same  $m/q$  ratio as cathode rays, which suggested they were one and the same. Nowadays, it would make sense simply to use the term "electron," and avoid the archaic "cathode ray" and "beta particle," but the old labels are still widely used, and it is unfortunately necessary for physics students to memorize all three names for the same thing.

At first, it seemed that neither alphas or gammas could be deflected in electric or magnetic fields, making it appear that neither was electrically charged. But soon Rutherford obtained a much more powerful magnet, and was able to use it to deflect the alphas but not the gammas. The alphas had a much larger value of  $m/q$  than the betas (about 4000 times greater), which was why they had been so hard to deflect. Gammas are uncharged, and were later found to be a form of light.

The  $m/q$  ratio of alpha particles turned out to be the same as those of two different types of ions,  $\text{He}^{++}$  (a helium atom with two missing electrons) and  $\text{H}_2^+$  (two hydrogen atoms bonded into a molecule, with one electron missing), so it seemed likely that they were one or the other of those. The diagram shows a simplified version of Rutherford's ingenious experiment proving that they were  $\text{He}^{++}$  ions. The gaseous element radon, an alpha emitter, was introduced into one half of a double glass chamber. The glass wall dividing



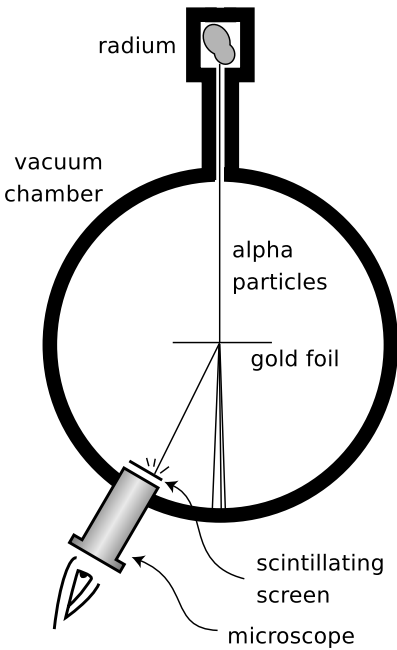
m / A simplified version of Rutherford's 1908 experiment, showing that alpha particles are doubly ionized helium atoms.



n / These pellets of uranium fuel will be inserted into the metal fuel rod and used in a nuclear reactor. The pellets emit alpha and beta radiation, which the gloves are thick enough to stop.



o / Ernest Rutherford (1871-1937).



p / Marsden and Rutherford's apparatus.

the chamber was made extremely thin, so that some of the rapidly moving alpha particles were able to penetrate it. The other chamber, which was initially evacuated, gradually began to accumulate a population of alpha particles (which would quickly pick up electrons from their surroundings and become electrically neutral). Rutherford then determined that it was helium gas that had appeared in the second chamber. Thus alpha particles were proved to be  $\text{He}^{++}$  ions. The nucleus was yet to be discovered, but in modern terms, we would describe a  $\text{He}^{++}$  ion as the nucleus of a He atom.

To summarize, here are the three types of radiation emitted by radioactive elements, and their descriptions in modern terms:

$\alpha$ particle	stopped by a few inches of air	He nucleus
$\beta$ particle	stopped by a piece of paper	electron
$\gamma$ ray	penetrates thick shielding	a type of light

### Discussion question

**A** Most sources of radioactivity emit alphas, betas, and gammas, not just one of the three. In the radon experiment, how did Rutherford know that he was studying the alphas?

### The planetary model

The stage was now set for the unexpected discovery that the positively charged part of the atom was a tiny, dense lump at the atom's center rather than the "cookie dough" of the raisin cookie model. By 1909, Rutherford was an established professor, and had students working under him. For a raw undergraduate named Marsden, he picked a research project he thought would be tedious but straightforward.

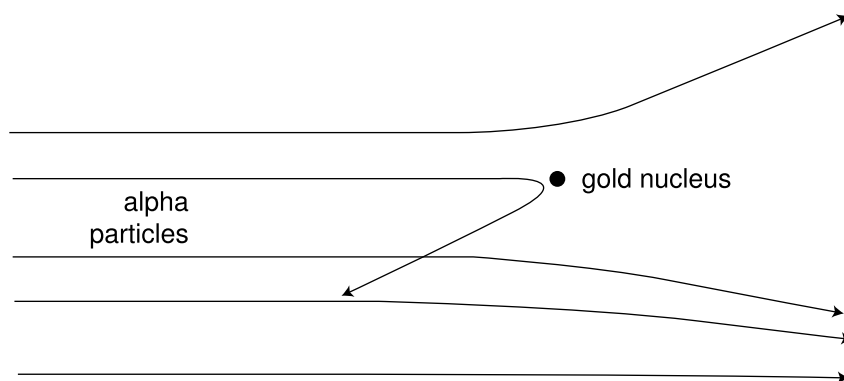
It was already known that although alpha particles would be stopped completely by a sheet of paper, they could pass through a sufficiently thin metal foil. Marsden was to work with a gold foil only 1000 atoms thick. (The foil was probably made by evaporating a little gold in a vacuum chamber so that a thin layer would be deposited on a glass microscope slide. The foil would then be lifted off the slide by submerging the slide in water.)

Rutherford had already determined in his previous experiments the speed of the alpha particles emitted by radium, a fantastic  $1.5 \times 10^7$  m/s. The experimenters in Rutherford's group visualized them as very small, very fast cannonballs penetrating the "cookie dough" part of the big gold atoms. A piece of paper has a thickness of a hundred thousand atoms or so, which would be sufficient to stop them completely, but crashing through a thousand would only slow them a little and turn them slightly off of their original paths.

Marsden's supposedly ho-hum assignment was to use the apparatus shown in figure p to measure how often alpha particles were deflected at various angles. A tiny lump of radium in a box emitted alpha

particles, and a thin beam was created by blocking all the alphas except those that happened to pass out through a tube. Typically deflected in the gold by only a small amount, they would reach a screen very much like the screen of a TV's picture tube, which would make a flash of light when it was hit. Here is the first example we have encountered of an experiment in which a beam of particles is detected one at a time. This was possible because each alpha particle carried so much kinetic energy; they were moving at about the same speed as the electrons in the Thomson experiment, but had ten thousand times more mass.

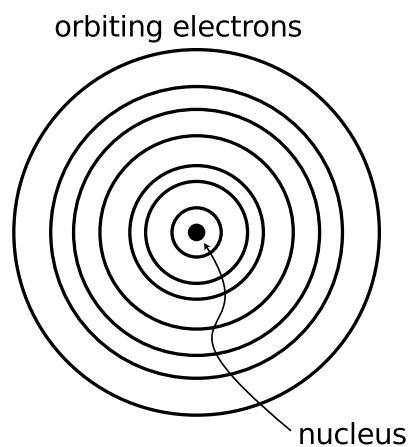
Marsden sat in a dark room, watching the apparatus hour after hour and recording the number of flashes with the screen moved to various angles. The rate of the flashes was highest when he set the screen at an angle close to the line of the alphas' original path, but if he watched an area farther off to the side, he would also occasionally see an alpha that had been deflected through a larger angle. After seeing a few of these, he got the crazy idea of moving the screen to see if even larger angles ever occurred, perhaps even angles larger than 90 degrees.



q / Alpha particles being scattered by a gold nucleus. On this scale, the gold atom is the size of a car, so all the alpha particles shown here are ones that just happened to come unusually close to the nucleus. For these exceptional alpha particles, the forces from the electrons are unimportant, because they are so much more distant than the nucleus.

The crazy idea worked: a few alpha particles were deflected through angles of up to 180 degrees, and the routine experiment had become an epoch-making one. Rutherford said, "We have been able to get some of the alpha particles coming backwards. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you." Explanations were hard to come by in the raisin cookie model. What intense electrical forces could have caused some of the alpha particles, moving at such astronomical





r / The planetary model of the atom.

speeds, to change direction so drastically? Since each gold atom was electrically neutral, it would not exert much force on an alpha particle outside it. True, if the alpha particle was very near to or inside of a particular atom, then the forces would not necessarily cancel out perfectly; if the alpha particle happened to come very close to a particular electron, the  $1/r^2$  form of the Coulomb force law would make for a very strong force. But Marsden and Rutherford knew that an alpha particle was 8000 times more massive than an electron, and it is simply not possible for a more massive object to rebound backwards from a collision with a less massive object while conserving momentum and energy. It might be possible in principle for a particular alpha to follow a path that took it very close to one electron, and then very close to another electron, and so on, with the net result of a large deflection, but careful calculations showed that such multiple “close encounters” with electrons would be millions of times too rare to explain what was actually observed.

At this point, Rutherford and Marsden dusted off an unpopular and neglected model of the atom, in which all the electrons orbited around a small, positively charged core or “nucleus,” just like the planets orbiting around the sun. All the positive charge and nearly all the mass of the atom would be concentrated in the nucleus, rather than spread throughout the atom as in the raisin cookie model. The positively charged alpha particles would be repelled by the gold atom’s nucleus, but most of the alphas would not come close enough to any nucleus to have their paths drastically altered. The few that did come close to a nucleus, however, could rebound backwards from a single such encounter, since the nucleus of a heavy gold atom would be fifty times more massive than an alpha particle. It turned out that it was not even too difficult to derive a formula giving the relative frequency of deflections through various angles, and this calculation agreed with the data well enough (to within 15%), considering the difficulty in getting good experimental statistics on the rare, very large angles.

What had started out as a tedious exercise to get a student started in science had ended as a revolution in our understanding of nature. Indeed, the whole thing may sound a little too much like a moralistic fable of the scientific method with overtones of the Horatio Alger genre. The skeptical reader may wonder why the planetary model was ignored so thoroughly until Marsden and Rutherford’s discovery. Is science really more of a sociological enterprise, in which certain ideas become accepted by the establishment, and other, equally plausible explanations are arbitrarily discarded? Some social scientists are currently ruffling a lot of scientists’ feathers with critiques very much like this, but in this particular case, there were very sound reasons for rejecting the planetary model. As you’ll learn in more detail later in this course, any charged particle that undergoes an acceleration dissipate energy in the form of light. In the

planetary model, the electrons were orbiting the nucleus in circles or ellipses, which meant they were undergoing acceleration, just like the acceleration you feel in a car going around a curve. They should have dissipated energy as light, and eventually they should have lost all their energy. Atoms don't spontaneously collapse like that, which was why the raisin cookie model, with its stationary electrons, was originally preferred. There were other problems as well. In the planetary model, the one-electron atom would have to be flat, which would be inconsistent with the success of molecular modeling with spherical balls representing hydrogen and atoms. These molecular models also seemed to work best if specific sizes were used for different atoms, but there is no obvious reason in the planetary model why the radius of an electron's orbit should be a fixed number. In view of the conclusive Marsden-Rutherford results, however, these became fresh puzzles in atomic physics, not reasons for disbelieving the planetary model.

#### *Some phenomena explained with the planetary model*

The planetary model may not be the ultimate, perfect model of the atom, but don't underestimate its power. It already allows us to visualize correctly a great many phenomena.

As an example, let's consider the distinctions among nonmetals, metals that are magnetic, and metals that are nonmagnetic. As shown in figure s, a metal differs from a nonmetal because its outermost electrons are free to wander rather than owing their allegiance to a particular atom. A metal that can be magnetized is one that is willing to line up the rotations of some of its electrons so that their axes are parallel. Recall that magnetic forces are forces made by moving charges; we have not yet discussed the mathematics and geometry of magnetic forces, but it is easy to see how random orientations of the atoms in the nonmagnetic substance would lead to cancellation of the forces.

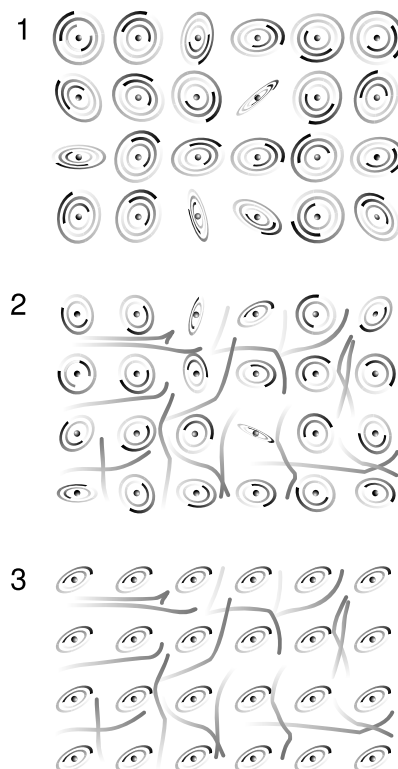
Even if the planetary model does not immediately answer such questions as why one element would be a metal and another a nonmetal, these ideas would be difficult or impossible to conceptualize in the raisin cookie model.

#### **Discussion question**

**A** In reality, charges of the same type repel one another and charges of different types are attracted. Suppose the rules were the other way around, giving repulsion between opposite charges and attraction between similar ones. What would the universe be like?

#### **Atomic number**

As alluded to in a discussion question in the previous section, scientists of this period had only a very approximate idea of how many units of charge resided in the nuclei of the various chemical elements. Although we now associate the number of units of nuclear



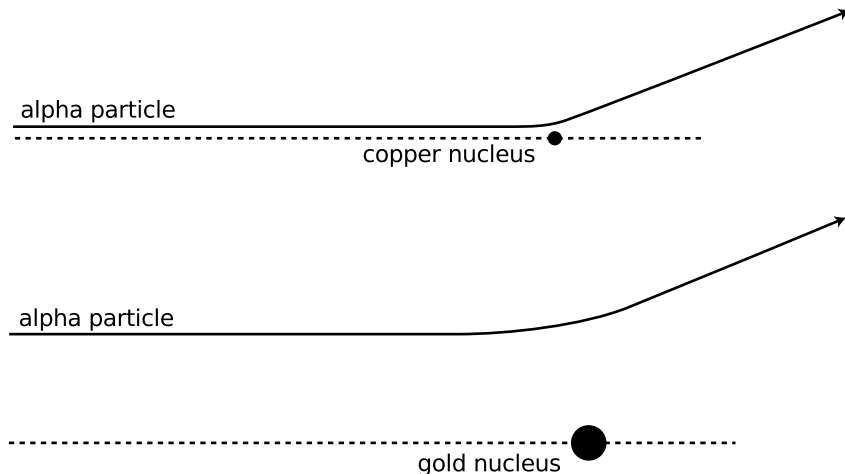
s / The planetary model applied to a nonmetal, 1, an unmagnetized metal, 2, and a magnetized metal, 3. Note that these figures are all simplified in several ways. For one thing, the electrons of an individual atom do not all revolve around the nucleus in the same plane. It is also very unusual for a metal to become so strongly magnetized that 100% of its atoms have their rotations aligned as shown in this figure.

t / A modern periodic table, labeled with atomic numbers. Mendelev's original table was upside-down compared to this one.

[illegible]

How did we finally find out? The riddle of the nuclear charges was at last successfully attacked using two different techniques, which gave consistent results. One set of experiments, involving x-rays, was performed by the young Henry Mosely, whose scientific brilliance was soon to be sacrificed in a battle between European imperialists over who would own the Dardanelles, during that pointless conflict

then known as the War to End All Wars, and now referred to as World War I.



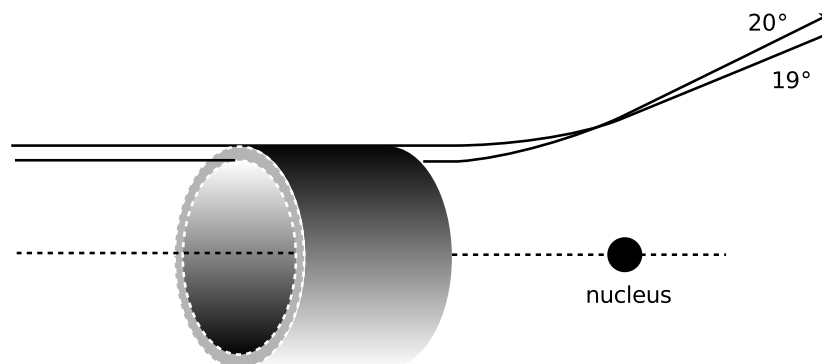
u / An alpha particle has to come much closer to the low-charged copper nucleus in order to be deflected through the same angle.

Since Mosely's analysis requires several concepts with which you are not yet familiar, we will instead describe the technique used by James Chadwick at around the same time. An added bonus of describing Chadwick's experiments is that they presaged the important modern technique of studying *collisions* of subatomic particles. In grad school, I worked with a professor whose thesis adviser's thesis adviser was Chadwick, and he related some interesting stories about the man. Chadwick was apparently a little nutty and a complete fanatic about science, to the extent that when he was held in a German prison camp during World War II, he managed to cajole his captors into allowing him to scrounge up parts from broken radios so that he could attempt to do physics experiments.

Chadwick's experiment worked like this. Suppose you perform two Rutherford-type alpha scattering measurements, first one with a gold foil as a target as in Rutherford's original experiment, and then one with a copper foil. It is possible to get large angles of deflection in both cases, but as shown in figure v, the alpha particle must be heading almost straight for the copper nucleus to get the same angle of deflection that would have occurred with an alpha that was much farther off the mark; the gold nucleus' charge is so much greater than the copper's that it exerts a strong force on the alpha particle even from far off. The situation is very much like that of a blindfolded person playing darts. Just as it is impossible to aim an alpha particle at an individual nucleus in the target, the blindfolded person cannot really aim the darts. Achieving a very close encounter with the copper atom would be akin to hitting an inner circle on the dartboard. It's much more likely that one would have the luck to hit the outer circle, which covers a greater number of square inches. By analogy, if you measure the frequency with which alphas are scattered by copper at some particular angle, say

between 19 and 20 degrees, and then perform the same measurement at the same angle with gold, you get a much higher percentage for gold than for copper.

v / An alpha particle must be headed for the ring on the front of the imaginary cylindrical pipe in order to produce scattering at an angle between 19 and 20 degrees. The area of this ring is called the “cross-section” for scattering at 19-20° because it is the cross-sectional area of a cut through the pipe.



In fact, the numerical ratio of the two nuclei’s charges can be derived from this same experimentally determined ratio. Using the standard notation  $Z$  for the atomic number (charge of the nucleus divided by  $e$ ), the following equation can be proved (example 4):

$$\frac{Z_{gold}^2}{Z_{copper}^2} = \frac{\text{number of alphas scattered by gold at } 19\text{-}20^\circ}{\text{number of alphas scattered by copper at } 19\text{-}20^\circ}$$

By making such measurements for targets constructed from all the elements, one can infer the ratios of all the atomic numbers, and since the atomic numbers of the light elements were already known, atomic numbers could be assigned to the entire periodic table. According to Mosely, the atomic numbers of copper, silver and platinum were 29, 47, and 78, which corresponded well with their positions on the periodic table. Chadwick’s figures for the same elements were 29.3, 46.3, and 77.4, with error bars of about 1.5 times the fundamental charge, so the two experiments were in good agreement.

The point here is absolutely not that you should be ready to plug numbers into the above equation for a homework or exam question! My overall goal in this chapter is to explain how we know what we know about atoms. An added bonus of describing Chadwick’s experiment is that the approach is very similar to that used in modern particle physics experiments, and the ideas used in the analysis are closely related to the now-ubiquitous concept of a “cross-section.” In the dartboard analogy, the cross-section would be the area of the circular ring you have to hit. The reasoning behind the invention of the term “cross-section” can be visualized as shown in figure v. In this language, Rutherford’s invention of the planetary model came from his unexpected discovery that there was a nonzero cross-section

for alpha scattering from gold at large angles, and Chadwick confirmed Mosely's determinations of the atomic numbers by measuring cross-sections for alpha scattering.

*Proof of the relationship between  $Z$  and scattering      example 4*

The equation above can be derived by the following not very rigorous proof. To deflect the alpha particle by a certain angle requires that it acquire a certain momentum component in the direction perpendicular to its original momentum. Although the nucleus's force on the alpha particle is not constant, we can pretend that it is approximately constant during the time when the alpha is within a distance equal to, say, 150% of its distance of closest approach, and that the force is zero before and after that part of the motion. (If we chose 120% or 200%, it shouldn't make any difference in the final result, because the final result is a ratio, and the effects on the numerator and denominator should cancel each other.) In the approximation of constant force, the change in the alpha's perpendicular momentum component is then equal to  $F\Delta t$ . The Coulomb force law says the force is proportional to  $Z/r^2$ . Although  $r$  does change somewhat during the time interval of interest, it's good enough to treat it as a constant number, since we're only computing the ratio between the two experiments' results. Since we are approximating the force as acting over the time during which the distance is not too much greater than the distance of closest approach, the time interval  $\Delta t$  must be proportional to  $r$ , and the sideways momentum imparted to the alpha,  $F\Delta t$ , is proportional to  $(Z/r^2)r$ , or  $Z/r$ . If we're comparing alphas scattered at the same angle from gold and from copper, then  $\Delta p$  is the same in both cases, and the proportionality  $\Delta p \propto Z/r$  tells us that the ones scattered from copper at that angle had to be headed in along a line closer to the central axis by a factor equaling  $Z_{\text{gold}}/Z_{\text{copper}}$ . If you imagine a "dartboard ring" that the alphas have to hit, then the ring for the gold experiment has the same proportions as the one for copper, but it is enlarged by a factor equal to  $Z_{\text{gold}}/Z_{\text{copper}}$ . That is, not only is the radius of the ring greater by that factor, but unlike the rings on a normal dartboard, the thickness of the outer ring is also greater in proportion to its radius. When you take a geometric shape and scale it up in size like a photographic enlargement, its area is increased in proportion to the square of the enlargement factor, so the area of the dartboard ring in the gold experiment is greater by a factor equal to  $(Z_{\text{gold}}/Z_{\text{copper}})^2$ . Since the alphas are aimed entirely randomly, the chances of an alpha hitting the ring are in proportion to the area of the ring, which proves the equation given above.

As an example of the modern use of scattering experiments and cross-section measurements, you may have heard of the recent experimental evidence for the existence of a particle called the top quark. Of the twelve subatomic particles currently believed to be the

smallest constituents of matter, six form a family called the quarks, distinguished from the other six by the intense attractive forces that make the quarks stick to each other. (The other six consist of the electron plus five other, more exotic particles.) The only two types of quarks found in naturally occurring matter are the “up quark” and “down quark,” which are what protons and neutrons are made of, but four other types were theoretically predicted to exist, for a total of six. (The whimsical term “quark” comes from a line by James Joyce reading “Three quarks for master Mark.”) Until recently, only five types of quarks had been proven to exist via experiments, and the sixth, the top quark, was only theorized. There was no hope of ever detecting a top quark directly, since it is radioactive, and only exists for a zillionth of a second before evaporating. Instead, the researchers searching for it at the Fermi National Accelerator Laboratory near Chicago measured cross-sections for scattering of nuclei off of other nuclei. The experiment was much like those of Rutherford and Chadwick, except that the incoming nuclei had to be boosted to much higher speeds in a particle accelerator. The resulting encounter with a target nucleus was so violent that both nuclei were completely demolished, but, as Einstein proved, energy can be converted into matter, and the energy of the collision creates a spray of exotic, radioactive particles, like the deadly shower of wood fragments produced by a cannon ball in an old naval battle. Among those particles were some top quarks. The cross-sections being measured were the cross-sections for the production of certain combinations of these secondary particles. However different the details, the principle was the same as that employed at the turn of the century: you smash things together and look at the fragments that fly off to see what was inside them. The approach has been compared to shooting a clock with a rifle and then studying the pieces that fly off to figure out how the clock worked.

### Discussion questions

**A** The diagram, showing alpha particles being deflected by a gold nucleus, was drawn with the assumption that alpha particles came in on lines at many different distances from the nucleus. Why wouldn't they all come in along the same line, since they all came out through the same tube?

**B** Why does it make sense that, as shown in the figure, the trajectories that result in  $19^\circ$  and  $20^\circ$  scattering cross each other?

**C** Rutherford knew the velocity of the alpha particles emitted by radium, and guessed that the positively charged part of a gold atom had a charge of about  $+100e$  (we now know it is  $+79e$ ). Considering the fact that some alpha particles were deflected by  $180^\circ$ , how could he then use conservation of energy to derive an upper limit on the size of a gold nucleus? (For simplicity, assume the size of the alpha particle is negligible compared to that of the gold nucleus, and ignore the fact that the gold nucleus recoils a little from the collision, picking up a little kinetic energy.)

## The structure of nuclei

### *The proton*

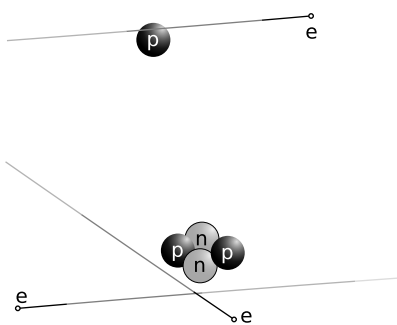
The fact that the nuclear charges were all integer multiples of  $e$  suggested to many physicists that rather than being a pointlike object, the nucleus might contain smaller particles having individual charges of  $+e$ . Evidence in favor of this idea was not long in arriving. Rutherford reasoned that if he bombarded the atoms of a very light element with alpha particles, the small charge of the target nuclei would give a very weak repulsion. Perhaps those few alpha particles that happened to arrive on head-on collision courses would get so close that they would physically crash into some of the target nuclei. An alpha particle is itself a nucleus, so this would be a collision between two nuclei, and a violent one due to the high speeds involved. Rutherford hit pay dirt in an experiment with alpha particles striking a target containing nitrogen atoms. Charged particles were detected flying out of the target like parts flying off of cars in a high-speed crash. Measurements of the deflection of these particles in electric and magnetic fields showed that they had the same charge-to-mass ratio as singly-ionized hydrogen atoms. Rutherford concluded that these were the conjectured singly-charged particles that held the charge of the nucleus, and they were later named protons. The hydrogen nucleus consists of a single proton, and in general, an element's atomic number gives the number of protons contained in each of its nuclei. The mass of the proton is about 1800 times greater than the mass of the electron.

### *The neutron*

It would have been nice and simple if all the nuclei could have been built only from protons, but that couldn't be the case. If you spend a little time looking at a periodic table, you will soon notice that although some of the atomic masses are very nearly integer multiples of hydrogen's mass, many others are not. Even where the masses are close whole numbers, the masses of an element other than hydrogen is always greater than its atomic number, not equal to it. Helium, for instance, has two protons, but its mass is four times greater than that of hydrogen.

Chadwick cleared up the confusion by proving the existence of a new subatomic particle. Unlike the electron and proton, which are electrically charged, this particle is electrically neutral, and he named it the neutron. Chadwick's experiment has been described in detail in section 14.2, but briefly the method was to expose a sample of the light element beryllium to a stream of alpha particles from a lump of radium. Beryllium has only four protons, so an alpha that happens to be aimed directly at a beryllium nucleus can actually hit it rather than being stopped short of a collision by electrical repulsion. Neutrons were observed as a new form of radiation emerging from the collisions, and Chadwick correctly inferred that they were previ-





w / Examples of the construction of atoms: hydrogen (top) and helium (bottom). On this scale, the electrons' orbits would be the size of a college campus.

ously unsuspected components of the nucleus that had been knocked out. As described earlier, Chadwick also determined the mass of the neutron; it is very nearly the same as that of the proton.

To summarize, atoms are made of three types of particles:

	<i>charge</i>	<i>mass in units of the proton's mass</i>	<i>location in atom</i>
proton	$+e$	1	in nucleus
neutron	0	1.001	in nucleus
electron	$-e$	$1/1836$	orbiting nucleus

The existence of neutrons explained the mysterious masses of the elements. Helium, for instance, has a mass very close to four times greater than that of hydrogen. This is because it contains two neutrons in addition to its two protons. The mass of an atom is essentially determined by the total number of neutrons and protons. The total number of neutrons plus protons is therefore referred to as the atom's *mass number*.

### Isotopes

We now have a clear interpretation of the fact that helium is close to four times more massive than hydrogen, and similarly for all the atomic masses that are close to an integer multiple of the mass of hydrogen. But what about copper, for instance, which had an atomic mass 63.5 times that of hydrogen? It didn't seem reasonable to think that it possessed an extra half of a neutron! The solution was found by measuring the mass-to-charge ratios of singly-ionized atoms (atoms with one electron removed). The technique is essentially that same as the one used by Thomson for cathode rays, except that whole atoms do not spontaneously leap out of the surface of an object as electrons sometimes do. Figure x shows an example of how the ions can be created and injected between the charged plates for acceleration.

Injecting a stream of copper ions into the device, we find a surprise — the beam splits into two parts! Chemists had elevated to dogma the assumption that all the atoms of a given element were identical, but we find that 69% of copper atoms have one mass, and 31% have another. Not only that, but both masses are very nearly integer multiples of the mass of hydrogen (63 and 65, respectively). Copper gets its chemical identity from the number of protons in its nucleus, 29, since chemical reactions work by electric forces. But apparently some copper atoms have  $63 - 29 = 34$  neutrons while others have  $65 - 29 = 36$ . The atomic mass of copper, 63.5, reflects the proportions of the mixture of the mass-63 and mass-65 varieties. The different mass varieties of a given element are called *isotopes* of that element.

Isotopes can be named by giving the mass number as a subscript to the left of the chemical symbol, e.g.,  $^{65}\text{Cu}$ . Examples:

	<i>protons</i>	<i>neutrons</i>	<i>mass number</i>
$^1\text{H}$	1	0	$0+1 = 1$
$^4\text{He}$	2	2	$2+2 = 4$
$^{12}\text{C}$	6	6	$6+6 = 12$
$^{14}\text{C}$	6	8	$6+8 = 14$
$^{262}\text{Ha}$	105	157	$105+157 = 262$

#### self-check B

Why are the positive and negative charges of the accelerating plates reversed in the isotope-separating apparatus compared to the Thomson apparatus? ▷ Answer, p. 978

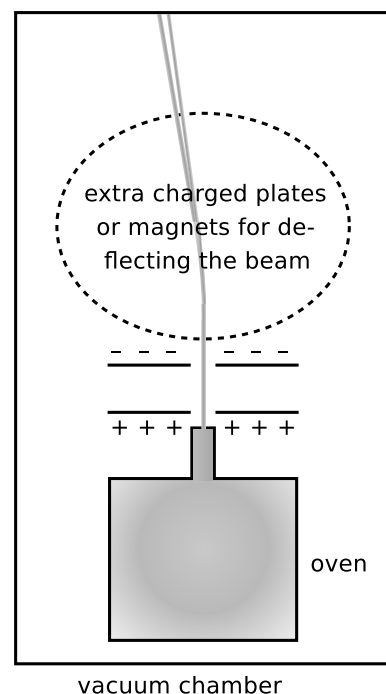
Chemical reactions are all about the exchange and sharing of electrons: the nuclei have to sit out this dance because the forces of electrical repulsion prevent them from ever getting close enough to make contact with each other. Although the protons do have a vitally important effect on chemical processes because of their electrical forces, the neutrons can have no effect on the atom's chemical reactions. It is not possible, for instance, to separate  $^{63}\text{Cu}$  from  $^{65}\text{Cu}$  by chemical reactions. This is why chemists had never realized that different isotopes existed. (To be perfectly accurate, different isotopes do behave slightly differently because the more massive atoms move more sluggishly and therefore react with a tiny bit less intensity. This tiny difference is used, for instance, to separate out the isotopes of uranium needed to build a nuclear bomb. The smallness of this effect makes the separation process a slow and difficult one, which is what we have to thank for the fact that nuclear weapons have not been built by every terrorist cabal on the planet.)

#### Sizes and shapes of nuclei

Matter is nearly all nuclei if you count by weight, but in terms of volume nuclei don't amount to much. The radius of an individual neutron or proton is very close to 1 fm ( $1\text{ fm}=10^{-15}\text{ m}$ ), so even a big lead nucleus with a mass number of 208 still has a diameter of only about 13 fm, which is ten thousand times smaller than the diameter of a typical atom. Contrary to the usual imagery of the nucleus as a small sphere, it turns out that many nuclei are somewhat elongated, like an American football, and a few have exotic asymmetric shapes like pears or kiwi fruits.

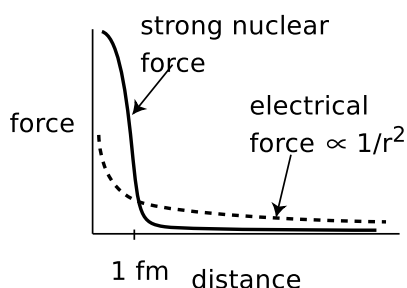
#### Discussion questions

**A** Suppose the entire universe was in a (very large) cereal box, and the nutritional labeling was supposed to tell a godlike consumer what percentage of the contents was nuclei. Roughly what would the percentage be like if the labeling was according to mass? What if it was by volume?



x / A version of the Thomson apparatus modified for measuring the mass-to-charge ratios of ions rather than electrons. A small sample of the element in question, copper in our example, is boiled in the oven to create a thin vapor. (A vacuum pump is continuously sucking on the main chamber to keep it from accumulating enough gas to stop the beam of ions.) Some of the atoms of the vapor are ionized by a spark or by ultraviolet light. Ions that wander out of the nozzle and into the region between the charged plates are then accelerated toward the top of the figure. As in the Thomson experiment, mass-to-charge ratios are inferred from the deflection of the beam.

y / A nuclear power plant at Cattenom, France. Unlike the coal and oil plants that supply most of the U.S.'s electrical power, a nuclear power plant like this one releases no pollution or greenhouse gases into the Earth's atmosphere, and therefore doesn't contribute to global warming. The white stuff puffing out of this plant is non-radioactive water vapor. Although nuclear power plants generate long-lived nuclear waste, this waste arguably poses much less of a threat to the biosphere than greenhouse gases would.



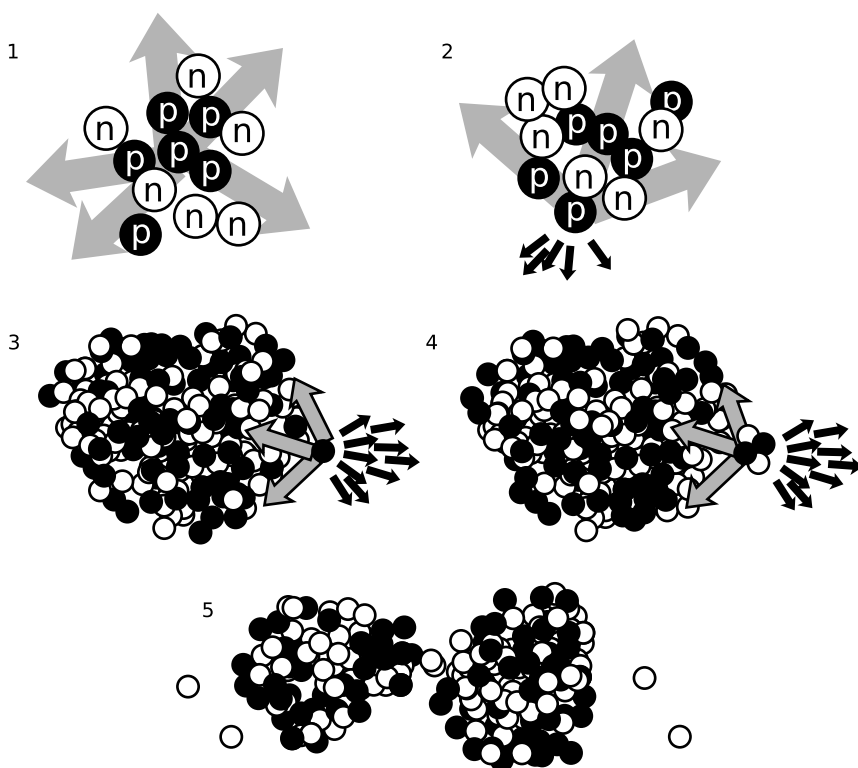
z / The strong nuclear force cuts off very sharply at a range of about 1 fm.

### The strong nuclear force, alpha decay and fission

Once physicists realized that nuclei consisted of positively charged protons and uncharged neutrons, they had a problem on their hands. The electrical forces among the protons are all repulsive, so the nucleus should simply fly apart! The reason all the nuclei in your body are not spontaneously exploding at this moment is that there is another force acting. This force, called the *strong nuclear force*, is always attractive, and acts between neutrons and neutrons, neutrons and protons, and protons and protons with roughly equal strength. The strong nuclear force does not have any effect on electrons, which is why it does not influence chemical reactions.

Unlike electric forces, whose strengths are given by the simple Coulomb force law, there is no simple formula for how the strong nuclear force depends on distance. Roughly speaking, it is effective over ranges of  $\sim 1$  fm, but falls off extremely quickly at larger distances (much faster than  $1/r^2$ ). Since the radius of a neutron or proton is about 1 fm, that means that when a bunch of neutrons and protons are packed together to form a nucleus, the strong nuclear force is effective only between neighbors.

Figure aa illustrates how the strong nuclear force acts to keep ordinary nuclei together, but is not able to keep very heavy nuclei from breaking apart. In aa/1, a proton in the middle of a carbon nucleus feels an attractive strong nuclear force (arrows) from each of its nearest neighbors. The forces are all in different directions, and tend to cancel out. The same is true for the repulsive electrical forces (not shown). In figure aa/2, a proton at the edge of the nucleus has neighbors only on one side, and therefore all the strong



aa / 1. The forces cancel. 2. The forces don't cancel. 3. In a heavy nucleus, the large number of electrical repulsions can add up to a force that is comparable to the strong nuclear attraction. 4. Alpha emission. 5. Fission.

nuclear forces acting on it are tending to pull it back in. Although all the electrical forces from the other five protons (dark arrows) are all pushing it out of the nucleus, they are not sufficient to overcome the strong nuclear forces.

In a very heavy nucleus, aa/3, a proton that finds itself near the edge has only a few neighbors close enough to attract it significantly via the strong nuclear force, but every other proton in the nucleus exerts a repulsive electrical force on it. If the nucleus is large enough, the total electrical repulsion may be sufficient to overcome the attraction of the strong force, and the nucleus may spit out a proton. Proton emission is fairly rare, however; a more common type of radioactive decay<sup>1</sup> in heavy nuclei is alpha decay, shown in aa/4. The imbalance of the forces is similar, but the chunk that is ejected is an alpha particle (two protons and two neutrons) rather than a single proton.

It is also possible for the nucleus to split into two pieces of roughly equal size, aa/5, a process known as fission. Note that in addition to the two large fragments, there is a spray of individual neutrons. In a nuclear fission bomb or a nuclear fission reactor, some of these

<sup>1</sup>Alpha decay is more common because an alpha particle happens to be a very stable arrangement of protons and neutrons.

neutrons fly off and hit other nuclei, causing them to undergo fission as well. The result is a chain reaction.

When a nucleus is able to undergo one of these processes, it is said to be radioactive, and to undergo radioactive decay. Some of the naturally occurring nuclei on earth are radioactive. The term “radioactive” comes from Becquerel’s image of rays radiating out from something, not from radio waves, which are a whole different phenomenon. The term “decay” can also be a little misleading, since it implies that the nucleus turns to dust or simply disappears – actually it is splitting into two new nuclei with the same total number of neutrons and protons, so the term “radioactive transformation” would have been more appropriate. Although the original atom’s electrons are mere spectators in the process of weak radioactive decay, we often speak loosely of “radioactive atoms” rather than “radioactive nuclei.”

### *Randomness in physics*

How does an atom decide when to decay? We might imagine that it is like a termite-infested house that gets weaker and weaker, until finally it reaches the day on which it is destined to fall apart. Experiments, however, have not succeeded in detecting such “ticking clock” hidden below the surface; the evidence is that all atoms of a given isotope are absolutely identical. Why, then, would one uranium atom decay today while another lives for another million years? The answer appears to be that it is entirely random. We can make general statements about the average time required for a certain isotope to decay, or how long it will take for half the atoms in a sample to decay (its half-life), but we can never predict the behavior of a particular atom.

This is the first example we have encountered of an inescapable randomness in the laws of physics. If this kind of randomness makes you uneasy, you’re in good company. Einstein’s famous quote is “...I am convinced that He [God] does not play dice.” Einstein’s distaste for randomness, and his association of determinism with divinity, goes back to the Enlightenment conception of the universe as a gigantic piece of clockwork that only had to be set in motion initially by the Builder. Physics had to be entirely rebuilt in the 20th century to incorporate the fundamental randomness of physics, and this modern revolution is the topic of chapters 33-26. In particular, we will delay the mathematical development of the half-life concept until then.

### **The weak nuclear force; beta decay**

All the nuclear processes we’ve discussed so far have involved rearrangements of neutrons and protons, with no change in the total number of neutrons or the total number of protons. Now consider the proportions of neutrons and protons in your body and in the

planet earth: neutrons and protons are roughly equally numerous in your body's carbon and oxygen nuclei, and also in the nickel and iron that make up most of the earth. The proportions are about 50-50. But, as discussed in more detail on p. 741, the only chemical elements produced in any significant quantities by the big bang<sup>2</sup> were hydrogen (about 90%) and helium (about 10%). If the early universe was almost nothing but hydrogen atoms, whose nuclei are protons, where did all those neutrons come from?

The answer is that there is another nuclear force, the weak nuclear force, that is capable of transforming neutrons into protons and vice-versa. Two possible reactions are

$$n \rightarrow p + e^- + \bar{\nu} \quad [\text{electron decay}]$$

and

$$p \rightarrow n + e^+ + \nu \quad . \quad [\text{positron decay}]$$

(There is also a third type called electron capture, in which a proton grabs one of the atom's electrons and they produce a neutron and a neutrino.)

Whereas alpha decay and fission are just a redivision of the previously existing particles, these reactions involve the destruction of one particle and the creation of three new particles that did not exist before.

There are three new particles here that you have never previously encountered. The symbol  $e^+$  stands for an antielectron, which is a particle just like the electron in every way, except that its electric charge is positive rather than negative. Antielectrons are also known as positrons. Nobody knows why electrons are so common in the universe and antielectrons are scarce. When an antielectron encounters an electron, they annihilate each other, producing gamma rays, and this is the fate of all the antielectrons that are produced by natural radioactivity on earth. Antielectrons are an example of antimatter. A complete atom of antimatter would consist of antiprotons, antielectrons, and antineutrons. Although individual particles of antimatter occur commonly in nature due to natural radioactivity and cosmic rays, only a few complete atoms of antihydrogen have ever been produced artificially.

The notation  $\nu$  stands for a particle called a neutrino, and  $\bar{\nu}$  means an antineutrino. Neutrinos and antineutrinos have no electric charge (hence the name).

We can now list all four of the known fundamental forces of physics:

- gravity

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<sup>2</sup>The evidence for the big bang theory of the origin of the universe was discussed in section 19.5.1.

- electromagnetism
- strong nuclear force
- weak nuclear force

The other forces we have learned about, such as friction and the normal force, all arise from electromagnetic interactions between atoms, and therefore are not considered to be fundamental forces of physics.

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#### *Decay of $^{212}\text{Pb}$*

*example 5*

As an example, consider the radioactive isotope of lead  $^{212}\text{Pb}$ . It contains 82 protons and 130 neutrons. It decays by the process  $n \rightarrow p + e^- + \bar{\nu}$ . The newly created proton is held inside the nucleus by the strong nuclear force, so the new nucleus contains 83 protons and 129 neutrons. Having 83 protons makes it the element bismuth, so it will be an atom of  $^{212}\text{Bi}$ .

In a reaction like this one, the electron flies off at high speed (typically close to the speed of light), and the escaping electrons are the things that make large amounts of this type of radioactivity dangerous. The outgoing electron was the first thing that tipped off scientists in the early 1900s to the existence of this type of radioactivity. Since they didn't know that the outgoing particles were electrons, they called them beta particles, and this type of radioactive decay was therefore known as beta decay. A clearer but less common terminology is to call the two processes electron decay and positron decay.

The neutrino or antineutrino emitted in such a reaction pretty much ignores all matter, because its lack of charge makes it immune to electrical forces, and it also remains aloof from strong nuclear interactions. Even if it happens to fly off going straight down, it is almost certain to make it through the entire earth without interacting with any atoms in any way. It ends up flying through outer space forever. The neutrino's behavior makes it exceedingly difficult to detect, and when beta decay was first discovered nobody realized that neutrinos even existed. We now know that the neutrino carries off some of the energy produced in the reaction, but at the time it seemed that the total energy afterwards (not counting the unsuspected neutrino's energy) was greater than the total energy before the reaction, violating conservation of energy. Physicists were getting ready to throw conservation of energy out the window as a basic law of physics when indirect evidence led them to the conclusion that neutrinos existed.

#### *The solar neutrino problem*

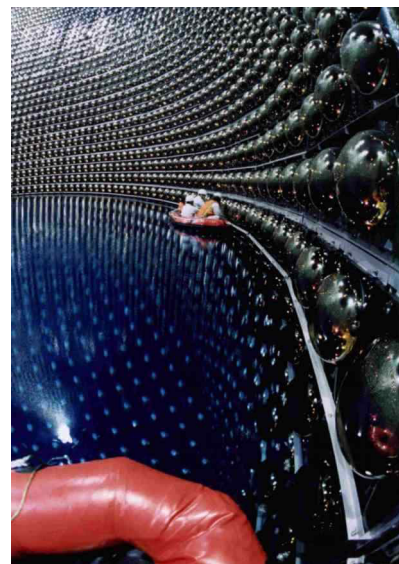
What about these neutrinos? Why haven't you heard of them before? It's not because they're rare — a billion neutrinos pass through

your body every microsecond, but until recently almost nothing was known about them. Produced as a side-effect of the nuclear reactions that power our sun and other stars, these ghostlike bits of matter are believed to be the most numerous particles in the universe. But they interact so weakly with ordinary matter that nearly all the neutrinos that enter the earth on one side will emerge from the other side of our planet without even slowing down.

Our first real peek at the properties of the elusive neutrino has come from a huge detector in a played-out Japanese zinc mine, figure ab. An international team of physicists outfitted the mineshaft with wall-to-wall light sensors, and then filled the whole thing with water so pure that you can see through it for a hundred meters, compared to only a few meters for typical tap water. Neutrinos stream through the 50 million liters of water continually, just as they flood everything else around us, and the vast majority never interact with a water molecule. A very small percentage, however, do annihilate themselves in the water, and the tiny flashes of light they produce can be detected by the beachball-sized vacuum tubes that line the darkened mineshaft. Most of the neutrinos around us come from the sun, but for technical reasons this type of water-based detector is more sensitive to the less common but more energetic neutrinos produced when cosmic ray particles strike the earth's atmosphere.

Neutrinos were already known to come in three “flavors,” which can be distinguished from each other by the particles created when they collide with matter. An “electron-flavored neutrino” creates an ordinary electron when it is annihilated, while the two other types create more exotic particles called mu and tau particles. Think of the three types of neutrinos as chocolate, vanilla, and strawberry. When you buy a chocolate ice cream cone, you expect that it will keep being chocolate as you eat it. The unexpected finding from the Japanese experiment is that some of the neutrinos are changing flavor between the time when they are produced by a cosmic ray and the moment when they wink out of existence in the water. It's as though your chocolate ice cream cone transformed itself magically into strawberry while your back was turned.

How did the physicists figure out the change in flavor? The experiment detects some neutrinos originating in the atmosphere above Japan, and also many neutrinos coming from distant parts of the earth. A neutrino created above the Atlantic Ocean arrives in Japan from underneath, and the experiment can distinguish these upward-traveling neutrinos from the downward-moving local variety. They found that the mixture of neutrinos coming from below was different from the mixture arriving from above, with some of the electron-flavored and tau-flavored neutrinos having apparently changed into mu-flavored neutrinos during their voyage through the earth. The ones coming from above didn't have time to change flavors on their



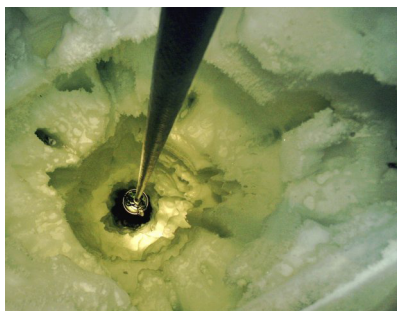
ab / This neutrino detector is in the process of being filled with ultrapure water.



much shorter journey.

This is interpreted as evidence that the neutrinos are constantly changing back and forth among the three flavors. On theoretical grounds, it is believed that such a vibration can only occur if neutrinos have mass. Only a rough estimate of the mass is possible at this point: it appears that neutrinos have a mass somewhere in the neighborhood of one billionth of the mass of an electron, or about  $10^{-39}$  kg.

If the neutrino's mass is so tiny, does it even matter? It matters to astronomers. Neutrinos are the only particles that can be used to probe certain phenomena. For example, they are the only direct probes we have for testing our models of the core of our own sun, which is the source of energy for all life on earth. Once astronomers have a good handle on the basic properties of the neutrino, they can start thinking seriously about using them for astronomy. As of 2006, the mass of the neutrino has been confirmed by an accelerator-based experiment, and neutrino observatories have been operating for a few years in Antarctica, using huge volumes of natural ice in the same way that the water was used in the Japanese experiment.



ac / A detector being lowered down a shaft at the IceCube neutrino telescope in Antarctica.

**A** In the reactions  $n \rightarrow p + e^- + \bar{\nu}$  and  $p \rightarrow n + e^+ + \nu$ , verify that charge is conserved. In beta decay, when one of these reactions happens to a neutron or proton within a nucleus, one or more gamma rays may also be emitted. Does this affect conservation of charge? Would it be possible for some extra electrons to be released without violating charge conservation?

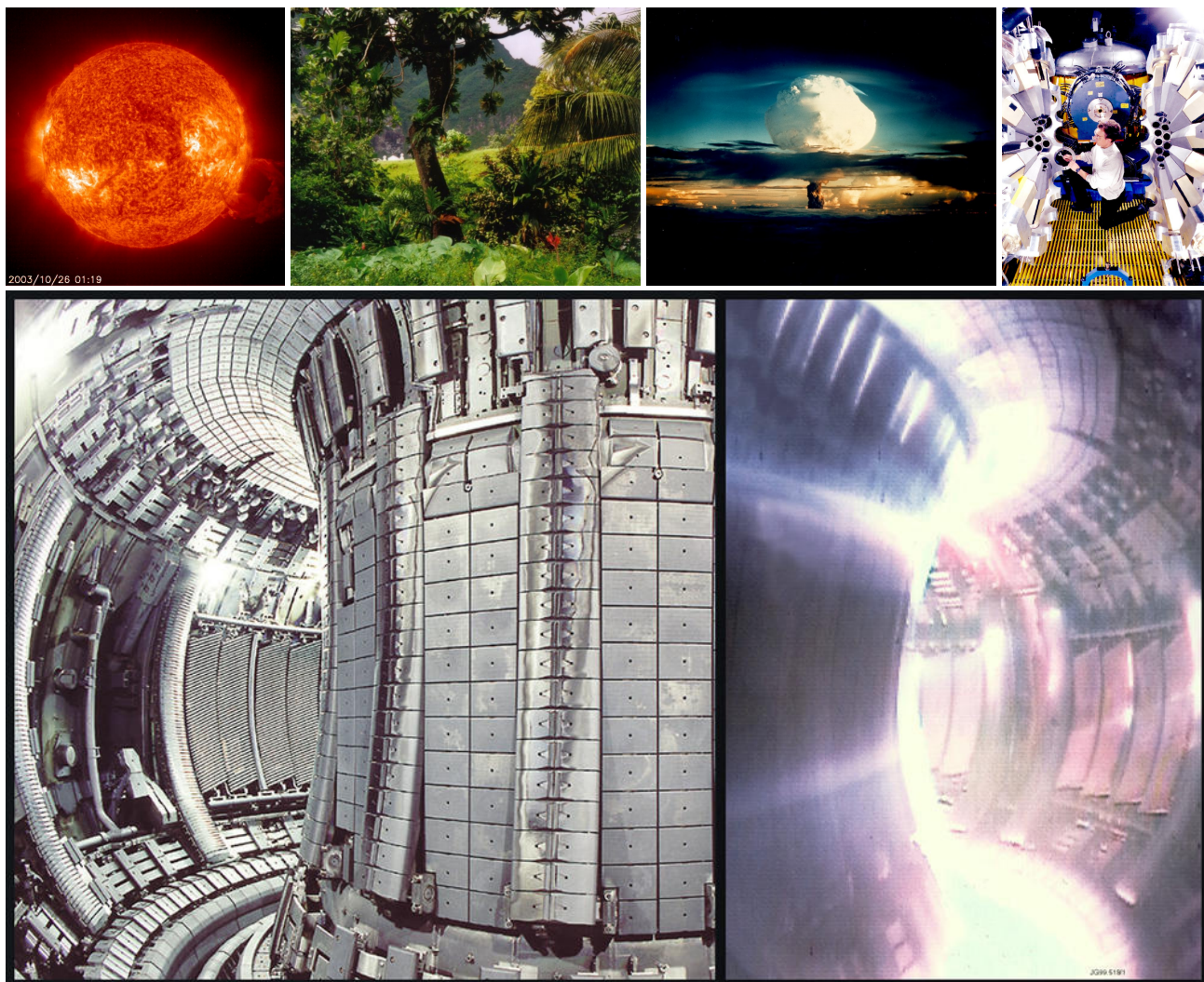
**B** When an antielectron and an electron annihilate each other, they produce two gamma rays. Is charge conserved in this reaction?

## Fusion

As we have seen, heavy nuclei tend to fly apart because each proton is being repelled by every other proton in the nucleus, but is only attracted by its nearest neighbors. The nucleus splits up into two parts, and as soon as those two parts are more than about 1 fm apart, the strong nuclear force no longer causes the two fragments to attract each other. The electrical repulsion then accelerates them, causing them to gain a large amount of kinetic energy. This release of kinetic energy is what powers nuclear reactors and fission bombs.

It might seem, then, that the lightest nuclei would be the most stable, but that is not the case. Let's compare an extremely light nucleus like  ${}^4\text{He}$  with a somewhat heavier one,  ${}^{16}\text{O}$ . A neutron or proton in  ${}^4\text{He}$  can be attracted by the three others, but in  ${}^{16}\text{O}$ , it might have five or six neighbors attracting it. The  ${}^{16}\text{O}$  nucleus is therefore more stable.

It turns out that the most stable nuclei of all are those around nickel



ad / 1. Our sun's source of energy is nuclear fusion, so nuclear fusion is also the source of power for all life on earth, including, 2, this rain forest in Fatu-Hiva. 3. The first release of energy by nuclear fusion through human technology was the 1952 Ivy Mike test at the Enewetak Atoll. 4. This array of gamma-ray detectors is called GAMMASPHERE. During operation, the array is closed up, and a beam of ions produced by a particle accelerator strikes a target at its center, producing nuclear fusion reactions. The gamma rays can be studied for information about the structure of the fused nuclei, which are typically varieties not found in nature. 5. Nuclear fusion promises to be a clean, inexhaustible source of energy. However, the goal of commercially viable nuclear fusion power has remained elusive, due to the engineering difficulties involved in magnetically containing a plasma (ionized gas) at a sufficiently high temperature and density. This photo shows the experimental JET reactor, with the device opened up on the left, and in action on the right.

and iron, having about 30 protons and 30 neutrons. Just as a nucleus that is too heavy to be stable can release energy by splitting apart into pieces that are closer to the most stable size, light nuclei can release energy if you stick them together to make bigger nuclei that are closer to the most stable size. Fusing one nucleus with another is called nuclear fusion. Nuclear fusion is what powers our sun and other stars.

## Nuclear energy and binding energies

In the same way that chemical reactions can be classified as exothermic (releasing energy) or endothermic (requiring energy to react), so nuclear reactions may either release or use up energy. The energies involved in nuclear reactions are greater by a huge factor. Thousands of tons of coal would have to be burned to produce as much energy as would be produced in a nuclear power plant by one kg of fuel.

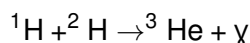
Although nuclear reactions that use up energy (endothermic reactions) can be initiated in accelerators, where one nucleus is rammed into another at high speed, they do not occur in nature, not even in the sun. The amount of kinetic energy required is simply not available.

To find the amount of energy consumed or released in a nuclear reaction, you need to know how much nuclear interaction energy,  $U_{nuc}$ , was stored or released. Experimentalists have determined the amount of nuclear energy stored in the nucleus of every stable element, as well as many unstable elements. This is the amount of mechanical work that would be required to pull the nucleus apart into its individual neutrons and protons, and is known as the nuclear binding energy.

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*A reaction occurring in the sun* *example 6*

The sun produces its energy through a series of nuclear fusion reactions. One of the reactions is



The excess energy is almost all carried off by the gamma ray (not by the kinetic energy of the helium-3 atom). The binding energies in units of pJ (picojoules) are:

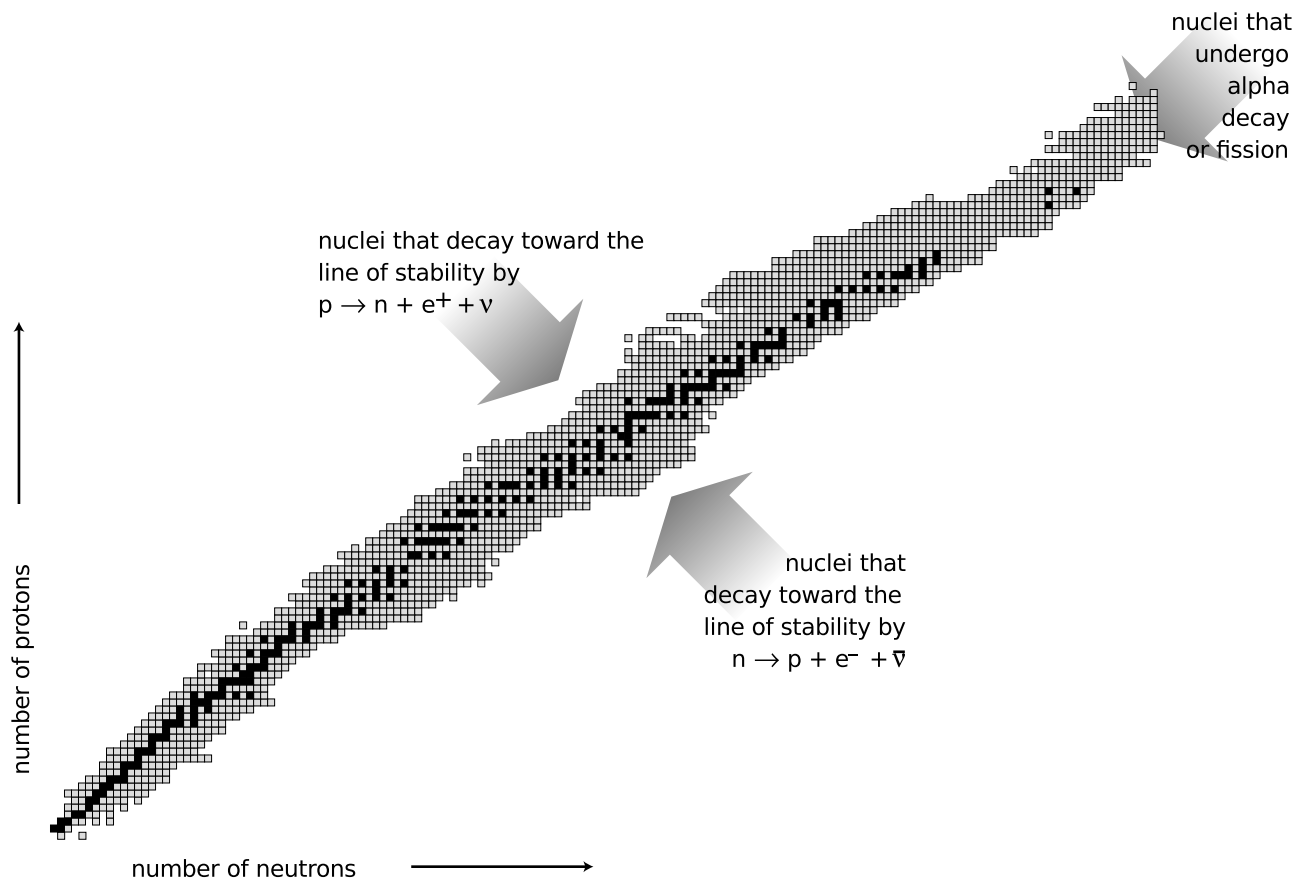
${}^1\text{H}$	0 J
${}^2\text{H}$	0.35593 pJ
${}^3\text{He}$	1.23489 pJ

The total initial nuclear energy is 0 pJ+0.35593 pJ, and the final nuclear energy is 1.23489 pJ, so by conservation of energy, the gamma ray must carry off 0.87896 pJ of energy. The gamma ray is then absorbed by the sun and converted to heat.

*self-check C*

Why is the binding energy of  ${}^1\text{H}$  exactly equal to zero?    ▷ Answer, p. 979

Figure ae is a compact way of showing the vast variety of the nuclei. Each box represents a particular number of neutrons and protons. The black boxes are nuclei that are stable, i.e., that would require an input of energy in order to change into another. The gray boxes show



ae / The known nuclei, represented on a chart of proton number versus neutron number. Note the two nuclei in the bottom row with zero protons. One is simply a single neutron. The other is a cluster of four neutrons. This “tetra-neutron” was reported, unexpectedly, to be a bound system in results from a 2002 experiment. The result is controversial. If correct, it implies the existence of a heretofore unsuspected type of matter, the neutron droplet, which we can think of as an atom with no protons or electrons.

all the unstable nuclei that have been studied experimentally. Some of these last for billions of years on the average before decaying and are found in nature, but most have much shorter average lifetimes, and can only be created and studied in the laboratory.

The curve along which the stable nuclei lie is called the line of stability. Nuclei along this line have the most stable proportion of neutrons to protons. For light nuclei the most stable mixture is about 50-50, but we can see that stable heavy nuclei have two or three times more neutrons than protons. This is because the electrical repulsions of all the protons in a heavy nucleus add up to a powerful force that would tend to tear it apart. The presence of a large number of neutrons increases the distances among the protons, and also increases the number of attractions due to the strong nuclear force.





af/A map showing levels of radiation near the site of the Chernobyl disaster. At the boundary of the most highly contaminated (bright red) areas, people would be exposed to about 13,000  $\mu\text{Sv}$  per year, or about four times the natural background level. In the pink areas, which are still densely populated, the exposure is comparable to the natural level found in a high-altitude city such as Denver.

## Biological effects of ionizing radiation

As a science educator, I find it frustrating that nowhere in the massive amount of journalism devoted to the Chernobyl disaster does one ever find any numerical statements about the amount of radiation to which people have been exposed. Anyone mentally capable of understanding sports statistics or weather reports ought to be able to understand such measurements, as long as something like the following explanatory text was inserted somewhere in the article:

Radiation exposure is measured in units of Sieverts (Sv). The average person is exposed to about 2000  $\mu\text{Sv}$  (microSieverts) each year from natural background sources.

With this context, people would be able to come to informed conclusions based on statements such as, “Children in Finland received an average dose of \_\_\_\_\_  $\mu\text{Sv}$  above natural background levels because of the Chernobyl disaster.”

What is a Sievert? It measures the amount of energy per kilogram deposited in the body by ionizing radiation, multiplied by a “quality factor” to account for the different health hazards posed by alphas, betas, gammas, neutrons, and other types of radiation. Only ionizing radiation is counted, since nonionizing radiation simply heats one’s body rather than killing cells or altering DNA. For instance, alpha particles are typically moving so fast that their kinetic energy is sufficient to ionize thousands of atoms, but it is possible for an alpha particle to be moving so slowly that it would not have enough kinetic energy to ionize even one atom.

Notwithstanding the pop culture images of the Incredible Hulk and Godzilla, it is not possible for a multicellular animal to become “mutated” as a whole. In most cases, a particle of ionizing radiation will not even hit the DNA, and even if it does, it will only affect the DNA of a single cell, not every cell in the animal’s body. Typically, that cell is simply killed, because the DNA becomes unable to function properly. Once in a while, however, the DNA may be altered so as to make that cell cancerous. For instance, skin cancer can be caused by UV light hitting a single skin cell in the body of a sunbather. If that cell becomes cancerous and begins reproducing uncontrollably, she will end up with a tumor twenty years later.

Other than cancer, the only other dramatic effect that can result from altering a single cell’s DNA is if that cell happens to be a sperm or ovum, which can result in nonviable or mutated offspring. Men are relatively immune to reproductive harm from radiation, because their sperm cells are replaced frequently. Women are more vulnerable because they keep the same set of ova as long as they live.

A whole-body exposure of 5,000,000  $\mu\text{Sv}$  will kill a person within a week or so. Luckily, only a small number of humans have ever been

exposed to such levels: one scientist working on the Manhattan Project, some victims of the Nagasaki and Hiroshima explosions, and 31 workers at Chernobyl. Death occurs by massive killing of cells, especially in the blood-producing cells of the bone marrow.

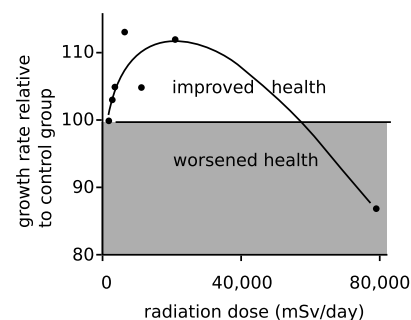
Lower levels, on the order of 1,000,000  $\mu\text{Sv}$ , were inflicted on some people at Nagasaki and Hiroshima. No acute symptoms result from this level of exposure, but certain types of cancer are significantly more common among these people. It was originally expected that the radiation would cause many mutations resulting in birth defects, but very few such inherited effects have been observed.

A great deal of time has been spent debating the effects of very low levels of ionizing radiation. The following table gives some sample figures.

maximum <i>beneficial</i> dose per day	$\sim 10,000 \mu\text{Sv}$
CT scan	$\sim 10,000 \mu\text{Sv}$
natural background per year	2,000-7,000 $\mu\text{Sv}$
health guidelines for exposure to a fetus	1,000 $\mu\text{Sv}$
flying from New York to Tokyo	150 $\mu\text{Sv}$
chest x-ray	50 $\mu\text{Sv}$

Note that the largest number, on the first line of the table, is the maximum *beneficial* dose. The most useful evidence comes from experiments in animals, which can intentionally be exposed to significant and well measured doses of radiation under controlled conditions. Experiments show that low levels of radiation activate cellular damage control mechanisms, increasing the health of the organism. For example, exposure to radiation up to a certain level makes mice grow faster; makes guinea pigs' immune systems function better against diphtheria; increases fertility in trout and mice; improves fetal mice's resistance to disease; increases the life-spans of flour beetles and mice; and reduces mortality from cancer in mice. This type of effect is called radiation hormesis.

There is also some evidence that in humans, small doses of radiation increase fertility, reduce genetic abnormalities, and reduce mortality from cancer. The human data, however, tend to be very poor compared to the animal data. Due to ethical issues, one cannot do controlled experiments in humans. For example, one of the best sources of information has been from the survivors of the Hiroshima and Nagasaki bomb blasts, but these people were also exposed to high levels of carcinogenic chemicals in the smoke from their burning cities; for comparison, firefighters have a heightened risk of cancer, and there are also significant concerns about cancer from the 9/11 attacks in New York. The direct empirical evidence about radiation hormesis in humans is therefore not good enough to tell us anything unambiguous,<sup>3</sup> and the most scientifically reasonable approach is to



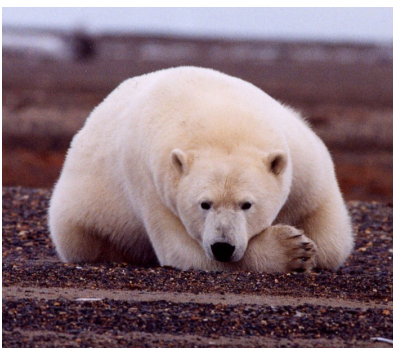
ag / A typical example of radiation hormesis: the health of mice is improved by low levels of radiation. In this study, young mice were exposed to fairly high levels of x-rays, while a control group of mice was not exposed. The mice were weighed, and their rate of growth was taken as a measure of their health. At levels below about 50,000  $\mu\text{Sv}$ , the radiation had a beneficial effect on the health of the mice, presumably by activating cellular damage control mechanisms. The two highest data points are statistically significant at the 99% level. The curve is a fit to a theoretical model. Redrawn from T.D. Luckey, *Hormesis with Ionizing Radiation*, CRC Press, 1980.

<sup>3</sup>For two opposing viewpoints, see Tubiana et al., "The Linear No-Threshold Relationship Is Inconsistent with Radiation Biologic and Experimental Data,"

assume that the results in animals also hold for humans: small doses of radiation in humans are beneficial, rather than harmful. However, a variety of cultural and historical factors have led to a situation in which public health policy is based on the assumption, known as “linear no-threshold” (LNT), that even tiny doses of radiation are harmful, and that the risk they carry is proportional to the dose. In other words, law and policy are made based on the assumption that the effects of radiation on humans are dramatically different than its effects on mice and guinea pigs. Even with the unrealistic assumption of LNT, one can still evaluate risks by comparing with natural background radiation. For example, we can see that the effect of a chest x-ray is about a hundred times smaller than the effect of spending a year in Colorado, where the level of natural background radiation from cosmic rays is higher than average, due to the high altitude. Dropping the implausible LNT assumption, we can see that the impact on one’s health of spending a year in Colorado is likely to be *positive*, because the excess radiation is below the maximum beneficial level.



ah / Wild Przewalski’s horses prosper in the Chernobyl area.



ai / Fossil fuels have done incomparably more damage to the environment than nuclear power ever has. Polar bears’ habitat is rapidly being destroyed by global warming.

In the late twentieth century, antinuclear activists largely succeeded in bringing construction of new nuclear power plants to a halt in the U.S. Ironically, we now know that the burning of fossil fuels, which leads to global warming, is a far more grave threat to the environment than even the Chernobyl disaster. A team of biologists writes: “During recent visits to Chernobyl, we experienced numerous sightings of moose (*Alces alces*), roe deer (*Capreol capreolus*), Russian wild boar (*Sus scrofa*), foxes (*Vulpes vulpes*), river otter (*Lutra canadensis*), and rabbits (*Lepus europaeus*) ... Diversity of flowers and other plants in the highly radioactive regions is impressive and equals that observed in protected habitats outside the zone ... The observation that typical human activity (industrialization, farming, cattle raising, collection of firewood, hunting, etc.) is more devastating to biodiversity and abundance of local flora and fauna than is the worst nuclear power plant disaster validates the negative impact the exponential growth of human populations has on wildlife.”<sup>4</sup> Nuclear power is the only source of energy that is sufficient to replace any significant percentage of energy from fossil fuels on the rapid schedule demanded by the speed at which global warming is progressing. People worried about the downside of nuclear energy might be better off putting their energy into issues related to nuclear weapons: the poor stewardship of the former Soviet Union’s

Radiology, 251 (2009) 13 and Little et al., “Risks Associated with Low Doses and Low Dose Rates of Ionizing Radiation: Why Linearity May Be (Almost) the Best We Can Do,” Radiology, 251 (2009) 6.

<sup>4</sup>Baker and Chesser, *Env. Toxicology and Chem.* 19 (1231) 2000. Similar effects have been seen at the Bikini Atoll, the site of a 1954 hydrogen bomb test. Although some species have disappeared from the area, the coral reef is in many ways healthier than similar reefs elsewhere, because humans have tended to stay away for fear of radiation (Richards et al., *Marine Pollution Bulletin* 56 (2008) 503).

warheads; nuclear proliferation in unstable states such as Pakistan; and the poor safety and environmental history of the superpowers' nuclear weapons programs, including the loss of several warheads in plane crashes, and the environmental disaster at the Hanford, Washington, weapons plant.

## The creation of the elements

### *Creation of hydrogen and helium in the Big Bang*

Did all the chemical elements we're made of come into being in the big bang?<sup>5</sup> Temperatures in the first microseconds after the big bang were so high that atoms and nuclei could not hold together at all. After things had cooled down enough for nuclei and atoms to exist, there was a period of about three minutes during which the temperature and density were high enough for fusion to occur, but not so high that atoms could hold together. We have a good, detailed understanding of the laws of physics that apply under these conditions, so theorists are able to say with confidence that the only element heavier than hydrogen that was created in significant quantities was helium.

### *We are stardust*

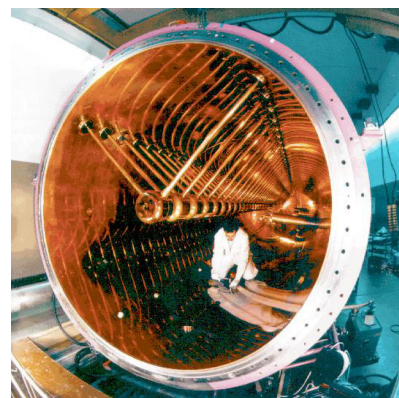
In that case, where did all the other elements come from? Astronomers came up with the answer. By studying the combinations of wavelengths of light, called spectra, emitted by various stars, they had been able to determine what kinds of atoms they contained. (We will have more to say about spectra at the end of this book.) They found that the stars fell into two groups. One type was nearly 100% hydrogen and helium, while the other contained 99% hydrogen and helium and 1% other elements. They interpreted these as two generations of stars. The first generation had formed out of clouds of gas that came fresh from the big bang, and their composition reflected that of the early universe. The nuclear fusion reactions by which they shine have mainly just increased the proportion of helium relative to hydrogen, without making any heavier elements. The members of the first generation that we see today, however, are only those that lived a long time. Small stars are more miserly with their fuel than large stars, which have short lives. The large stars of the first generation have already finished their lives. Near the end of its lifetime, a star runs out of hydrogen fuel and undergoes a series of violent and spectacular reorganizations as it fuses heavier and heavier elements. Very large stars finish this sequence of events by undergoing supernova explosions, in which some of their material is flung off into the void while the rest collapses into an exotic object such as a black hole or neutron star.

The second generation of stars, of which our own sun is an example,

<sup>5</sup>The evidence for the big bang theory of the origin of the universe was discussed in book 3 of this series.



aj / The Crab Nebula is a remnant of a supernova explosion. Almost all the elements our planet is made of originated in such explosions.



ak / Construction of the UNILAC accelerator in Germany, one of whose uses is for experiments to create very heavy artificial elements. In such an experiment, fusion products recoil through a device called SHIP (not shown) that separates them based on their charge-to-mass ratios — it is essentially just a scaled-up version of Thomson's apparatus. A typical experiment runs for several months, and out of the billions of fusion reactions induced during this time, only one or two may result in the production of superheavy atoms. In all the rest, the fused nucleus breaks up immediately. SHIP is used to identify the small number of "good" reactions and separate them from this intense background.



condensed out of clouds of gas that had been enriched in heavy elements due to supernova explosions. It is those heavy elements that make up our planet and our bodies.

#### *Artificial synthesis of heavy elements*

Elements up to uranium, atomic number 92, were created by these astronomical processes. Beyond that, the increasing electrical repulsion of the protons leads to shorter and shorter half-lives. Even if a supernova a billion years ago did create some quantity of an element such as Berkelium, number 97, there would be none left in the Earth's crust today. The heaviest elements have all been created by artificial fusion reactions in accelerators. As of 2010, the heaviest element that has been created is 118.

Although the creation of a new element, i.e., an atom with a novel number of protons, has historically been considered a glamorous accomplishment, to the nuclear physicist the creation of an atom with a hitherto unobserved number of neutrons is equally important. The greatest neutron number reached so far is 179. One tantalizing goal of this type of research is the theoretical prediction that there might be an island of stability beyond the previously explored tip of the chart of the nuclei shown on p. 26.4.8. Just as certain numbers of electrons lead to the chemical stability of the noble gases (helium, neon, argon, ...), certain numbers of neutrons and protons lead to a particularly stable packing of orbits. Calculations dating back to the 1960's have hinted that there might be relatively stable nuclei having approximately 114 protons and 184 neutrons. The isotopes of elements 114 and 116 that have been produced so far have had half-lives in the second or millisecond range. This may not seem like very long, but lifetimes in the microsecond range are more typical for the superheavy elements that have previously been discovered. There is even speculation that certain superheavy isotopes would be stable enough to be produced in quantities that could for instance be weighed and used in chemical reactions.

#### **Discussion questions**

**A** Should the quality factor for neutrinos be very small, because they mostly don't interact with your body?

**B** Would an alpha source be likely to cause different types of cancer depending on whether the source was external to the body or swallowed in contaminated food? What about a gamma source?

### **26.5 Relativistic mass and energy**

The radioactive decay processes described in this chapter do not conserve mass. For example, you're probably reading this in a building that has smoke detectors containing the isotope  $^{241}\text{Am}$ , an alpha emitter. The decay products are a helium atom plus an atom of  $^{237}\text{Np}$ . In units of  $10^{-27}$  kg, the masses involved are:

$^{241}\text{Am}$	400.28421
$^{237}\text{Np}$	393.62768
$^4\text{He}$	6.64647

The final state has a total mass of 400.27415 of these units, meaning a loss of 0.01006. This is an example of Einstein's famous  $E = mc^2$  at work: some mass has been converted into energy. In fact this type of mass-energy conversion is not just a property of nuclear decay. It is an example of a much wider set of phenomena in relativity. Let's see how quantities like mass, force, momentum, and energy behave relativistically.

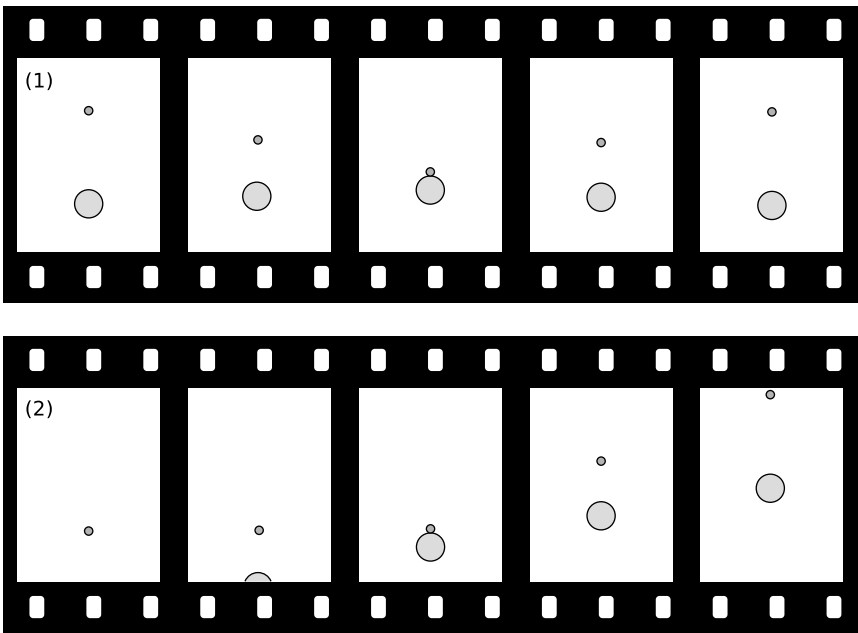
## Momentum

Consider the following scheme for traveling faster than the speed of light. The basic idea can be demonstrated by dropping a ping-pong ball and a baseball stacked on top of each other like a snowman. They separate slightly in mid-air, and the baseball therefore has time to hit the floor and rebound before it collides with the ping-pong ball, which is still on the way down. The result is a surprise if you haven't seen it before: the ping-pong ball flies off at high speed and hits the ceiling! A similar fact is known to people who investigate the scenes of accidents involving pedestrians. If a car moving at 90 kilometers per hour hits a pedestrian, the pedestrian flies off at nearly double that speed, 180 kilometers per hour. Now suppose the car was moving at 90 percent of the speed of light. Would the pedestrian fly off at 180% of  $c$ ?

To see why not, we have to back up a little and think about where this speed-doubling result comes from. For any collision, there is a special frame of reference, the center-of-mass frame, in which the two colliding objects approach each other, collide, and rebound with their velocities reversed. In the center-of-mass frame, the total momentum of the objects is zero both before and after the collision.

Figure al/1 shows such a frame of reference for objects of very unequal mass. Before the collision, the large ball is moving relatively slowly toward the top of the page, but because of its greater mass, its momentum cancels the momentum of the smaller ball, which is moving rapidly in the opposite direction. The total momentum is zero. After the collision, the two balls just reverse their directions of motion. We know that this is the right result for the outcome of the collision because it conserves both momentum and kinetic energy, and everything not forbidden is compulsory, i.e., in any experiment, there is only one possible outcome, which is the one that obeys all the conservation laws.

al / An unequal collision, viewed in the center-of-mass frame, 1, and in the frame where the small ball is initially at rest, 2. The motion is shown as it would appear on the film of an old-fashioned movie camera, with an equal amount of time separating each frame from the next. Film 1 was made by a camera that tracked the center of mass, film 2 by one that was initially tracking the small ball, and kept on moving at the same speed after the collision.



self-check D

How do we know that momentum and kinetic energy are conserved in figure al/1? ▷ Answer, p. 979

Let’s make up some numbers as an example. Say the small ball is 1 kg, the big one 6 kg. In frame 1, let’s make the velocities as follows:

	before the collision	after the collision
◦	-0.6	0.6
●	0.1	-0.1

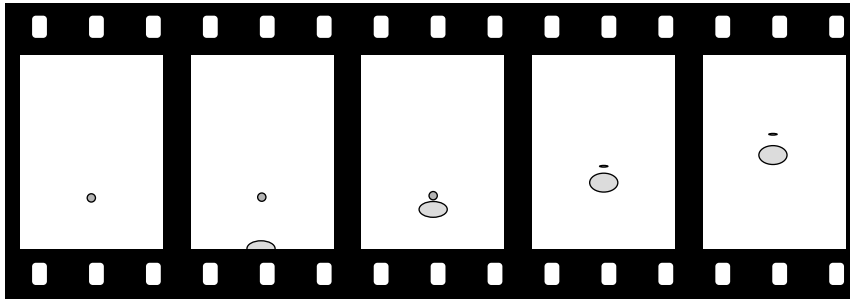
Figure al/2 shows the same collision in a frame of reference where the small ball was initially at rest. To find all the velocities in this frame, we just add 0.6 to all the ones in the previous table.

	before the collision	after the collision
◦	0	1.2
●	0.7	0.5

In this frame, as expected, the small ball flies off with a velocity, 1.2, that is almost twice the initial velocity of the big ball, 0.7. In this example the ratio of the two balls’ masses was 6, but if the ratio of the masses is made larger and larger, the ratio of the velocities gets closer and closer to 2.

If all those velocities were in meters per second, then that’s exactly what would happen. But what if all these velocities were in units of the speed of light? Now it’s no longer a good approximation just to add velocities. We need to combine them according to the relativistic rules. For instance, in problem 1 on p. 675 you showed that combining a velocity of 0.6 times the speed of light with another velocity of 0.6 results in 0.88, not 1.2. The results are very different:

	before the collision	after the collision
◦	0	0.88
●	0.67	0.51



am / A 6-kg ball moving at 88% of the speed of light hits a 1-kg ball. The balls appear foreshortened due to the relativistic distortion of space.

We can interpret this as follows. Figure al/1 is one in which the big ball is moving fairly slowly. This is very nearly the way the scene would be seen by an ant standing on the big ball. According to an observer in frame am, however, both balls are moving at nearly the speed of light after the collision. Because of this, the balls appear foreshortened, but the distance between the two balls is also shortened. To this observer, it seems that the small ball isn't pulling away from the big ball very fast.

Now here's what's interesting about all this. The outcome shown in figure al/2 was supposed to be the only one possible, the only one that satisfied both conservation of energy and conservation of momentum. So how can the *different* result shown in figure am be possible? The answer is that relativistically, momentum must not equal  $mv$ . The old, familiar definition is only an approximation that's valid at low speeds. If we observe the behavior of the small ball in figure am, it looks as though it somehow had some extra inertia. It's as though a football player tried to knock another player down without realizing that the other guy had a three-hundred-pound bag full of lead shot hidden under his uniform — he just doesn't seem to react to the collision as much as he should. This extra inertia is described<sup>6</sup> by redefining momentum as

$$p = m\gamma v \quad .$$

At very low velocities,  $\gamma$  is close to 1, and the result is very nearly  $mv$ , as demanded by the correspondence principle. But at very high velocities,  $\gamma$  gets very big — the small ball in figure am has a  $\gamma$  of 2.1, and therefore has more than double the inertia that we would expect nonrelativistically.

This also explains the answer to another paradox often posed by beginners at relativity. Suppose you keep on applying a steady force to an object that's already moving at  $0.9999c$ . Why doesn't it just

<sup>6</sup>See p. 753 for a proof.

keep on speeding up past  $c$ ? The answer is that force is the rate of change of momentum. At  $0.9999c$ , an object already has a  $\gamma$  of 71, and therefore has already sucked up 71 times the momentum you'd expect at that speed. As its velocity gets closer and closer to  $c$ , its  $\gamma$  approaches infinity. To move at  $c$ , it would need an infinite momentum, which could only be caused by an infinite force.

### Equivalence of mass and energy

Now we're ready to see why mass and energy must be equivalent as claimed in the famous  $E = mc^2$ . So far we've only considered collisions in which none of the kinetic energy is converted into any other form of energy, such as heat or sound. Let's consider what happens if a blob of putty moving at velocity  $v$  hits another blob that is initially at rest, sticking to it. The nonrelativistic result is that to obey conservation of momentum the two blobs must fly off together at  $v/2$ . Half of the initial kinetic energy has been converted to heat.<sup>7</sup>

Relativistically, however, an interesting thing happens. A hot object has more momentum than a cold object! This is because the relativistically correct expression for momentum is  $m\gamma v$ , and the more rapidly moving atoms in the hot object have higher values of  $\gamma$ . In our collision, the final combined blob must therefore be moving a little more slowly than the expected  $v/2$ , since otherwise the final momentum would have been a little greater than the initial momentum. To an observer who believes in conservation of momentum and knows only about the overall motion of the objects and not about their heat content, the low velocity after the collision would seem to be the result of a magical change in the mass, as if the mass of two combined, hot blobs of putty was more than the sum of their individual masses.

We know that the masses of all the atoms in the blobs must be the same as they always were. The change is due to the change in  $\gamma$  with heating, not to a change in mass. The heat energy, however, seems to be acting as if it was equivalent to some extra mass. If the quantity of heat is  $E$ , then it turns out that the extra mass  $m$  is such that  $E = mc^2$  (proof, p. 752).

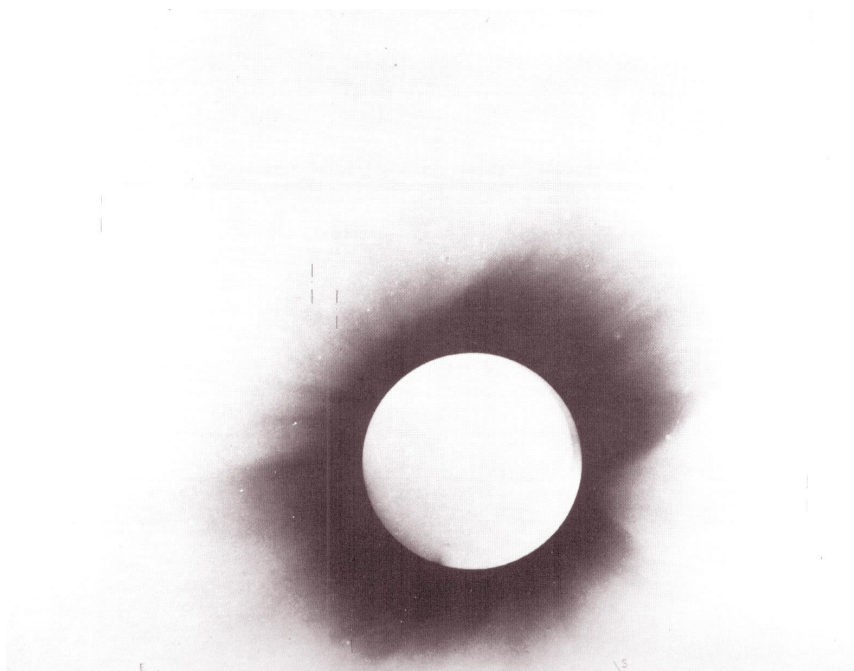
But this whole argument was based on the fact that heat is a form of kinetic energy at the atomic level. Would  $E = mc^2$  apply to other forms of energy as well? Suppose a rocket ship contains some electrical energy stored in a battery. If we believed that  $E = mc^2$  applied to forms of kinetic energy but not to electrical energy, then we would have to believe that the pilot of the rocket could slow the ship down by using the battery to run a heater! This would

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<sup>7</sup>A double-mass object moving at half the speed does not have the same kinetic energy. Kinetic energy depends on the square of the velocity, so cutting the velocity in half reduces the energy by a factor of 1/4, which, multiplied by the doubled mass, makes 1/2 the original energy.

not only be strange, but it would violate the principle of relativity, because the result of the experiment would be different depending on whether the ship was at rest or not. The only logical conclusion is that all forms of energy are equivalent to mass. Running the heater then has no effect on the motion of the ship, because the total energy in the ship was unchanged; one form of energy (electrical) was simply converted to another (heat).

The equation  $E = mc^2$  tells us how much energy is equivalent to how much mass: the conversion factor is the square of the speed of light,  $c$ . Since  $c$  a big number, you get a really really big number when you multiply it by itself to get  $c^2$ . This means that even a small amount of mass is equivalent to a very large amount of energy.



ao / Example 7.

#### *Gravity bending light*

*example 7*

Gravity is a universal attraction between things that have mass, and since the energy in a beam of light is equivalent to a some very small amount of mass, we expect that light will be affected by gravity, although the effect should be very small. The first important experimental confirmation of relativity came in 1919 when stars next to the sun during a solar eclipse were observed to have shifted a little from their ordinary position. (If there was no eclipse, the glare of the sun would prevent the stars from being observed.) Starlight had been deflected by the sun's gravity. Figure ao is a photographic negative, so the circle that appears bright is actually the dark face of the moon, and the dark area is really the bright

## LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less  
Agog Over Results of Eclipse  
Observations.

### EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed  
or Were Calculated to be,  
but Nobody Need Worry.

### A BOOK FOR 12 WISE MEN

No More in All the World Could  
Comprehend It, Said Einstein When  
His Daring Publishers Accepted It.

an / A New York Times headline from November 10, 1919, describing the observations discussed in example 7.

corona of the sun. The stars, marked by lines above and below then, appeared at positions slightly different than their normal ones.

#### *Black holes*

*example 8*

A star with sufficiently strong gravity can prevent light from leaving. Quite a few black holes have been detected via their gravitational forces on neighboring stars or clouds of gas and dust.

You've learned about conservation of mass and conservation of energy, but now we see that they're not even separate conservation laws. As a consequence of the theory of relativity, mass and energy are equivalent, and are not separately conserved — one can be converted into the other. Imagine that a magician waves his wand, and changes a bowl of dirt into a bowl of lettuce. You'd be impressed, because you were expecting that both dirt and lettuce would be conserved quantities. Neither one can be made to vanish, or to appear out of thin air. However, there are processes that can change one into the other. A farmer changes dirt into lettuce, and a compost heap changes lettuce into dirt. At the most fundamental level, lettuce and dirt aren't really different things at all; they're just collections of the same kinds of atoms — carbon, hydrogen, and so on. Because mass and energy are like two different sides of the same coin, we may speak of mass-energy, a single conserved quantity, found by adding up all the mass and energy, with the appropriate conversion factor:  $E + mc^2$ .

#### *A rusting nail*

*example 9*

▷ An iron nail is left in a cup of water until it turns entirely to rust. The energy released is about 0.5 MJ. In theory, would a sufficiently precise scale register a change in mass? If so, how much?

▷ The energy will appear as heat, which will be lost to the environment. The total mass-energy of the cup, water, and iron will indeed be lessened by 0.5 MJ. (If it had been perfectly insulated, there would have been no change, since the heat energy would have been trapped in the cup.) The speed of light is  $c = 3 \times 10^8$  meters per second, so converting to mass units, we have

$$\begin{aligned} m &= \frac{E}{c^2} \\ &= \frac{0.5 \times 10^6 \text{ J}}{(3 \times 10^8 \text{ m/s})^2} \\ &= 6 \times 10^{-12} \text{ kilograms} \end{aligned}$$

The change in mass is too small to measure with any practical technique. This is because the square of the speed of light is such a large number.

### Electron-positron annihilation

example 10

Natural radioactivity in the earth produces positrons, which are like electrons but have the opposite charge. A form of antimatter, positrons annihilate with electrons to produce gamma rays, a form of high-frequency light. Such a process would have been considered impossible before Einstein, because conservation of mass and energy were believed to be separate principles, and this process eliminates 100% of the original mass. The amount of energy produced by annihilating 1 kg of matter with 1 kg of antimatter is

$$\begin{aligned} E &= mc^2 \\ &= (2 \text{ kg}) (3.0 \times 10^8 \text{ m/s})^2 \\ &= 2 \times 10^{17} \text{ J} \end{aligned}$$

which is on the same order of magnitude as a day's energy consumption for the entire world's population!

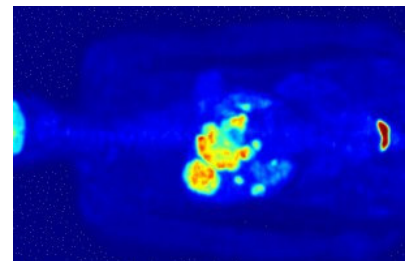
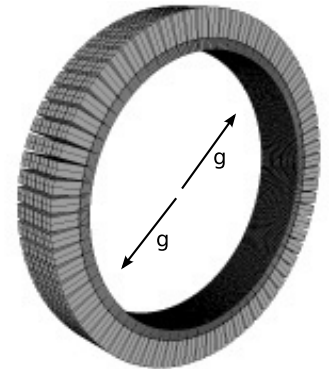
Positron annihilation forms the basis for the medical imaging technique called a PET (positron emission tomography) scan, in which a positron-emitting chemical is injected into the patient and mapped by the emission of gamma rays from the parts of the body where it accumulates.

One commonly hears some misinterpretations of  $E = mc^2$ , one being that the equation tells us how much kinetic energy an object would have if it was moving at the speed of light. This wouldn't make much sense, both because the equation for kinetic energy has 1/2 in it,  $KE = (1/2)mv^2$ , and because a material object can't be made to move at the speed of light. However, this naturally leads to the question of just how much mass-energy a moving object has. We know that when the object is at rest, it has no kinetic energy, so its mass-energy is simply equal to the energy-equivalent of its mass,  $mc^2$ ,

$$\mathbb{E} = mc^2 \text{ when } v = 0$$

where the symbol  $\mathbb{E}$  stands for mass-energy. (You can write this symbol yourself by writing an E, and then adding an extra line to it. Have fun!) The point of using the new symbol is simply to remind ourselves that we're talking about relativity, so an object at rest has  $\mathbb{E} = mc^2$ , not  $E = 0$  as we'd assume in classical physics.

Suppose we start accelerating the object with a constant force. A constant force means a constant rate of transfer of momentum, but  $p = mv$  approaches infinity as  $v$  approaches  $c$ , so the object will only get closer and closer to the speed of light, but never reach it. Now what about the work being done by the force? The force keeps doing work and doing work, which means that we keep on using up energy. Mass-energy is conserved, so the energy being expended must equal the increase in the object's mass-energy. We can continue



ap / Top: A PET scanner. Middle: Each positron annihilates with an electron, producing two gamma-rays that fly off back-to-back. When two gamma rays are observed simultaneously in the ring of detectors, they are assumed to come from the same annihilation event, and the point at which they were emitted must lie on the line connecting the two detectors. Bottom: A scan of a person's torso. The body has concentrated the radioactive tracer around the stomach, indicating an abnormal medical condition.



this process for as long as we like, and the amount of mass-energy will increase without limit. We therefore conclude that an object's mass-energy approaches infinity as its speed approaches the speed of light,

$$\mathbb{E} \rightarrow \infty \text{ when } v \rightarrow c \quad .$$

Now that we have some idea what to expect, what is the actual equation for the mass-energy? As proved in my book *Simple Nature*, it is

$$\mathbb{E} = m\gamma c^2 \quad .$$

*self-check E*

Verify that this equation has the two properties we wanted. ▷

Answer, p. 979

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*KE compared to  $mc^2$  at low speeds* *example 11*

▷ An object is moving at ordinary nonrelativistic speeds. Compare its kinetic energy to the energy  $mc^2$  it has purely because of its mass.

▷ The speed of light is a very big number, so  $mc^2$  is a huge number of joules. The object has a gigantic amount of energy because of its mass, and only a relatively small amount of additional kinetic energy because of its motion.

Another way of seeing this is that at low speeds,  $\gamma$  is only a tiny bit greater than 1, so  $\mathbb{E}$  is only a tiny bit greater than  $mc^2$ .

---

*The correspondence principle for mass-energy* *example 12*

▷ Show that the equation  $\mathbb{E} = m\gamma c^2$  obeys the correspondence principle.

▷ As we accelerate an object from rest, its mass-energy becomes greater than its resting value. We interpret this excess mass-energy as the object's kinetic energy,

$$\begin{aligned} KE &= \mathbb{E}(v) - \mathbb{E}(v = 0) \\ &= m\gamma c^2 - mc^2 \\ &= m(\gamma - 1)c^2 \quad . \end{aligned}$$

In example 1 on page 640, we found  $\gamma \approx 1 + v^2/2c^2$ , so

$$\begin{aligned} KE &\approx m\left(1 + \frac{v^2}{2c^2} - 1\right)c^2 \\ &= \frac{1}{2}mv^2 \quad , \end{aligned}$$

which is the nonrelativistic expression. As demanded by the correspondence principle, relativity agrees with nonrelativistic physics at speeds that are small compared to the speed of light.

## 26.6 ★ Proofs

In section 26.5 I gave physical arguments to the effect that relativistic momentum should be greater than  $mv$  and that an energy  $E$  should be equivalent relativistically to some amount of mass  $m$ . In this section I'll prove that the relativistic equations are as claimed:  $p = \gamma mv$  and  $E = mc^2$ . The structure of the proofs is essentially the same as in two famous 1905 papers by Einstein, "On the electrodynamics of moving bodies" and "Does the inertia of a body depend upon its energy content?" If you're interested in reading these arguments as Einstein originally wrote them, you can find English translations at [www.fourmilab.ch](http://www.fourmilab.ch). We start off by proving two preliminary results relating to Doppler shifts.

### Transformation of the fields in a light wave

On p. 669 I showed that when a light wave is observed in two different frames in different states of motion parallel to the wave's direction of motion, the frequency is observed to be Doppler-shifted by a factor  $D(v) = \sqrt{(1-v)/(1+v)}$ , where  $c = 1$  and  $v$  is the relative velocity of the two frames. But a change in frequency is not the only change we expect. We also expect the *intensity* of the wave to change, since a combination of electric and magnetic fields observed in one frame of reference becomes some other set of fields in a different frame (p. 646). There are equations that express this transformation from  $\mathbf{E}$  and  $\mathbf{B}$  to  $\mathbf{E}'$  and  $\mathbf{B}'$ , but they're a little complicated, so instead we'll just determine what happens in the special case of an electromagnetic wave.

Since the transformation of  $\mathbf{E}$  and  $\mathbf{B}$  to  $\mathbf{E}'$  and  $\mathbf{B}'$  is a universal thing, we're free to imagine that the wave was created in any way we wish. Suppose that it was created by a uniform sheet of charge in the  $x$ - $y$  plane, oscillating in the  $y$  direction with amplitude  $A$  and frequency  $f$ . This will clearly produce electromagnetic waves propagating in the  $+z$  and  $-z$  directions, and by an argument similar to that of problem 7 on p. 625, we know that these waves' intensity will not fall off at all with distance from the sheet. Since magnetic fields are produced by currents, and the currents produced by the motion of the sheet are proportional to  $Af$ , the amplitude of the magnetic field in the wave is proportional to  $Af$ . The oscillating magnetic field induces an electric field, and since electromagnetic waves always have  $E = Bc$ , the oscillating part of the electric field is also proportional of  $Af$ .

An observer moving away from the sheet sees a sheet that is both oscillating more slowly ( $f$  is Doppler-shifted to  $fD$ ) and receding. But the recession has no effect, because the fields don't fall off with distance. Also,  $A$  stays the same, because the Lorentz transformation has no effect on lengths perpendicular to the relative motion of the two frames. Since the fields are proportional to  $Af$ , the fields

seen by the receding observer are attenuated by a factor of  $D$ .

### Transformation of the volume of a light wave

Since the fields in an electromagnetic wave are changed by a factor of  $D$  when we change frames, we might expect that the wave's energy would change by a factor of  $D^2$ . But the square of the field only gives the energy per unit volume, and the volume changes as well. The following argument shows that the volume increases by a factor of  $1/D$ .

If an electromagnetic wave-train has duration  $\Delta t$ , we already know that its duration changes by a factor of  $1/D$  when we change to a different frame of reference. But the speed of light is the same for all observers, so if the length of the wave-train is  $\Delta z$ , all observers must agree on the value of  $\Delta z/\Delta t$ , and  $1/D$  must also be the factor by which  $\Delta z$  scales up.<sup>8</sup> Since the Lorentz transformation doesn't change  $\Delta x$  or  $\Delta y$ , the volume of the wave-train is also increased by a factor of  $1/D$ .

### Transformation of the energy of a light wave

Combining the two preceding results, we find that when we change frames of reference, the energy *density* (per unit volume) of a light wave changes by a factor of  $D^2$ , but the volume changes by  $1/D$ , so the result is that the wave's energy changes by a factor of  $D$ . In Einstein's words, "It is remarkable that the energy and the frequency of a [wave-train] vary with the state of motion of the observer in accordance with the same law," i.e., that both scale by the same factor  $D$ . Einstein had a reason to be especially interested in this fact. In the same "miracle year" of 1905, he also published a paper in which he hypothesized that light had both particle and wave properties, with the energy  $E$  of a light-particle related to the frequency  $f$  of the corresponding light-wave by  $E = hf$ , where  $h$  was a constant. (More about this in ch. 34.) If  $E$  and  $f$  had not both scaled by the same factor, then the relation  $E = hf$  could not have held in all frames of reference.

$$E = mc^2$$

Suppose that a material object O, initially at rest, emits two light rays, each with energy  $E$ , in the  $+z$  and  $-z$  directions. O could be a lantern with windows on opposite sides, or it could be an electron and an antielectron annihilating each other to produce a pair of gamma rays. In this frame, O loses energy  $2E$  and the light rays

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<sup>8</sup>At first glance, one might think that this length-scaling factor would simply be  $\gamma$ , and that the volume would be reduced rather than increased. But  $\gamma$  is only the scale-down factor for the length of a thing compared to that thing's length in a frame where it is at rest. Light waves don't have a frame in which they're at rest. One can also see this from the geometry of figure r on p. 670. The diagram is completely symmetric with respect to its treatment of time and space, so if we flip it across its diagonal, interchanging the roles of  $z$  and  $t$ , we obtain the same result for the wave-train's spatial extent  $\Delta z$ .

gain  $2E$ , so energy is conserved.

We now switch to a new frame of reference moving at a certain velocity  $v$  in the  $z$  direction relative to the original frame. We assume that  $O$ 's energy is different in this frame, but that the change in its energy amounts to multiplication by some unitless factor  $x$ , which depends only on  $v$ , since there is nothing else it could depend on that could allow us to form a unitless quantity. In this frame the light rays have energies  $ED(v)$  and  $ED(-v)$ . If conservation of energy is to hold in the new frame as it did in the old, we must have  $2xE = ED(v) + ED(-v)$ . After some algebra, we find  $x = 1/\sqrt{1-v^2}$ . In other words, an object with energy  $E$  in its rest frame has energy  $\gamma E$  in a frame moving at velocity  $v$  relative to the first one. Since  $\gamma$  is never zero, it follows that even an object at rest has some nonzero energy. We define this energy-at-rest as its mass, i.e.,  $E = m$  in units where  $c = 1$ .

$$P = m\gamma v$$

Defining an object's energy-at-rest as its mass only works if this same mass is also a valid measure of inertia. More specifically, we should be able to use this mass to construct a self-consistent logical system in which (1) momentum is conserved, (2) conservation of momentum holds in all frames of reference, and (3)  $p \approx mv$  for  $v \ll c$ , satisfying the correspondence principle.

Let a material object  $P$ , at rest and having mass  $2E$ , be completely annihilated, creating two beams of light, each with energy  $E$ , flying off in opposite directions. A real-world example would be if  $P$  consisted of an electron and an antielectron. As shown on p. 668, light has momentum. Because beams of light can be split up or recombined without violating conservation of momentum, a light wave's momentum must be proportional to its energy,  $|p| = yE$ , where the constant of proportionality  $y$  is found in problem 12 on p. 759 but not needed here. Let the momentum of a material object be  $mvx$ , where our goal is to prove  $x = \gamma$ . In this frame of reference,  $v = 0$ , and conservation of momentum follows by symmetry.

We now change to a new frame of reference, moving at some speed  $v$  along the line of emission of the two light rays. In this frame, conservation of momentum requires  $2Evx = yE/D - yED$ . We therefore have  $vx/y = (1/D - D)/2$ , which can be shown with a little algebra to equal  $v\gamma$ . Since only  $x$  can depend on  $v$ , not  $y$ , and the correspondence principle requires  $x \approx 1$  for  $v \ll c$ , we find that  $x = \gamma$ , as claimed.

Problem 15 on p. 761 checks that this result also works correctly for a system consisting of material particles.

## 26.7 ★ Tachyons

Is  $c$  really the ultimate speed limit in the universe, or does this rule have any loopholes? We've discussed three arguments against velocities greater than  $c$ :

1. Velocities don't add linearly, and the combination of two velocities less than  $c$  always gives another velocity less than  $c$  (p. 661 and problem 1 on page 675). Therefore no continuous process of acceleration can boost an object from a speed less than  $c$  to one greater than  $c$ .
2. Signals propagating at speeds greater than  $c$  would violate causality (figure f, p. 661).
3. For  $m \neq 0$ , the equation  $E = \gamma mc^2$  gives  $\infty$  at  $v = c$ , since  $\gamma$  becomes infinite.

A loophole in argument 1 is that we could have an object that had always existed at  $v > c$ , so that there was never any need to start it from  $v < c$ . Hypothetical subatomic particles like this are referred to as tachyons, from the Greek word for “fast.” From the time when it was created to the time it was destroyed, a tachyon would always be moving at  $v > c$ .

Argument 2 can be evaded simply by assuming that causality is violated. This seems to allow paradoxes such as the grandfather paradox, in which a person goes back in time and kills his own grandfather as a child — making it impossible for the murderer to have been born and committed the murder. There are however various ways in which violation of causality could be allowed without causing a lack of mathematical self-consistency; for a readable summary, see [plato.stanford.edu/entries/time-travel-phys](http://plato.stanford.edu/entries/time-travel-phys).

Argument 3 tells us that tachyons should never be able to travel *at*  $c$ , only at  $v > c$ . Furthermore,  $\gamma$  is an imaginary number for  $v > c$ . If we want the mass-energy to be a real number, that means that a tachyon's mass would have to be an imaginary number. A tachyon *decelerates* when energy is added to it, approaching  $c$  from above as its energy approaches infinity.

Do tachyons exist in nature? In September 2011, an experiment<sup>9</sup> measured the speed of neutrinos with energies of about  $3 \times 10^{-9}$  J, and found that it was greater than  $c$  by about 20 parts per million. Both skepticism and excitement followed. In February 2012, the research group announced that the result had been a mistake and was due to a loose cable somewhere in the bowels of the gigantic experiment.

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<sup>9</sup>[arxiv.org/abs/1109.4897](http://arxiv.org/abs/1109.4897)

## Summary

### Selected vocabulary

alpha particle . .	a form of radioactivity consisting of helium nuclei
beta particle . . .	a form of radioactivity consisting of electrons
gamma ray . . . .	a form of radioactivity consisting of a very high-frequency form of light
proton . . . . .	a positively charged particle, one of the types that nuclei are made of
neutron . . . . .	an uncharged particle, the other types that nuclei are made of
isotope . . . . .	one of the possible varieties of atoms of a given element, having a certain number of neutrons
atomic number .	the number of protons in an atom's nucleus; determines what element it is
atomic mass . . .	the mass of an atom
mass number . .	the number of protons plus the number of neutrons in a nucleus; approximately proportional to its atomic mass
strong nuclear force . . . . .	the force that holds nuclei together against electrical repulsion
weak nuclear force . . . . .	the force responsible for beta decay
beta decay . . . .	the radioactive decay of a nucleus via the reaction $n \rightarrow p + e^- + \bar{\nu}$ or $p \rightarrow n + e^+ + \nu$ ; so called because an electron or antielectron is also known as a beta particle
alpha decay . . .	the radioactive decay of a nucleus via emission of an alpha particle
fission . . . . .	the radioactive decay of a nucleus by splitting into two parts
fusion . . . . .	a nuclear reaction in which two nuclei stick together to form one bigger nucleus
$\mu\text{Sv}$ . . . . .	a unit for measuring a person's exposure to radioactivity

### Notation

$e^-$ . . . . .	an electron
$e^+$ . . . . .	an antielectron; just like an electron, but with positive charge
$n$ . . . . .	a neutron
$p$ . . . . .	a proton
$\nu$ . . . . .	a neutrino
$\bar{\nu}$ . . . . .	an antineutrino
$E$ . . . . .	mass-energy

## Other terminology and notation

$Z$ . . . . .	atomic number (number of protons in a nucleus)
$N$ . . . . .	number of neutrons in a nucleus
$A$ . . . . .	mass number ( $N + Z$ )

## Summary

Quantization of charge: Millikan's oil drop experiment showed that the total charge of an object could only be an integer multiple of a basic unit of charge ( $e$ ). This supported the idea the the "flow" of electrical charge was the motion of tiny particles rather than the motion of some sort of mysterious electrical fluid.

Einstein's analysis of Brownian motion was the first definitive proof of the existence of atoms. Thomson's experiments with vacuum tubes demonstrated the existence of a new type of microscopic particle with a very small ratio of mass to charge. Thomson correctly interpreted these as building blocks of matter even smaller than atoms: the first discovery of subatomic particles. These particles are called electrons.

The above experimental evidence led to the first useful model of the interior structure of atoms, called the raisin cookie model. In the raisin cookie model, an atom consists of a relatively large, massive, positively charged sphere with a certain number of negatively charged electrons embedded in it.

Rutherford and Marsden observed that some alpha particles from a beam striking a thin gold foil came back at angles up to 180 degrees. This could not be explained in the then-favored raisin-cookie model of the atom, and led to the adoption of the planetary model of the atom, in which the electrons orbit a tiny, positively-charged nucleus. Further experiments showed that the nucleus itself was a cluster of positively-charged protons and uncharged neutrons.

Radioactive nuclei are those that can release energy. The most common types of radioactivity are alpha decay (the emission of a helium nucleus), beta decay (the transformation of a neutron into a proton or vice-versa), and gamma decay (the emission of a type of very-high-frequency light). Stars are powered by nuclear fusion reactions, in which two light nuclei collide and form a bigger nucleus, with the release of energy.

Human exposure to ionizing radiation is measured in units of microsieverts ( $\mu\text{Sv}$ ). The typical person is exposed to about 2000  $\mu\text{Sv}$  worth of natural background radiation per year.

## Exploring further

**The First Three Minutes**, Steven Weinberg. This book describes the first three minutes of the universe's existence.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

**1** Use the nutritional information on some packaged food to make an order-of-magnitude estimate of the amount of chemical energy stored in one atom of food, in units of joules. Assume that a typical atom has a mass of  $10^{-26}$  kg. This constitutes a rough estimate of the amounts of energy there are on the atomic scale. [See chapter 1 for help on how to do order-of-magnitude estimates. Note that a nutritional “calorie” is really a kilocalorie.] ✓

**2** The nuclear process of beta decay by electron capture is described parenthetically on p. 731. The reaction is  $p + e^- \rightarrow n + \nu$ .  
(a) Show that charge is conserved in this reaction.  
(b) Explain why electron capture doesn’t occur in hydrogen atoms. (If it did, matter wouldn’t exist!) ▷ Solution, p. 975

**3**  $^{234}\text{Pu}$  decays either by electron decay or by alpha decay. (A given  $^{234}\text{Pu}$  nucleus may do either one; it’s random.) What are the isotopes created as products of these two modes of decay?

**4** (a) Recall that the gravitational energy of two gravitationally interacting spheres is given by  $PE = -Gm_1m_2/r$ , where  $r$  is the center-to-center distance. What would be the analogous equation for two electrically interacting spheres? Justify your choice of a plus or minus sign on physical grounds, considering attraction and repulsion.

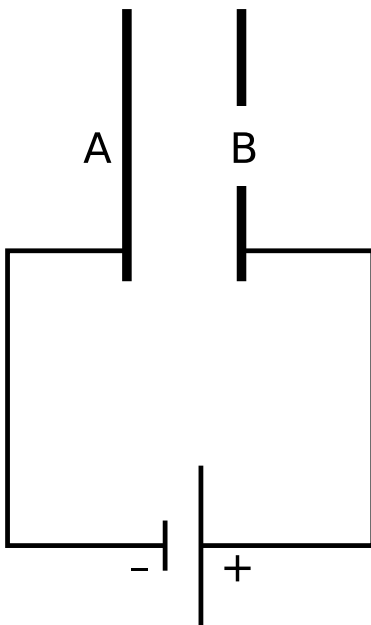
(b) Use this expression to estimate the energy required to pull apart a raisin-cookie atom of the one-electron type, assuming a radius of  $10^{-10}$  m.

(c) Compare this with the result of problem 1. ✓

**5** A neon light consists of a long glass tube full of neon, with metal caps on the ends. Positive charge is placed on one end of the tube, negative on the other. The electric forces generated can be strong enough to strip electrons off of a certain number of neon atoms. Assume for simplicity that only one electron is ever stripped off of any neon atom. When an electron is stripped off of an atom, both the electron and the neon atom (now an ion) have electric charge, and they are accelerated by the forces exerted by the charged ends of the tube. (They do not feel any significant forces from the other ions and electrons within the tube, because only a tiny minority of neon atoms ever gets ionized.) Light is finally produced when ions are reunited with electrons. Give a numerical comparison of the magnitudes and directions of the accelerations of the electrons and ions. [You may need some data from page 996.] ✓

**6** If you put two hydrogen atoms near each other, they will feel an attractive force, and they will pull together to form a molecule.



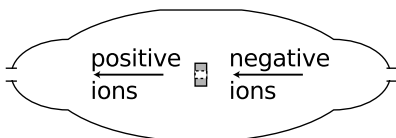


Problem 7.

(Molecules consisting of two hydrogen atoms are the normal form of hydrogen gas.) How is this possible, since each is electrically neutral? Shouldn't the attractive and repulsive forces all cancel out exactly? Use the raisin cookie model. (Students who have taken chemistry often try to use fancier models to explain this, but if you can't explain it using a simple model, you probably don't understand the fancy model as well as you thought you did!) It's not so easy to prove that the force should actually be attractive rather than repulsive, so just concentrate on explaining why it doesn't necessarily have to vanish completely.

**7** The figure shows a simplified diagram of an electron gun such as the one used in an old-fashioned TV tube. Electrons that spontaneously emerge from the negative electrode (cathode) are then accelerated to the positive electrode, which has a hole in it. (Once they emerge through the hole, they will slow down. However, if the two electrodes are fairly close together, this slowing down is a small effect, because the attractive and repulsive forces experienced by the electron tend to cancel.) (a) If the voltage difference between the electrodes is  $\Delta V$ , what is the velocity of an electron as it emerges at B? (Assume its initial velocity, at A, is negligible.) (b) Evaluate your expression numerically for the case where  $\Delta V = 10$  kV, and compare to the speed of light. ▷ Solution, p. 976

**8** The figure shows a simplified diagram of a device called a tandem accelerator, used for accelerating beams of ions up to speeds on the order of 1% of the speed of light. The nuclei of these ions collide with the nuclei of atoms in a target, producing nuclear reactions for experiments studying the structure of nuclei. The outer shell of the accelerator is a conductor at zero voltage (i.e., the same voltage as the Earth). The electrode at the center, known as the "terminal," is at a high positive voltage, perhaps millions of volts. Negative ions with a charge of  $-1$  unit (i.e., atoms with one extra electron) are produced offstage on the right, typically by chemical reactions with cesium, which is a chemical element that has a strong tendency to give away electrons. Relatively weak electric and magnetic forces are used to transport these  $-1$  ions into the accelerator, where they are attracted to the terminal. Although the center of the terminal has a hole in it to let the ions pass through, there is a very thin carbon foil there that they must physically penetrate. Passing through the foil strips off some number of electrons, changing the atom into a positive ion, with a charge of  $+n$  times the fundamental charge. Now that the atom is positive, it is repelled by the terminal, and accelerates some more on its way out of the accelerator. (a) Find the velocity,  $v$ , of the emerging beam of positive ions, in terms of  $n$ , their mass  $m$ , the terminal voltage  $V$ , and fundamental constants. Neglect the small change in mass caused by the loss of electrons in the stripper foil. ✓



Problem 8.

(b) To fuse protons with protons, a minimum beam velocity of about

11% of the speed of light is required. What terminal voltage would be needed in this case? ✓

**9** In example 3 on page 641, I remarked that accelerating a macroscopic (i.e., not microscopic) object to close to the speed of light would require an unreasonable amount of energy. Suppose that the starship Enterprise from Star Trek has a mass of  $8.0 \times 10^7$  kg, about the same as the Queen Elizabeth 2. Compute the kinetic energy it would have to have if it was moving at half the speed of light. Compare with the total energy content of the world's nuclear arsenals, which is about  $10^{21}$  J. ✓

**10** (a) A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and undergoes beta decay (which you may want to review). The masses of the particles involved are as follows:

neutron	$1.67495 \times 10^{-27}$ kg
proton	$1.67265 \times 10^{-27}$ kg
electron	$0.00091 \times 10^{-27}$ kg
antineutrino	$< 10^{-35}$ kg

Find the energy released in the decay of a free neutron. ✓

(b) Neutrons and protons make up essentially all of the mass of the ordinary matter around us. We observe that the universe around us has no free neutrons, but lots of free protons (the nuclei of hydrogen, which is the element that 90% of the universe is made of). We find neutrons only inside nuclei along with other neutrons and protons, not on their own.

If there are processes that can convert neutrons into protons, we might imagine that there could also be proton-to-neutron conversions, and indeed such a process does occur sometimes in nuclei that contain both neutrons and protons: a proton can decay into a neutron, a positron, and a neutrino. A positron is a particle with the same properties as an electron, except that its electrical charge is positive. A neutrino, like an antineutrino, has negligible mass.

Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive, and you wouldn't exist!)

**11** (a) Find a relativistic equation for the velocity of an object in terms of its mass and momentum (eliminating  $\gamma$ ). ✓

(b) Show that your result is approximately the same as the classical value,  $p/m$ , at low velocities.

(c) Show that very large momenta result in speeds close to the speed of light. ★

**12** An object moving at a speed very close to the speed of light is referred to as ultrarelativistic. Ordinarily (luckily) the only ultrarelativistic objects in our universe are subatomic particles, such

as cosmic rays or particles that have been accelerated in a particle accelerator.

- (a) What kind of number is  $\gamma$  for an ultrarelativistic particle?
- (b) Repeat example 11 on page 750, but instead of very low, non-relativistic speeds, consider ultrarelativistic speeds.
- (c) Find an equation for the ratio  $E/p$ . The speed may be relativistic, but don't assume that it's ultrarelativistic. ✓
- (d) Simplify your answer to part c for the case where the speed is ultrarelativistic. ✓
- (e) We can think of a beam of light as an ultrarelativistic object — it certainly moves at a speed that's sufficiently close to the speed of light! Suppose you turn on a one-watt flashlight, leave it on for one second, and then turn it off. Compute the momentum of the recoiling flashlight, in units of  $\text{kg}\cdot\text{m/s}$ . (Cf. p. 668.) ✓
- (f) Discuss how your answer in part e relates to the correspondence principle.

**13** As discussed in chapter 19, the speed at which a disturbance travels along a string under tension is given by  $v = \sqrt{T/\mu}$ , where  $\mu$  is the mass per unit length, and  $T$  is the tension.

- (a) Suppose a string has a density  $\rho$ , and a cross-sectional area  $A$ . Find an expression for the maximum tension that could possibly exist in the string without producing  $v > c$ , which is impossible according to relativity. Express your answer in terms of  $\rho$ ,  $A$ , and  $c$ . The interpretation is that relativity puts a limit on how strong any material can be. ✓
- (b) Every substance has a tensile strength, defined as the force per unit area required to break it by pulling it apart. The tensile strength is measured in units of  $\text{N/m}^2$ , which is the same as the pascal (Pa), the mks unit of pressure. Make a numerical estimate of the maximum tensile strength allowed by relativity in the case where the rope is made out of ordinary matter, with a density on the same order of magnitude as that of water. (For comparison, kevlar has a tensile strength of about  $4 \times 10^9$  Pa, and there is speculation that fibers made from carbon nanotubes could have values as high as  $6 \times 10^{10}$  Pa.) ✓
- (c) A black hole is a star that has collapsed and become very dense, so that its gravity is too strong for anything ever to escape from it. For instance, the escape velocity from a black hole is greater than  $c$ , so a projectile can't be shot out of it. Many people, when they hear this description of a black hole in terms of an escape velocity greater than  $c$ , wonder why it still wouldn't be possible to extract an object from a black hole by other means than launching it out as a projectile. For example, suppose we lower an astronaut into a black hole on a rope, and then pull him back out again. Why might this not work?

**14** (a) A charged particle is surrounded by a uniform electric field. Starting from rest, it is accelerated by the field to speed  $v$  after

traveling a distance  $d$ . Now it is allowed to continue for a further distance  $3d$ , for a total displacement from the start of  $4d$ . What speed will it reach, assuming classical physics?

(b) Find the relativistic result for the case of  $v = c/2$ .

**15** Problem 15 on p. 363 (with the solution given in the back of the book) demonstrates that in Newtonian mechanics, conservation of momentum is the necessary and sufficient condition if conservation of energy is to hold in all frames of reference. The purpose of this problem is to generalize that idea to relativity (in one dimension).

Let a system contain two interacting particles, each with unit mass. Then if energy is conserved in a particular frame, we must have  $\mathfrak{Y}_1 + \mathfrak{Y}_2 = \mathfrak{Y}'_1 + \mathfrak{Y}'_2$ , where the primes indicate the quantities after interaction. Suppose that we now change to a new frame, in motion relative to the first one at a velocity  $\epsilon$  that is much less than 1 (in units where  $c = 1$ ). The velocities all change according to the result of problem 21 on p. 680. Show that energy is conserved in the new frame if and only if momentum is conserved.

Hints: (1) Since  $\epsilon$  is small, you can take  $1/(1+\epsilon) \approx 1-\epsilon$ . (2) Rather than directly using the result of problem 21, it is easier to eliminate the velocities in favor of the corresponding Doppler-shift factors  $D$ , which simply multiply when the velocities are combined. (3) The identity  $v\mathfrak{Y} = (1/D - D)/2$  is handy here. ★

## Exercise 26: Sports in Slowlightland

In Slowlightland, the speed of light is  $20 \text{ mi/hr} = 32 \text{ km/hr} = 9 \text{ m/s}$ . Think of an example of how relativistic effects would work in sports. Things can get very complex very quickly, so try to think of a simple example that focuses on just one of the following effects:

- relativistic momentum
- relativistic kinetic energy
- relativistic addition of velocities (See problem 21, with the answer given on p. 975.)
- time dilation and length contraction
- Doppler shifts of light (See section 24.7.)
- equivalence of mass and energy
- time it takes for light to get to an athlete's eye
- deflection of light rays by gravity

# Chapter 27

## General relativity

What you've learned so far about relativity is known as the special theory of relativity, which is compatible with three of the four known forces of nature: electromagnetism, the strong nuclear force, and the weak nuclear force. Gravity, however, can't be shoehorned into the special theory. In order to make gravity work, Einstein had to generalize relativity. The resulting theory is known as the general theory of relativity.<sup>1</sup>

### 27.1 Our universe isn't Euclidean

Euclid proved thousands of years ago that the angles in a triangle add up to  $180^\circ$ . But what does it really mean to "prove" this? Euclid proved it based on certain assumptions (his five postulates), listed in the margin of this page. But how do we know that the postulates are true?

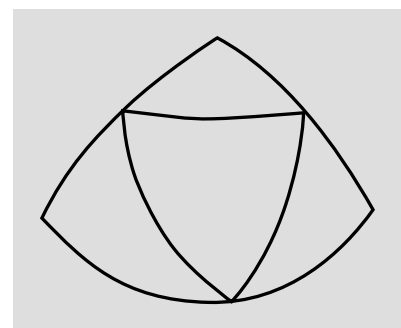
Only by observation can we tell whether any of Euclid's statements are correct characterizations of how space actually behaves in our universe. If we draw a triangle on paper with a ruler and measure its angles with a protractor, we will quickly verify to pretty good precision that the sum is close to  $180^\circ$ . But of course we already knew that space was at least *approximately* Euclidean. If there had been any gross error in Euclidean geometry, it would have been detected in Euclid's own lifetime. The correspondence principle tells us that if there is going to be any deviation from Euclidean geometry, it must be small under ordinary conditions.

To improve the precision of the experiment, we need to make sure that our ruler is very straight. One way to check would be to sight along it by eye, which amounts to comparing its straightness to that of a ray of light. For that matter, we might as well throw the physical ruler in the trash and construct our triangle out of three laser beams. To avoid effects from the air we should do the experiment in outer space. Doing it in space also has the advantage of allowing us to make the triangle very large; as shown in figure a, the discrepancy

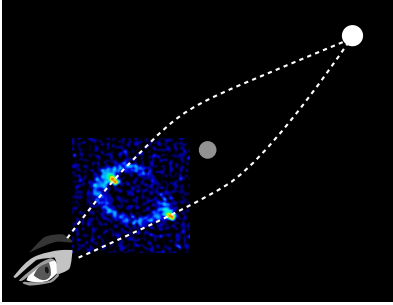
<sup>1</sup>Einstein originally described the distinction between the two theories by saying that the special theory applied to nonaccelerating frames of reference, while the general one allowed any frame at all. The modern consensus is that Einstein was misinterpreting his own theory, and that special relativity actually handles accelerating frames just fine.

*Postulates of Euclidean geometry:*

1. Two points determine a line.
2. Line segments can be extended.
3. A unique circle can be constructed given any point as its center and any line segment as its radius.
4. All right angles are equal to one another.
5. Given a line and a point not on the line, no more than one line can be drawn through the point and parallel to the given line.



a / Noneuclidean effects, such as the discrepancy from  $180^\circ$  in the sum of the angles of a triangle, are expected to be proportional to area. Here, a noneuclidean equilateral triangle is cut up into four smaller equilateral triangles, each with  $1/4$  the area. As proved in problem 1, the discrepancy is quadrupled when the area is quadrupled.



b / An Einstein's ring. The distant object is a quasar, MG1131+0456, and the one in the middle is an unknown object, possibly a supermassive black hole. The intermediate object's gravity focuses the rays of light from the distant one. Because the entire arrangement lacks perfect axial symmetry, the ring is nonuniform; most of its brightness is concentrated in two lumps on opposite sides.

from  $180^\circ$  is expected to be proportional to the area of the triangle.

But we already know that light rays are bent by gravity. We expect it based on  $E = mc^2$ , which tells us that the energy of a light ray is equivalent to a certain amount of mass, and furthermore it has been verified experimentally by the deflection of starlight by the sun (example 7, p. 747). We therefore know that our universe is noneuclidean, and we gain the further insight that the level of deviation from Euclidean behavior depends on gravity.

Since the noneuclidean effects are bigger when the system being studied is larger, we expect them to be especially important in the study of cosmology, where the distance scales are very large.

#### *Einstein's ring*

#### *example 1*

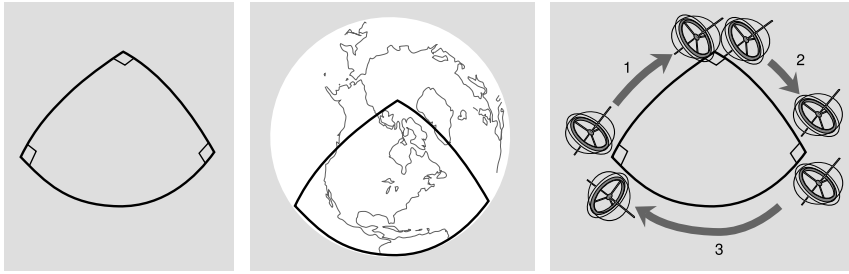
An Einstein's ring, figure b, is formed when there is a chance alignment of a distant source with a closer gravitating body. This type of gravitational lensing is direct evidence for the noneuclidean nature of space. The two light rays are lines, and they violate Euclid's first postulate, that two points determine a line.

One could protest that effects like these are just an imperfection of the light rays as physical models of straight lines. Maybe the noneuclidean effects would go away if we used something better and straighter than a light ray. But we don't know of anything straighter than a light ray. Furthermore, we observe that all measuring devices, not just optical ones, report the same noneuclidean behavior.

### **Curvature**

An example of such a non-optical measurement is the Gravity Probe B satellite, figure d, which was launched into a polar orbit in 2004 and operated until 2010. The probe carried four gyroscopes made of quartz, which were the most perfect spheres ever manufactured, varying from sphericity by no more than about 40 atoms. Each gyroscope floated weightlessly in a vacuum, so that its rotation was perfectly steady. After 5000 orbits, the gyroscopes had reoriented themselves by about  $2 \times 10^{-3}^\circ$  relative to the distant stars. This effect cannot be explained by Newtonian physics, since no torques acted on them. It was, however, exactly as predicted by Einstein's theory of general relativity. It becomes easier to see why such an effect would be expected due to the noneuclidean nature of space if we characterize euclidean geometry as the geometry of a flat plane as opposed to a curved one. On a curved surface like a sphere, figure c, Euclid's fifth postulate fails, and it's not hard to see that we can get triangles for which the sum of the angles is not  $180^\circ$ . By transporting a gyroscope all the way around the edges of such a triangle and back to its starting point, we change its orientation.

The triangle in figure c has angles that add up to more than  $180^\circ$ .



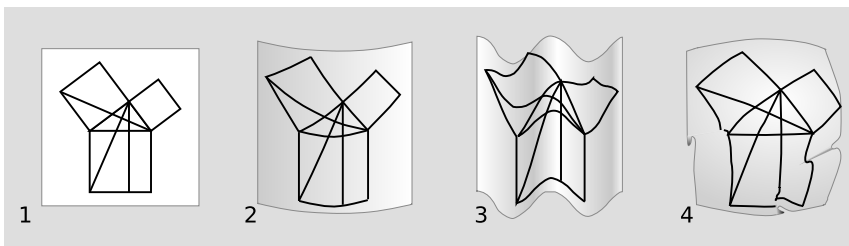
c / *Left*: A 90-90-90 triangle. Its angles add up to more than  $180^\circ$ . *Middle*: The triangle “pops” off the page visually. We intuitively want to visualize it as lying on a curved surface such as the earth’s. *Right*: A gyroscope carried smoothly around its perimeter ends up having changed its orientation when it gets back to its starting point.

This type of curvature is referred to as positive. It is also possible to have negative curvature, as in figure e.

In general relativity, curvature isn’t just something caused by gravity. Gravity *is* curvature, and the curvature involves both space and time, as may become clearer once you get to figure k. Thus the distinction between special and general relativity is that general relativity handles curved spacetime, while special relativity is restricted to the case where spacetime is flat.

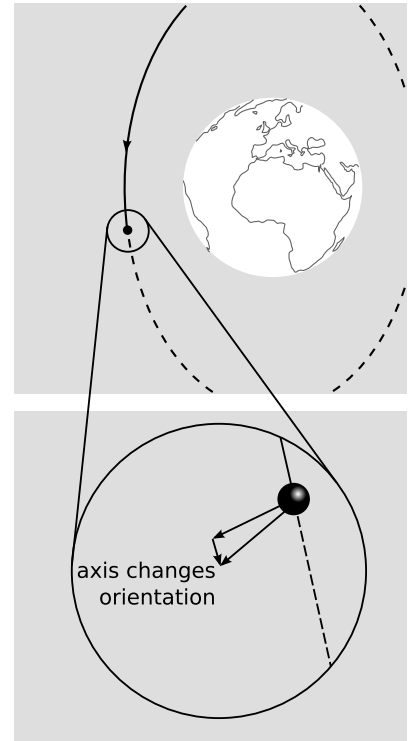
### Curvature doesn’t require higher dimensions

Although we often visualize curvature by imagining embedding a two-dimensional surface in a three-dimensional space, that’s just an aid in visualization. There is no evidence for any additional dimensions, nor is it necessary to hypothesize them in order to let spacetime be curved as described in general relativity.

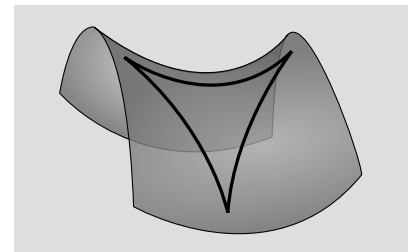


f / Only measurements from within the plane define whether the plane is curved. It could look curved when drawn embedded in three dimensions, but nevertheless still be intrinsically flat.

Put yourself in the shoes of a two-dimensional being living in a two-dimensional space. Euclid’s postulates all refer to constructions that can be performed using a compass and an unmarked straightedge. If this being can physically verify them all as descriptions of the



d / Gravity Probe B was in a polar orbit around the earth. As in the right panel of figure c, the orientation of the gyroscope changes when it is carried around a curve and back to its starting point. Because the effect was small, it was necessary to let it accumulate over the course of 5000 orbits in order to make it detectable.



e / A triangle in a space with negative curvature has angles that add to less than  $180^\circ$ .



space she inhabits, then she knows that her space is Euclidean, and that propositions such as the Pythagorean theorem are physically valid in her universe. But the diagram in f/1 illustrating the proof of the Pythagorean theorem in Euclid's *Elements* (proposition I.47) is equally valid if the page is rolled onto a cylinder, 2, or formed into a wavy corrugated shape, 3. These types of curvature, which can be achieved without tearing or crumpling the surface, are not real to her. They are simply side-effects of visualizing her two-dimensional universe as if it were embedded in a hypothetical third dimension — which doesn't exist in any sense that is empirically verifiable to her. Of the curved surfaces in figure f, only the sphere, 4, has curvature that she can measure; the diagram can't be plastered onto the sphere without folding or cutting and pasting.

So the observation of curvature doesn't imply the existence of extra dimensions, nor does embedding a space in a higher-dimensional one so that it looks curvy always mean that there will be any curvature detectable from within the lower-dimensional space.

## 27.2 The equivalence principle

### Universality of free-fall

Although light rays and gyroscopes seem to agree that space is curved in a gravitational field, it's always conceivable that we could find something else that would disagree. For example, suppose that there is a new and improved ray called the StraightRay<sup>TM</sup>. The StraightRay is like a light ray, but when we construct a triangle out of StraightRays, we always get the Euclidean result for the sum of the angles. We would then have to throw away general relativity's whole idea of describing gravity in terms of curvature. One good way of making a StraightRay would be if we had a supply of some kind of exotic matter — call it FloatyStuff<sup>TM</sup> — that had the ordinary amount of inertia, but was completely unaffected by gravity. We could then shoot a stream of FloatyStuff particles out of a nozzle at nearly the speed of light and make a StraightRay.

Normally when we release a material object in a gravitational field, it experiences a force  $mg$ , and then by Newton's second law its acceleration is  $a = F/m = mg/m = g$ . The  $m$ 's cancel, which is the reason that everything falls with the same acceleration (in the absence of other forces such as air resistance). The universality of this behavior is what allows us to interpret the gravity geometrically in general relativity. For example, the Gravity Probe B gyroscopes were made out of quartz, but if they had been made out of something else, it wouldn't have mattered. But if we had access to some FloatyStuff, the geometrical picture of gravity would fail, because the " $m$ " that described its susceptibility to gravity would be a different " $m$ " than the one describing its inertia.

The question of the existence or nonexistence of such forms of matter turns out to be related to the question of what kinds of motion are relative. Let's say that alien gangsters land in a flying saucer, kidnap you out of your back yard, konk you on the head, and take you away. When you regain consciousness, you're locked up in a sealed cabin in their spaceship. You pull your keychain out of your pocket and release it, and you observe that it accelerates toward the floor with an acceleration that seems quite a bit slower than what you're used to on earth, perhaps a third of a gee. There are two possible explanations for this. One is that the aliens have taken you to some other planet, maybe Mars, where the strength of gravity is a third of what we have on earth. The other is that your keychain didn't really accelerate at all: you're still inside the flying saucer, which is accelerating at a third of a gee, so that it was really the deck that accelerated up and hit the keys.

There is absolutely no way to tell which of these two scenarios is actually the case — unless you happen to have a chunk of FloatyStuff in your other pocket. If you release the FloatyStuff and it hovers above the deck, then you're on another planet and experiencing genuine gravity; your keychain responded to the gravity, but the FloatyStuff didn't. But if you release the FloatyStuff and see it hit the deck, then the flying saucer is accelerating through outer space.

The nonexistence of FloatyStuff in our universe is called the *equivalence principle*. If the equivalence principle holds, then an acceleration (such as the acceleration of the flying saucer) is always equivalent to a gravitational field, and no observation can ever tell the difference without reference to something external. (And suppose you did have some external reference point — how would you know whether *it* was accelerating?) If the equivalence principle fails, then general relativity's geometrical interpretation of gravity as a form of curvature fails, and we need to find a new theory of gravity.

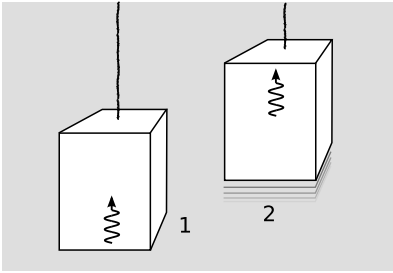
#### The artificial horizon

#### example 2

The pilot of an airplane cannot always easily tell which way is up. The horizon may not be level simply because the ground has an actual slope, and in any case the horizon may not be visible if the weather is foggy. One might imagine that the problem could be solved simply by hanging a pendulum and observing which way it pointed, but by the equivalence principle the pendulum cannot tell the difference between a gravitational field and an acceleration of the aircraft relative to the ground — nor can any other accelerometer, such as the pilot's inner ear. For example, when the plane is turning to the right, accelerometers will be tricked into believing that “down” is down and to the left. To get around this problem, airplanes use a device called an artificial horizon, which is essentially a gyroscope. The gyroscope has to be initialized when the plane is known to be oriented in a horizontal plane. No



g / An artificial horizon.



h/c. A ray of light is emitted upward from the floor of the elevator. The elevator accelerates upward. 2. By the time the light is detected at the ceiling, the elevator has changed its velocity, so the light is detected with a Doppler shift.



Figure 27.10 Pound and Rebka at the top and bottom of the tower.

gyroscope is perfect, so over time it will drift. For this reason the instrument also contains an accelerometer, and the gyroscope is always forced into agreement with the accelerometer's average output over the preceding several minutes. If the plane is flown in circles for several minutes, the artificial horizon will be fooled into indicating that the wrong direction is vertical.

### Gravitational Doppler shifts and time dilation

An interesting application of the equivalence principle is the explanation of gravitational time dilation. As described on p. 632, experiments show that a clock at the top of a mountain runs faster than one down at its foot.

To calculate this effect, we make use of the fact that the gravitational field in the area around the mountain is equivalent to an acceleration. Suppose we're in an elevator accelerating upward with acceleration  $a$ , and we shoot a ray of light from the floor up toward the ceiling, at height  $h$ . The time  $\Delta t$  it takes the light ray to get to the ceiling is about  $h/c$ , and by the time the light ray reaches the ceiling, the elevator has sped up by  $v = a\Delta t = ah/c$ , so we'll see a red-shift in the ray's frequency. Since  $v$  is small compared to  $c$ , we don't need to use the fancy Doppler shift equation from section 24.7; we can just approximate the Doppler shift factor as  $1 - v/c \approx 1 - ah/c^2$ . By the equivalence principle, we should expect that if a ray of light starts out low down and then rises up through a gravitational field  $g$ , its frequency will be Doppler shifted by a factor of  $1 - gh/c^2$ . This effect was observed in a famous experiment carried out by Pound and Rebka in 1959. Gamma-rays were emitted at the bottom of a 22.5-meter tower at Harvard and detected at the top with the Doppler shift predicted by general relativity. (See problem 4.)

In the mountain-valley experiment, the frequency of the clock in the valley therefore appears to be running too slowly by a factor of  $1 - gh/c^2$  when it is compared via radio with the clock at the top of the mountain. We conclude that time runs more slowly when one is lower down in a gravitational field, and the slow-down factor between two points is given by  $1 - gh/c^2$ , where  $h$  is the difference in height.

We have built up a picture of light rays interacting with gravity. To confirm that this makes sense, recall that we have already observed on p. 668 and in problem 12 on p. 759 that light has momentum. The equivalence principle says that whatever has inertia must also participate in gravitational interactions. Therefore light waves must have weight, and must lose energy when they rise through a gravitational field (cf. p. 752).

## Local flatness

The noneuclidean nature of spacetime produces effects that grow in proportion to the area of the region being considered. Interpreting such effects as evidence of curvature, we see that this connects naturally to the idea that curvature is undetectable from close up. For example, the curvature of the earth's surface is not normally noticeable to us in everyday life. Locally, the earth's surface is flat, and the same is true for spacetime.

Local flatness turns out to be another way of stating the equivalence principle. In a variation on the alien-abduction story, suppose that you regain consciousness aboard the flying saucer and find yourself weightless. If the equivalence principle holds, then you have no way of determining from local observations, inside the saucer, whether you are actually weightless in deep space, or simply free-falling in apparent weightlessness, like the astronauts aboard the International Space Station. That means that locally, we can always adopt a free-falling frame of reference in which there is no gravitational field at all. If there is no gravity, then special relativity is valid, and we can treat our local region of spacetime as being approximately flat.

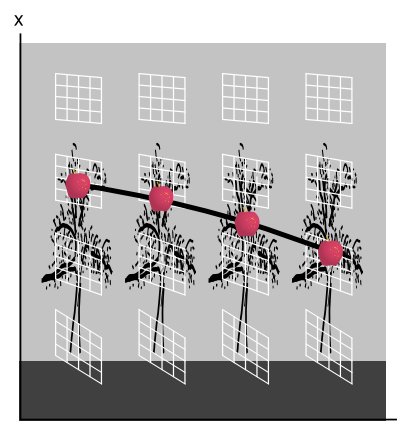
In figure k, an apple falls out of a tree. Its path is a “straight” line in spacetime, in the same sense that the equator is a “straight” line on the earth's surface.

## Inertial frames

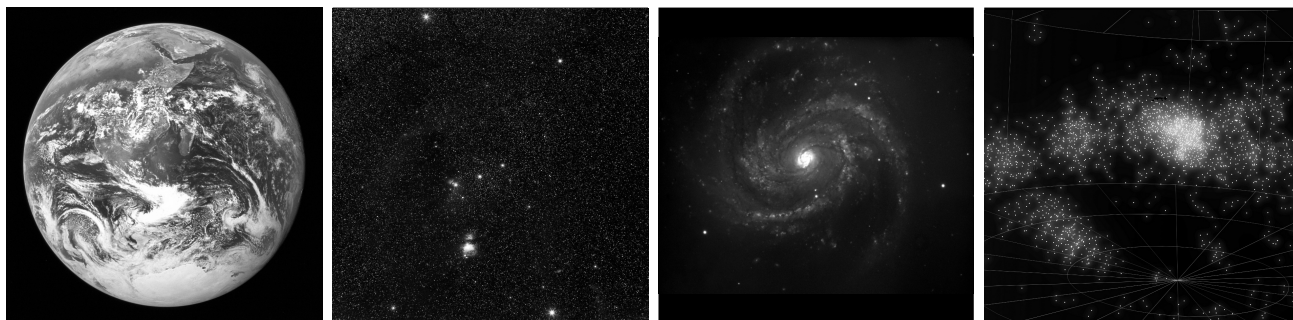
In Newtonian mechanics, we have a distinction between inertial and noninertial frames of reference. An inertial frame according to Newton is one that has a constant velocity vector relative to the stars. But what if the stars themselves are accelerating due to a gravitational force from the rest of the galaxy? We could then take the galaxy's center of mass as defining an inertial frame, but what if something else is acting on the galaxy?



j / The earth is flat — locally.



k / Spacetime is locally flat.



l / Wouldn't it be nice if we could define the meaning of a Newtonian inertial frame of reference? Newton makes it sound easy: to define an inertial frame, just find some object that is not accelerating because it is not being acted on by any external forces. But what object would we use? The earth? The “fixed stars?” Our galaxy? Our supercluster of galaxies? All of these are accelerating — relative to something.

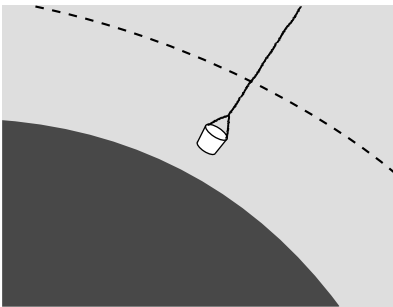
If we had some FloatyStuff, we could resolve the whole question. FloatyStuff isn't affected by gravity, so if we release a sample of it in mid-air, it will continue on a trajectory that defines a perfect Newtonian inertial frame. (We'd better have it on a tether, because otherwise the earth's rotation will carry the earth out from under it.) But if the equivalence principle holds, then Newton's definition of an inertial frame is fundamentally flawed.

There is a different definition of an inertial frame that works better in relativity. A Newtonian inertial frame was defined by an object that isn't subject to any forces, gravitational or otherwise. In general relativity, we instead define an inertial frame using an object that isn't influenced by anything other than gravity. By this definition, a free-falling rock defines an inertial frame, but this book sitting on your desk does not.

## 27.3 Black holes

The observations described so far showed only small effects from curvature. To get a big effect, we should look at regions of space in which there are strong gravitational fields. The prime example is a black hole. The best studied examples are two objects in our own galaxy: Cygnus X-1, which is believed to be a black hole with about ten times the mass of our sun, and Sagittarius A\*, an object near the center of our galaxy with about four million solar masses. (See problem 14, p. 261 for how we know Sagittarius A\*'s mass.)

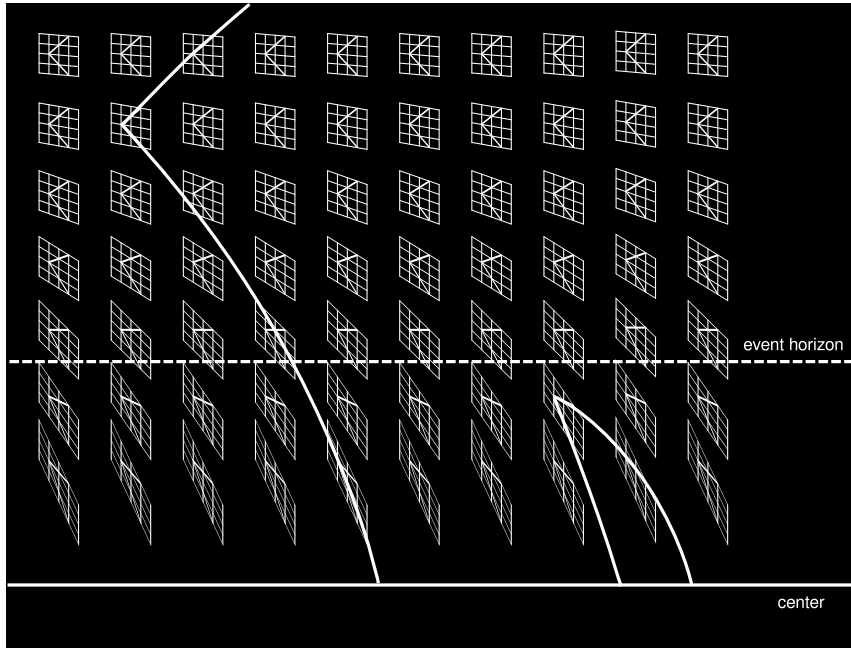
Although a black hole is a relativistic object, we can gain some insight into how it works by applying Newtonian physics. As shown in problem 21 on p. 333, for spherical a body of mass  $M$ , there is an escape velocity  $v = \sqrt{2GM/r}$ , which is the minimum velocity that we would need to give to a projectile shot from a distance  $r$  so that it would never fall back down. If  $r$  is small enough, the escape velocity will be greater than  $c$ , so that even a ray of light can never escape.



m / Matter is lifted out of a Newtonian black hole with a bucket. The dashed line represents the point at which the escape velocity equals the speed of light. For a real, relativistic black hole, this is impossible.

We can now make an educated guess as to what this means without having to study all the mathematics of general relativity. In relativity,  $c$  isn't really the speed of light, it's really to be thought of as a restriction on how fast cause and effect can propagate through space. This suggests the correct interpretation, which is that for an object compact enough to be a black hole, there is no way for an event at a distance closer than  $r$  to have an effect on an event far away. There is an invisible, spherical boundary with radius  $r$ , called the event horizon, and the region within that boundary is cut off from the rest of the universe in terms of cause and effect. If you wanted to explore that region, you could drop into it while wearing a space-suit — but it would be a one-way trip, because you could never get back out to report on what you had seen.

In the Newtonian description of a black hole, matter could be lifted out of a black hole, m. Would this be possible with a real-world black hole, which is relativistic rather than Newtonian? No, because the bucket is causally separated from the outside universe. No rope would be strong enough for this job (problem 13, p. 760).



n / The equivalence principle tells us that spacetime locally has the same structure as in special relativity, so we can draw the familiar parallelogram of  $x - t$  coordinates at each point near the black hole. Superimposed on each little grid is a pair of lines representing motion at the speed of light in both directions, inward and outward. Because spacetime is curved, these lines do not appear to be at 45-degree angles, but to an observer in that region, they would appear to be. When light rays are emitted inward and outward from a point outside the event horizon, one escapes and one plunges into the black hole. On this diagram, they look like they are decelerating and accelerating, but local observers comparing them to their own coordinate grids would always see them as moving at exactly  $c$ . When rays are emitted from a point *inside* the event horizon, neither escapes; the distortion is so severe that “outward” is really inward.

One misleading aspect of the Newtonian analysis is that it encourages us to imagine that a light ray trying to escape from a black hole will slow down, stop, and then fall back in. This can’t be right, because we know that any observer who sees a light ray flying by always measures its speed to be  $c$ . This was true in special relativity, and by the equivalence principle we can be assured that the same is true *locally* in general relativity. Figure n shows what would really happen.

Although the light rays in figure n don’t speed up or slow down, they do experience gravitational Doppler shifts. If a light ray is emitted from just above the event horizon, then it will escape to an infinite distance, but it will suffer an extreme Doppler shift toward low frequencies. A distant observer also has the option of interpreting this as a gravitational time dilation that greatly lowers the frequency of the oscillating electric charges that produced the ray. If the point of emission is made closer and closer to the horizon, the frequency and energy (see p. 752) measured by a distant observer approach zero, making the ray impossible to observe.

## Information paradox

Black holes have some disturbing implications for the kind of universe that in the Age of the Enlightenment was imagined to have been set in motion initially and then left to run forever like clockwork.

Newton's laws have built into them the implicit assumption that omniscience is possible, at least in principle. For example, Newton's definition of an inertial frame of reference leads to an infinite regress, as described on p. 769. For Newton this isn't a problem, because in principle an omniscient observer can know the location of every mass in the universe. In this conception of the cosmos, there are no theoretical limits on human knowledge, only practical ones; if we could gather sufficiently precise data about the state of the universe at one time, and if we could carry out all the calculations to extrapolate into the future, then we could know everything that would ever happen. (See the famous quote by Laplace on p. 18.)

But the existence of event horizons surrounding black holes makes it impossible for any observer to be omniscient; only an observer inside a particular horizon can see what's going on inside that horizon.

Furthermore, a black hole has at its center an infinitely dense point, called a singularity, containing all its mass, and this implies that information can be destroyed and made inaccessible to *any* observer at all. For example, suppose that astronaut Alice goes on a suicide mission to explore a black hole, free-falling in through the event horizon. She has a certain amount of time to collect data and satisfy her intellectual curiosity, but then she impacts the singularity and is compacted into a mathematical point. Now astronaut Betty decides that she will never be satisfied unless the secrets revealed to Alice are known to her as well — and besides, she was Alice's best friend, and she wants to know whether Alice had any last words. Betty can jump through the horizon, but she can never know Alice's last words, nor can any other observer who jumps in after Alice does.

This destruction of information is known as the black hole information paradox, and it's referred to as a paradox because quantum physics (ch. 33-36) has built into its DNA the requirement that information is never lost in this sense.

## Formation

Around 1960, as black holes and their strange properties began to be better understood and more widely discussed, many physicists who found these issues distressing comforted themselves with the belief that black holes would never really form from realistic initial conditions, such as the collapse of a massive star. Their skepticism was not entirely unreasonable, since it is usually very hard in astronomy to hit a gravitating target, the reason being that conservation of angular momentum tends to make the projectile swing past. (See



problem 13 on p. 397 for a quantitative analysis.) For example, if we wanted to drop a space probe into the sun, we would have to extremely precisely stop its sideways orbital motion so that it would drop almost exactly straight in. Once a star started to collapse, the theory went, and became relatively compact, it would be such a small target that further infalling material would be unlikely to hit it, and the process of collapse would halt. According to this point of view, theorists who had calculated the collapse of a star into a black hole had been oversimplifying by assuming a star that was initially perfectly spherical and nonrotating. Remove the unrealistically perfect symmetry of the initial conditions, and a black hole would never actually form.

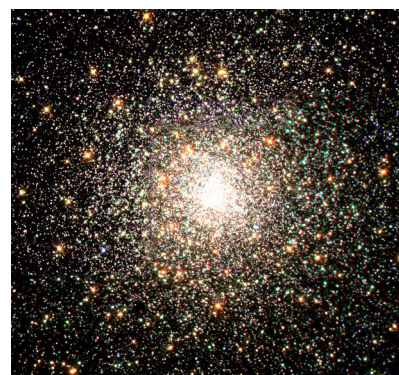
But Roger Penrose proved in 1964 that this was wrong. In fact, once an object collapses to a certain density, the Penrose singularity theorem guarantees mathematically that it will collapse further until a singularity is formed, and this singularity is surrounded by an event horizon. Since the brightness of an object like Sagittarius A\* is far too low to be explained unless it has an event horizon (the interstellar gas flowing into it would glow due to frictional heating), we can be certain that there really is a singularity at its core.

## 27.4 Cosmology

### The Big Bang

Section 19.5.1 presented the evidence, discovered by Hubble, that the universe is expanding in the aftermath of the Big Bang: when we observe the light from distant galaxies, it is always Doppler-shifted toward the red end of the spectrum, indicating that no matter what direction we look in the sky, everything is rushing away from us. This seems to go against the modern attitude, originated by Copernicus, that we and our planet do not occupy a special place in the universe. Why is everything rushing away from *our* planet in particular? But general relativity shows that this anti-Copernican conclusion is wrong. General relativity describes space not as a rigidly defined background but as something that can curve and stretch, like a sheet of rubber. We imagine all the galaxies as existing on the surface of such a sheet, which then expands uniformly. The space between the galaxies (but not the galaxies themselves) grows at a steady rate, so that any observer, inhabiting any galaxy, will see every other galaxy as receding. There is therefore no privileged or special location in the universe.

We might think that there would be another kind of special place, which would be the one at which the Big Bang happened. Maybe someone has put a brass plaque there? But general relativity doesn't describe the Big Bang as an explosion that suddenly occurred in a preexisting background of time and space. According to general



o / In Newtonian contexts, physicists and astronomers had a correct intuition that it's hard for things to collapse gravitationally. This star cluster has been around for about 15 billion years, but it hasn't collapsed into a black hole. If any individual star happens to head toward the center, conservation of angular momentum tends to cause it to swing past and fly back out. The Penrose singularity theorem tells us that this Newtonian intuition is wrong when applied to an object that has collapsed past a certain point.



relativity, space itself came into existence at the Big Bang, and the hot, dense matter of the early universe was uniformly distributed everywhere. The Big Bang happened everywhere at once.

Observations show that the universe is very uniform on large scales, and for ease of calculation, the first physical models of the expanding universe were constructed with perfect uniformity. In these models, the Big Bang was a singularity. This singularity can't even be included as an event in spacetime, so that time itself only exists after the Big Bang. A Big Bang singularity also creates an even more acute version of the black hole information paradox. Whereas matter and information disappear *into* a black hole singularity, stuff pops *out* of a Big Bang singularity, and there is no physical principle that could predict what it would be.

As with black holes, there was considerable skepticism about whether the existence of an initial singularity in these models was an artifact of the unrealistically perfect uniformity assumed in the models. Perhaps in the real universe, extrapolation of all the paths of the galaxies backward in time would show them missing each other by millions of light-years. But in 1972 Stephen Hawking proved a variant on the Penrose singularity theorem that applied to Big Bang singularities. By the Hawking singularity theorem, the level of uniformity we see in the present-day universe is more than sufficient to prove that a Big Bang singularity must have existed.

### **The cosmic censorship hypothesis**

It might not be too much of a philosophical jolt to imagine that information was spontaneously created in the Big Bang. Setting up the initial conditions of the entire universe is traditionally the prerogative of God, not the laws of physics. But there is nothing fundamental in general relativity that forbids the existence of other singularities that act like the Big Bang, being information producers rather than information consumers. As John Earman of the University of Pittsburgh puts it, anything could pop out of such a singularity, including green slime or your lost socks. This would eliminate any hope of finding a universal set of laws of physics that would be able to make a prediction given any initial situation.

That would be such a devastating defeat for the enterprise of physics that in 1969 Penrose proposed an alternative, humorously named the “cosmic censorship hypothesis,” which states that every singularity in our universe, other than the Big Bang, is hidden behind an event horizon. Therefore if green slime spontaneously pops out of one, there is limited impact on the predictive ability of physics, since the slime can never have any causal effect on the outside world. A singularity that is not modestly cloaked behind an event horizon is referred to as a naked singularity. Nobody has yet been able to prove the cosmic censorship hypothesis.

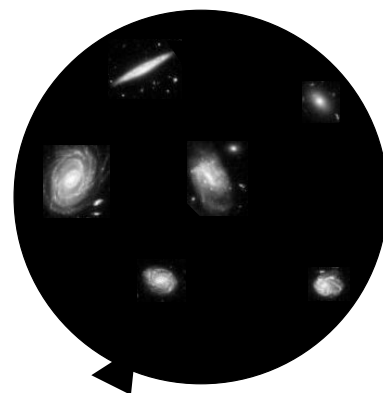
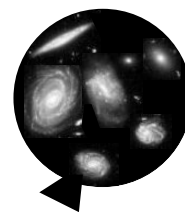
## The advent of high-precision cosmology

We expect that if there is matter in the universe, it should have gravitational fields, and in the rubber-sheet analogy this should be represented as a curvature of the sheet. Instead of a flat sheet, we can have a spherical balloon, so that cosmological expansion is like inflating it with more and more air. It is also possible to have negative curvature, as in figure e on p. 765. All three of these are valid, possible cosmologies according to relativity. The positive-curvature type happens if the average density of matter in the universe is above a certain critical level, the negative-curvature one if the density is below that value.

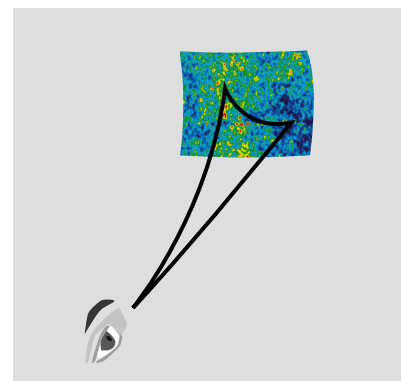
To find out which type of universe we inhabit, we could try to take a survey of the matter in the universe and determine its average density. Historically, it has been very difficult to do this, even to within an order of magnitude. Most of the matter in the universe probably doesn't emit light, making it difficult to detect. Astronomical distance scales are also very poorly calibrated against absolute units such as the SI.

Instead, we measure the universe's curvature, and infer the density of matter from that. It turns out that we can do this by observing the cosmic microwave background (CMB) radiation, which is the light left over from the brightly glowing early universe, which was dense and hot. As the universe has expanded, light waves that were in flight have expanded their wavelengths along with it. This afterglow of the big bang was originally visible light, but after billions of years of expansion it has shifted into the microwave radio part of the electromagnetic spectrum. The CMB is not perfectly uniform, and this turns out to give us a way to measure the universe's curvature. Since the CMB was emitted when the universe was only about 400,000 years old, any vibrations or disturbances in the hot hydrogen and helium gas that filled space in that era would only have had time to travel a certain distance, limited by the speed of sound. We therefore expect that no feature in the CMB should be bigger than a certain known size. In a universe with negative spatial curvature, the sum of the interior angles of a triangle is less than the Euclidean value of 180 degrees. Therefore if we observe a variation in the CMB over some angle, the distance between two points on the sky is actually greater than would have been inferred from Euclidean geometry. The opposite happens if the curvature is positive.

This observation was done by the 1989-1993 COBE probe, and its 2001-2009 successor, the Wilkinson Microwave Anisotropy Probe. The result is that the angular sizes are almost exactly *equal* to what they should be according to Euclidean geometry. We therefore infer that the universe is very close to having zero average spatial curvature on the cosmological scale, and this tells us that its average



p / An expanding universe with positive spatial curvature can be imagined as a balloon being blown up. Every galaxy's distance from every other galaxy increases, but no galaxy is the center of the expansion.



q / The angular scale of fluctuations in the cosmic microwave background can be used to infer the curvature of the universe.

density must be within about 0.5% of the critical value. The years since COBE and WMAP mark the advent of an era in which cosmology has gone from being a field of estimates and rough guesses to a high-precision science.

If one is inclined to be skeptical about the seemingly precise answers to the mysteries of the cosmos, there are consistency checks that can be carried out. In the bad old days of low-precision cosmology, estimates of the age of the universe ranged from 10 billion to 20 billion years, and the low end was inconsistent with the age of the oldest star clusters. This was believed to be a problem either for observational cosmology or for the astrophysical models used to estimate the ages of the clusters: “You can’t be older than your ma.” Current data have shown that the low estimates of the age were incorrect, so consistency is restored. (The best figure for the age of the universe is currently  $13.8 \pm 0.1$  billion years.)

### **Dark energy and dark matter**

Not everything works out so smoothly, however. One surprise, discussed in section 10.6, is that the universe’s expansion is not currently slowing down, as had been expected due to the gravitational attraction of all the matter in it. Instead, it is currently speeding up. This is attributed to a variable in Einstein’s equations, long assumed to be zero, which represents a universal gravitational repulsion of space itself, occurring even when there is no matter present. The current term for this is “dark energy,” although the impressive-sounding term is really just a label for our ignorance about what causes it.

Another surprise comes from attempts to model the formation of the elements during the era shortly after the Big Bang, before the formation of the first stars (section 26.4.10). The observed relative abundances of hydrogen, helium, and deuterium ( $^2\text{H}$ ) cannot be reconciled with the density of low-velocity matter inferred from the observational data. If the inferred mass density were entirely due to normal matter (i.e., matter whose mass consisted mostly of protons and neutrons), then nuclear reactions in the dense early universe should have proceeded relatively efficiently, leading to a much higher ratio of helium to hydrogen, and a much lower abundance of deuterium. The conclusion is that most of the matter in the universe must be made of an unknown type of exotic matter, known generically as “dark matter.” We are in the ironic position of knowing that precisely 96% of the universe is something other than atoms, but knowing essentially nothing about what that something is. As of 2011, there are four experiments, all performed at the bottom of mineshafts to eliminate background radiation, that have accumulated data in a search for direct detection of dark matter. Of these, two claim to have detected dark matter, while two claim negative results that appear to contradict the two positive

ones. A June 2011 review of the data is given by Hooper and Kelso, [arxiv.org/abs/1106.1066](http://arxiv.org/abs/1106.1066).

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** Prove, as claimed in the caption of figure a on p. 763, that  $S - 180^\circ = 4(s - 180^\circ)$ , where  $S$  is the sum of the angles of the large equilateral triangle and  $s$  is the corresponding sum for one of the four small ones. ▷ Solution, p. 976

**2** If a two-dimensional being lived on the surface of a cone, would it say that its space was curved, or not?

**3** (a) Verify that the equation  $1 - gh/c^2$  for the gravitational Doppler shift and gravitational time dilation has units that make sense. (b) Does this equation satisfy the correspondence principle?

**4** (a) Calculate the Doppler shift to be expected in the Pound-Rebka experiment described on p. 768. (b) Verify that the gravitational time dilation in the Iijima mountain-valley experiment (p. 632) is approximately as predicted by general relativity. (The authors found agreement with theory to within 5%. I found that when I analyzed the graph I got considerably worse agreement than that, although the numbers were still in the right ballpark. I suspect the discrepancy is because analyzing the results from the graph requires making a number of simplifications, e.g., not taking into account the time that the traveling clock spent being driven up and down the mountain.)

**5** The International Space Station orbits at an altitude of about 350 km and a speed of about 8000 m/s relative to the ground. Compare the gravitational and kinematic time dilations. Over all, does time run faster on the ISS than on the ground, or more slowly?

**6** Section 27.3 presented a Newtonian estimate of how compact an object would have to be in order to be a black hole. Although this estimate is not really right, it turns out to give the right answer to within about a factor of 2. To roughly what size would the earth have to be compressed in order to become a black hole?

**7** Clock A sits on a desk. Clock B is tossed up in the air from the same height as the desk and then comes back down. Compare the elapsed times. ▷ Hint, p. 973   ▷ Solution, p. 976

**8** The angular defect  $d$  of a triangle (measured in radians) is defined as  $s - \pi$ , where  $s$  is the sum of the interior angles. The angular defect is proportional to the area  $A$  of the triangle. Consider the geometry measured by a two-dimensional being who lives on the surface of a sphere of radius  $R$ . First find some triangle on the sphere whose area and angular defect are easy to calculate. Then determine the general equation for  $d$  in terms of  $A$  and  $R$ . ✓

# Optics





## Chapter 28

# The ray model of light

Ads for one Macintosh computer bragged that it could do an arithmetic calculation in less time than it took for the light to get from the screen to your eye. We find this impressive because of the contrast between the speed of light and the speeds at which we interact with physical objects in our environment. Perhaps it shouldn't surprise us, then, that Newton succeeded so well in explaining the motion of objects, but was far less successful with the study of light.

The climax of our study of electricity and magnetism was discovery that light is an electromagnetic wave. Knowing this, however, is not the same as knowing everything about eyes and telescopes. In fact, the full description of light as a wave can be rather cumbersome. We will instead spend most of our treatment of optics making use of a simpler model of light, the ray model, which does a fine job in most practical situations. Not only that, but we will even backtrack a little and start with a discussion of basic ideas about light and vision that predated the discovery of electromagnetic waves.



## 28.1 The nature of light

### The cause and effect relationship in vision

Despite its title, this chapter is far from your first look at light. That familiarity might seem like an advantage, but most people have never thought carefully about light and vision. Even smart people who have thought hard about vision have come up with incorrect ideas. The ancient Greeks, Arabs and Chinese had theories of light and vision, all of which were mostly wrong, and all of which were accepted for thousands of years.

One thing the ancients did get right is that there is a distinction between objects that emit light and objects that don't. When you see a leaf in the forest, it's because three different objects are doing their jobs: the leaf, the eye, and the sun. But luminous objects like the sun, a flame, or the filament of a light bulb can be seen by the eye without the presence of a third object. Emission of light is often, but not always, associated with heat. In modern times, we are familiar with a variety of objects that glow without being heated, including fluorescent lights and glow-in-the-dark toys.

How do we see luminous objects? The Greek philosophers Pythagoras (b. ca. 560 BC) and Empedocles of Acragas (b. ca. 492 BC), who unfortunately were very influential, claimed that when you looked at a candle flame, the flame and your eye were both sending out some kind of mysterious stuff, and when your eye's stuff collided with the candle's stuff, the candle would become evident to your sense of sight.

Bizarre as the Greek "collision of stuff theory" might seem, it had a couple of good features. It explained why both the candle and your eye had to be present for your sense of sight to function. The theory could also easily be expanded to explain how we see nonluminous objects. If a leaf, for instance, happened to be present at the site of the collision between your eye's stuff and the candle's stuff, then the leaf would be stimulated to express its green nature, allowing you to perceive it as green.

Modern people might feel uneasy about this theory, since it suggests that greenness exists only for our seeing convenience, implying a human precedence over natural phenomena. Nowadays, people would expect the cause and effect relationship in vision to be the other way around, with the leaf doing something to our eye rather than our eye doing something to the leaf. But how can you tell? The most common way of distinguishing cause from effect is to determine which happened first, but the process of seeing seems to occur too quickly to determine the order in which things happened. Certainly there is no obvious time lag between the moment when you move your head and the moment when your reflection in the mirror moves.

Today, photography provides the simplest experimental evidence

that nothing has to be emitted from your eye and hit the leaf in order to make it “greenify.” A camera can take a picture of a leaf even if there are no eyes anywhere nearby. Since the leaf appears green regardless of whether it is being sensed by a camera, your eye, or an insect’s eye, it seems to make more sense to say that the leaf’s greenness is the cause, and something happening in the camera or eye is the effect.

### **Light is a thing, and it travels from one point to another.**

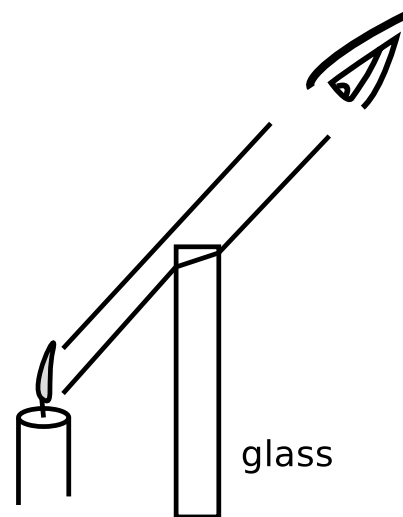
Another issue that few people have considered is whether a candle’s flame simply affects your eye directly, or whether it sends out light which then gets into your eye. Again, the rapidity of the effect makes it difficult to tell what’s happening. If someone throws a rock at you, you can see the rock on its way to your body, and you can tell that the person affected you by sending a material substance your way, rather than just harming you directly with an arm motion, which would be known as “action at a distance.” It is not easy to do a similar observation to see whether there is some “stuff” that travels from the candle to your eye, or whether it is a case of action at a distance.

Newtonian physics includes both action at a distance (e.g., the earth’s gravitational force on a falling object) and contact forces such as the normal force, which only allow distant objects to exert forces on each other by shooting some substance across the space between them (e.g., a garden hose spraying out water that exerts a force on a bush).

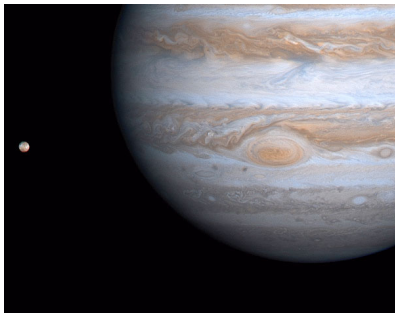
One piece of evidence that the candle sends out stuff that travels to your eye is that as in figure a, intervening transparent substances can make the candle appear to be in the wrong location, suggesting that light is a thing that can be bumped off course. Many people would dismiss this kind of observation as an optical illusion, however. (Some optical illusions are purely neurological or psychological effects, although some others, including this one, turn out to be caused by the behavior of light itself.)

A more convincing way to decide in which category light belongs is to find out if it takes time to get from the candle to your eye; in Newtonian physics, action at a distance is supposed to be instantaneous. The fact that we speak casually today of “the speed of light” implies that at some point in history, somebody succeeded in showing that light did not travel infinitely fast. Galileo tried, and failed, to detect a finite speed for light, by arranging with a person in a distant tower to signal back and forth with lanterns. Galileo uncovered his lantern, and when the other person saw the light, he uncovered his lantern. Galileo was unable to measure any time lag that was significant compared to the limitations of human reflexes.

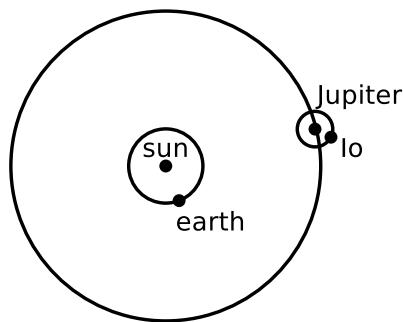
The first person to prove that light’s speed was finite, and to deter-



a / Light from a candle is bumped off course by a piece of glass. Inserting the glass causes the apparent location of the candle to shift. The same effect can be produced by taking off your eyeglasses and looking at which you see near the edge of the lens, but a flat piece of glass works just as well as a lens for this purpose.



b / An image of Jupiter and its moon Io (left) from the Cassini probe.



c / The earth is moving toward Jupiter and Io. Since the distance is shrinking, it is taking less and less time for the light to get to us from Io, and Io appears to circle Jupiter more quickly than normal. Six months later, the earth will be on the opposite side of the sun, and receding from Jupiter and Io, so Io will appear to revolve around Jupiter more slowly.

mine it numerically, was Ole Roemer, in a series of measurements around the year 1675. Roemer observed Io, one of Jupiter's moons, over a period of several years. Since Io presumably took the same amount of time to complete each orbit of Jupiter, it could be thought of as a very distant, very accurate clock. A practical and accurate pendulum clock had recently been invented, so Roemer could check whether the ratio of the two clocks' cycles, about 42.5 hours to 1 orbit, stayed exactly constant or changed a little. If the process of seeing the distant moon was instantaneous, there would be no reason for the two to get out of step. Even if the speed of light was finite, you might expect that the result would be only to offset one cycle relative to the other. The earth does not, however, stay at a constant distance from Jupiter and its moons. Since the distance is changing gradually due to the two planets' orbital motions, a finite speed of light would make the "Io clock" appear to run faster as the planets drew near each other, and more slowly as their separation increased. Roemer did find a variation in the apparent speed of Io's orbits, which caused Io's eclipses by Jupiter (the moments when Io passed in front of or behind Jupiter) to occur about 7 minutes early when the earth was closest to Jupiter, and 7 minutes late when it was farthest. Based on these measurements, Roemer estimated the speed of light to be approximately  $2 \times 10^8$  m/s, which is in the right ballpark compared to modern measurements of  $3 \times 10^8$  m/s. (I'm not sure whether the fairly large experimental error was mainly due to imprecise knowledge of the radius of the earth's orbit or limitations in the reliability of pendulum clocks.)

### Light can travel through a vacuum.

Many people are confused by the relationship between sound and light. Although we use different organs to sense them, there are some similarities. For instance, both light and sound are typically emitted in all directions by their sources. Musicians even use visual metaphors like "tone color," or "a bright timbre" to describe sound. One way to see that they are clearly different phenomena is to note their very different velocities. Sure, both are pretty fast compared to a flying arrow or a galloping horse, but as we have seen, the speed of light is so great as to appear instantaneous in most situations. The speed of sound, however, can easily be observed just by watching a group of schoolchildren a hundred feet away as they clap their hands to a song. There is an obvious delay between when you see their palms come together and when you hear the clap.

The fundamental distinction between sound and light is that sound is an oscillation in air pressure, so it requires air (or some other medium such as water) in which to travel. Today, we know that outer space is a vacuum, so the fact that we get light from the sun, moon and stars clearly shows that air is not necessary for the propagation of light.

## Discussion questions

- A** If you observe thunder and lightning, you can tell how far away the storm is. Do you need to know the speed of sound, of light, or of both?
- B** When phenomena like X-rays and cosmic rays were first discovered, suggest a way one could have tested whether they were forms of light.
- C** Why did Roemer only need to know the radius of the earth's orbit, not Jupiter's, in order to find the speed of light?

## 28.2 Interaction of light with matter

### Absorption of light

The reason why the sun feels warm on your skin is that the sunlight is being absorbed, and the light energy is being transformed into heat energy. The same happens with artificial light, so the net result of leaving a light turned on is to heat the room. It doesn't matter whether the source of the light is hot, like the sun, a flame, or an incandescent light bulb, or cool, like a fluorescent bulb. (If your house has electric heat, then there is absolutely no point in fastidiously turning off lights in the winter; the lights will help to heat the house at the same dollar rate as the electric heater.)

This process of heating by absorption is entirely different from heating by thermal conduction, as when an electric stove heats spaghetti sauce through a pan. Heat can only be conducted through matter, but there is vacuum between us and the sun, or between us and the filament of an incandescent bulb. Also, heat conduction can only transfer heat energy from a hotter object to a colder one, but a cool fluorescent bulb is perfectly capable of heating something that had already started out being warmer than the bulb itself.

### How we see nonluminous objects

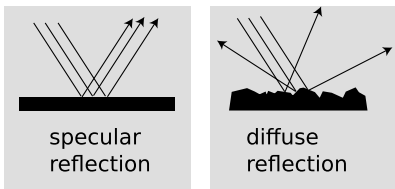
Not all the light energy that hits an object is transformed into heat. Some is reflected, and this leads us to the question of how we see nonluminous objects. If you ask the average person how we see a light bulb, the most likely answer is "The light bulb makes light, which hits our eyes." But if you ask how we see a book, they are likely to say "The bulb lights up the room, and that lets me see the book." All mention of light actually entering our eyes has mysteriously disappeared.

Most people would disagree if you told them that light was reflected from the book to the eye, because they think of reflection as something that mirrors do, not something that a book does. They associate reflection with the formation of a reflected image, which does not seem to appear in a piece of paper.

Imagine that you are looking at your reflection in a nice smooth piece of aluminum foil, fresh off the roll. You perceive a face, not a piece of metal. Perhaps you also see the bright reflection of a lamp



d / Two self-portraits of the author, one taken in a mirror and one with a piece of aluminum foil.



e / Specular and diffuse reflection.

over your shoulder behind you. Now imagine that the foil is just a little bit less smooth. The different parts of the image are now a little bit out of alignment with each other. Your brain can still recognize a face and a lamp, but it's a little scrambled, like a Picasso painting. Now suppose you use a piece of aluminum foil that has been crumpled up and then flattened out again. The parts of the image are so scrambled that you cannot recognize an image. Instead, your brain tells you you're looking at a rough, silvery surface.

Mirror-like reflection at a specific angle is known as specular reflection, and random reflection in many directions is called diffuse reflection. Diffuse reflection is how we see nonluminous objects. Specular reflection only allows us to see images of objects other than the one doing the reflecting. In top part of figure d, imagine that the rays of light are coming from the sun. If you are looking down at the reflecting surface, there is no way for your eye-brain system to tell that the rays are not really coming from a sun down below you.

Figure f shows another example of how we can't avoid the conclusion that light bounces off of things other than mirrors. The lamp is one I have in my house. It has a bright bulb, housed in a completely opaque bowl-shaped metal shade. The only way light can get out of the lamp is by going up out of the top of the bowl. The fact that I can read a book in the position shown in the figure means that light must be bouncing off of the ceiling, then bouncing off of the book, then finally getting to my eye.

This is where the shortcomings of the Greek theory of vision become glaringly obvious. In the Greek theory, the light from the bulb and my mysterious "eye rays" are both supposed to go to the book, where they collide, allowing me to see the book. But we now have a total of four objects: lamp, eye, book, and ceiling. Where does the ceiling come in? Does it also send out its own mysterious "ceiling rays," contributing to a three-way collision at the book? That would just be too bizarre to believe!

The differences among white, black, and the various shades of gray in between is a matter of what percentage of the light they absorb and what percentage they reflect. That's why light-colored clothing is more comfortable in the summer, and light-colored upholstery in a car stays cooler than dark upholstery.

## Numerical measurement of the brightness of light

We have already seen that the physiological sensation of loudness relates to the sound's intensity (power per unit area), but is not directly proportional to it. If sound A has an intensity of  $1 \text{ nW/m}^2$ , sound B is  $10 \text{ nW/m}^2$ , and sound C is  $100 \text{ nW/m}^2$ , then the increase in loudness from B to C is perceived to be the same as the increase from A to B, not ten times greater. That is, the sensation of loudness is logarithmic.

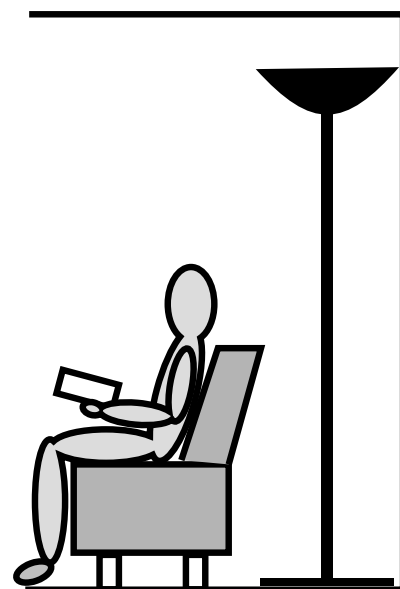
The same is true for the brightness of light. Brightness is related to power per unit area, but the psychological relationship is a logarithmic one rather than a proportionality. For doing physics, it's the power per unit area that we're interested in. The relevant unit is  $\text{W/m}^2$ . One way to determine the brightness of light is to measure the increase in temperature of a black object exposed to the light. The light energy is being converted to heat energy, and the amount of heat energy absorbed in a given amount of time can be related to the power absorbed, using the known heat capacity of the object. More practical devices for measuring light intensity, such as the light meters built into some cameras, are based on the conversion of light into electrical energy, but these meters have to be calibrated somehow against heat measurements.

### Discussion questions

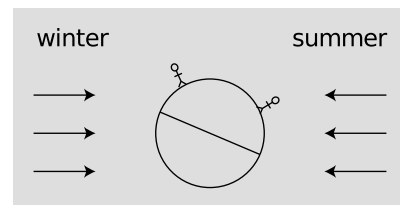
**A** The curtains in a room are drawn, but a small gap lets light through, illuminating a spot on the floor. It may or may not also be possible to see the beam of sunshine crossing the room, depending on the conditions. What's going on?

**B** Laser beams are made of light. In science fiction movies, laser beams are often shown as bright lines shooting out of a laser gun on a spaceship. Why is this scientifically incorrect?

**C** A documentary film-maker went to Harvard's 1987 graduation ceremony and asked the graduates, on camera, to explain the cause of the seasons. Only two out of 23 were able to give a correct explanation, but you now have all the information needed to figure it out for yourself, assuming you didn't already know. The figure shows the earth in its winter and summer positions relative to the sun. Hint: Consider the units used to measure the brightness of light, and recall that the sun is lower in the sky in winter, so its rays are coming in at a shallower angle.



f / Light bounces off of the ceiling, then off of the book.



g / Discussion question C.

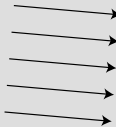
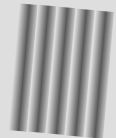

## 28.3 The ray model of light

### Models of light

Note how I've been casually diagramming the motion of light with pictures showing light rays as lines on the page. More formally, this is known as the ray model of light. The ray model of light seems natural once we convince ourselves that light travels through space, and observe phenomena like sunbeams coming through holes in clouds. Having already been introduced to the concept of light

as an electromagnetic wave, you know that the ray model is not the ultimate truth about light, but the ray model is simpler, and in any case science always deals with models of reality, not the ultimate nature of reality. The following table summarizes three models of light.

h / Three models of light.

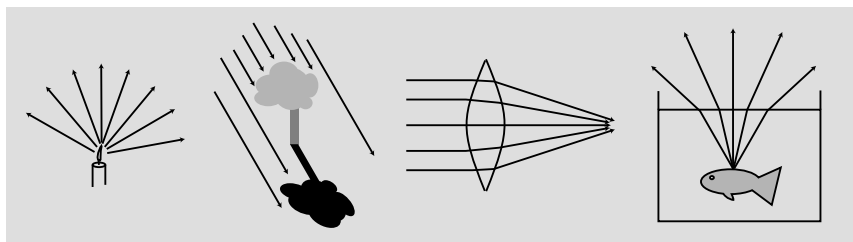
ray model		Advantage: Simplicity.
wave model		Advantage: Color is described naturally in terms of wavelength. Required in order to explain the interaction of light with material objects of sizes comparable to or smaller than a wavelength of light.
particle model		Required in order to explain the interaction of light with individual atoms. At the atomic level, it becomes apparent that a beam of light has a certain graininess to it.

The ray model is a generic one. By using it we can discuss the path taken by the light, without committing ourselves to any specific description of what it is that is moving along that path. We will use the nice simple ray model for most of our treatment of optics, and with it we can analyze a great many devices and phenomena. Not until chapter 32 will we concern ourselves specifically with wave optics, although in the intervening chapters I will sometimes analyze the same phenomenon using both the ray model and the wave model.

Note that the statements about the applicability of the various models are only rough guides. For instance, wave interference effects are often detectable, if small, when light passes around an obstacle that is quite a bit bigger than a wavelength. Also, the criterion for when we need the particle model really has more to do with energy scales than distance scales, although the two turn out to be related.

The alert reader may have noticed that the wave model is required at scales smaller than a wavelength of light (on the order of a micrometer for visible light), and the particle model is demanded on the atomic scale or lower (a typical atom being a nanometer or so in size). This implies that at the smallest scales we need *both* the wave model and the particle model. They appear incompatible, so how can we simultaneously use both? The answer is that they are not as incompatible as they seem. Light is both a wave and a particle,

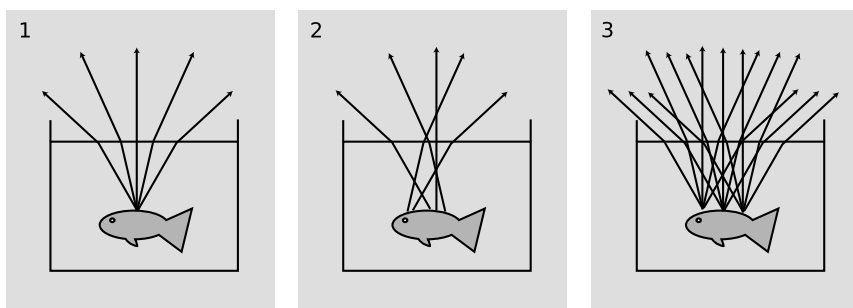
but a full understanding of this apparently nonsensical statement is a topic for chapter 34.



i / Examples of ray diagrams.

## Ray diagrams

Without even knowing how to use the ray model to calculate anything numerically, we can learn a great deal by drawing ray diagrams. For instance, if you want to understand how eyeglasses help you to see in focus, a ray diagram is the right place to start. Many students under-utilize ray diagrams in optics and instead rely on rote memorization or plugging into formulas. The trouble with memorization and plug-ins is that they can obscure what's really going on, and it is easy to get them wrong. Often the best plan is to do a ray diagram first, then do a numerical calculation, then check that your numerical results are in reasonable agreement with what you expected from the ray diagram.



j / 1. Correct. 2. Incorrect: implies that diffuse reflection only gives one ray from each reflecting point. 3. Correct, but unnecessarily complicated

Figure j shows some guidelines for using ray diagrams effectively. The light rays bend when they pass out through the surface of the water (a phenomenon that we'll discuss in more detail later). The rays appear to have come from a point above the goldfish's actual location, an effect that is familiar to people who have tried spearfishing.

- A stream of light is not really confined to a finite number of narrow lines. We just draw it that way. In j/1, it has been necessary to choose a finite number of rays to draw (five), rather than the theoretically infinite number of rays that will diverge from that point.



- There is a tendency to conceptualize rays incorrectly as objects. In his *Optics*, Newton goes out of his way to caution the reader against this, saying that some people “consider ... the refraction of ... rays to be the bending or breaking of them in their passing out of one medium into another.” But a ray is a record of the path traveled by light, not a physical thing that can be bent or broken.
- In theory, rays may continue infinitely far into the past and future, but we need to draw lines of finite length. In j/1, a judicious choice has been made as to where to begin and end the rays. There is no point in continuing the rays any farther than shown, because nothing new and exciting is going to happen to them. There is also no good reason to start them earlier, before being reflected by the fish, because the direction of the diffusely reflected rays is random anyway, and unrelated to the direction of the original, incoming ray.
- When representing diffuse reflection in a ray diagram, many students have a mental block against drawing many rays fanning out from the same point. Often, as in example j/2, the problem is the misconception that light can only be reflected in one direction from one point.
- Another difficulty associated with diffuse reflection, example j/3, is the tendency to think that in addition to drawing many rays coming out of one point, we should also be drawing many rays coming from many points. In j/1, drawing many rays coming out of one point gives useful information, telling us, for instance, that the fish can be seen from any angle. Drawing many sets of rays, as in j/3, does not give us any more useful information, and just clutters up the picture in this example. The only reason to draw sets of rays fanning out from more than one point would be if different things were happening to the different sets.

### Discussion question

**A** Suppose an intelligent tool-using fish is spear-hunting for humans. Draw a ray diagram to show how the fish has to correct its aim. Note that although the rays are now passing from the air to the water, the same rules apply: the rays are closer to being perpendicular to the surface when they are in the water, and rays that hit the air-water interface at a shallow angle are bent the most.

## 28.4 Geometry of specular reflection

To change the motion of a material object, we use a force. Is there any way to exert a force on a beam of light? Experiments show that electric and magnetic fields do not deflect light beams, so apparently light has no electric charge. Light also has no mass, so

until the twentieth century it was believed to be immune to gravity as well. Einstein predicted that light beams would be very slightly deflected by strong gravitational fields, and he was proved correct by observations of rays of starlight that came close to the sun, but obviously that's not what makes mirrors and lenses work!

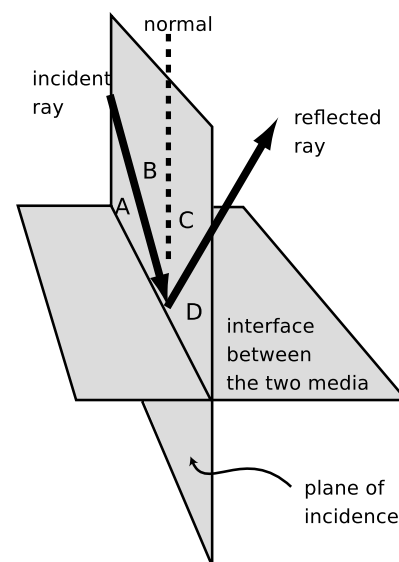
If we investigate how light is reflected by a mirror, we will find that the process is horribly complex, but the final result is surprisingly simple. What actually happens is that the light is made of electric and magnetic fields, and these fields accelerate the electrons in the mirror. Energy from the light beam is momentarily transformed into extra kinetic energy of the electrons, but because the electrons are accelerating they re-radiate more light, converting their kinetic energy back into light energy. We might expect this to result in a very chaotic situation, but amazingly enough, the electrons move together to produce a new, reflected beam of light, which obeys two simple rules:

- The angle of the reflected ray is the same as that of the incident ray.
- The reflected ray lies in the plane containing the incident ray and the normal (perpendicular) line. This plane is known as the plane of incidence.

The two angles can be defined either with respect to the normal, like angles B and C in the figure, or with respect to the reflecting surface, like angles A and D. There is a convention of several hundred years' standing that one measures the angles with respect to the normal, but the rule about equal angles can logically be stated either as  $B=C$  or as  $A=D$ .

The phenomenon of reflection occurs only at the boundary between two media, just like the change in the speed of light that passes from one medium to another. As we have seen in chapter 20, this is the way all waves behave.

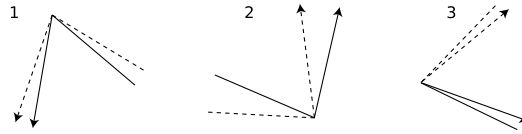
Most people are surprised by the fact that light can be reflected back from a less dense medium. For instance, if you are diving and you look up at the surface of the water, you will see a reflection of yourself.



k / The geometry of specular reflection.

### self-check A

Each of these diagrams is supposed to show two different rays being reflected from the same point on the same mirror. Which are correct, and which are incorrect?



▷ Answer, p. 979

## Reversibility of light rays

The fact that specular reflection displays equal angles of incidence and reflection means that there is a symmetry: if the ray had come in from the right instead of the left in the figure above, the angles would have looked exactly the same. This is not just a pointless detail about specular reflection. It's a manifestation of a very deep and important fact about nature, which is that the laws of physics do not distinguish between past and future. Cannonballs and planets have trajectories that are equally natural in reverse, and so do light rays. This type of symmetry is called time-reversal symmetry.

Typically, time-reversal symmetry is a characteristic of any process that does not involve heat. For instance, the planets do not experience any friction as they travel through empty space, so there is no frictional heating. We should thus expect the time-reversed versions of their orbits to obey the laws of physics, which they do. In contrast, a book sliding across a table does generate heat from friction as it slows down, and it is therefore not surprising that this type of motion does not appear to obey time-reversal symmetry. A book lying still on a flat table is never observed to spontaneously start sliding, sucking up heat energy and transforming it into kinetic energy.

Similarly, the only situation we've observed so far where light does not obey time-reversal symmetry is absorption, which involves heat. Your skin absorbs visible light from the sun and heats up, but we never observe people's skin to glow, converting heat energy into visible light. People's skin does glow in infrared light, but that doesn't mean the situation is symmetric. Even if you absorb infrared, you don't emit visible light, because your skin isn't hot enough to glow in the visible spectrum.

These apparent heat-related asymmetries are not actual asymmetries in the laws of physics. The interested reader may wish to learn more about this from optional chapter 16 on thermodynamics.

### Ray tracing on a computer

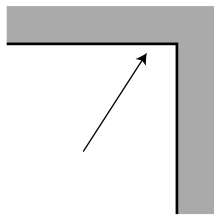
### example 1

A number of techniques can be used for creating artificial visual scenes in computer graphics. Figure I shows such a scene, which was created by the brute-force technique of simply constructing a very detailed ray diagram on a computer. This technique requires a great deal of computation, and is therefore too slow to

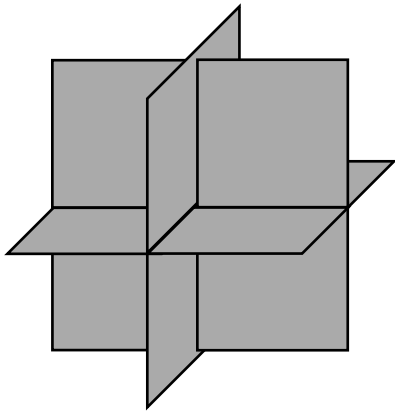
be used for video games and computer-animated movies. One trick for speeding up the computation is to exploit the reversibility of light rays. If one was to trace every ray emitted by every illuminated surface, only a tiny fraction of those would actually end up passing into the virtual “camera,” and therefore almost all of the computational effort would be wasted. One can instead start a ray at the camera, trace it backward in time, and see where it would have come from. With this technique, there is no wasted effort.



1 / This photorealistic image of a nonexistent countertop was produced completely on a computer, by computing a complicated ray diagram.



m / Discussion question B.



n / Discussion question C.

### Discussion questions

**A** If a light ray has a velocity vector with components  $c_x$  and  $c_y$ , what will happen when it is reflected from a surface that lies along the  $y$  axis? Make sure your answer does not imply a change in the ray's speed.

**B** Generalizing your reasoning from discussion question A, what will happen to the velocity components of a light ray that hits a corner, as shown in the figure, and undergoes two reflections?

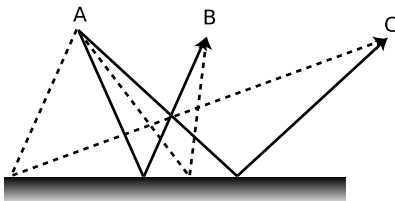
**C** Three pieces of sheet metal arranged perpendicularly as shown in the figure form what is known as a radar corner. Let's assume that the radar corner is large compared to the wavelength of the radar waves, so that the ray model makes sense. If the radar corner is bathed in radar rays, at least some of them will undergo three reflections. Making a further generalization of your reasoning from the two preceding discussion questions, what will happen to the three velocity components of such a ray? What would the radar corner be useful for?

## 28.5 ★ The principle of least time for reflection

We had to choose between an unwieldy explanation of reflection at the atomic level and a simpler geometric description that was not as fundamental. There is a third approach to describing the interaction of light and matter which is very deep and beautiful. Emphasized by the twentieth-century physicist Richard Feynman, it is called the principle of least time, or Fermat's principle.

Let's start with the motion of light that is not interacting with matter at all. In a vacuum, a light ray moves in a straight line. This can be rephrased as follows: of all the conceivable paths light could follow from P to Q, the only one that is physically possible is the path that takes the least time.

What about reflection? If light is going to go from one point to another, being reflected on the way, the quickest path is indeed the one with equal angles of incidence and reflection. If the starting and ending points are equally far from the reflecting surface, o, it's not hard to convince yourself that this is true, just based on symmetry. There is also a tricky and simple proof, shown in figure p, for the more general case where the points are at different distances from the surface.



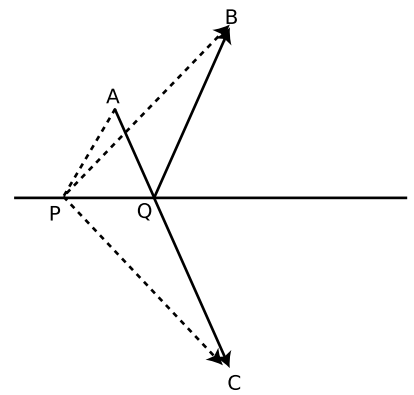
o / The solid lines are physically possible paths for light rays traveling from A to B and from A to C. They obey the principle of least time. The dashed lines do not obey the principle of least time, and are not physically possible.

Not only does the principle of least time work for light in a vacuum and light undergoing reflection, we will also see in a later chapter that it works for the bending of light when it passes from one medium into another.

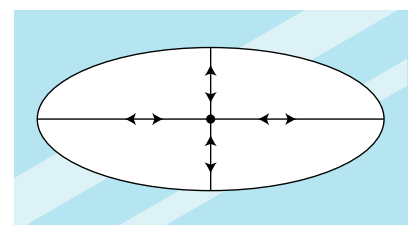
Although it is beautiful that the entire ray model of light can be reduced to one simple rule, the principle of least time, it may seem a little spooky to speak as if the ray of light is intelligent, and has carefully planned ahead to find the shortest route to its destination. How does it know in advance where it's going? What if we moved the mirror while the light was en route, so conditions along its planned path were not what it "expected?" The answer is that the principle of least time is really a shortcut for finding certain results of the wave model of light, which is the topic of the last chapter of this book.

There are a couple of subtle points about the principle of least time. First, the path does not have to be the quickest of all possible paths; it only needs to be quicker than any path that differs infinitesimally from it. In figure p, for instance, light could get from A to B either by the reflected path AQB or simply by going straight from A to B. Although AQB is not the shortest possible path, it cannot be shortened by changing it infinitesimally, e.g., by moving Q a little to the right or left. On the other hand, path APB is physically impossible, because it is possible to improve on it by moving point P infinitesimally to the right.

It's not quite right to call this the principle of *least* time. In figure q, for example, the four physically possible paths by which a ray can return to the center consist of two shortest-time paths and two longest-time paths. Strictly speaking, we should refer to the *principle of least or greatest time*, but most physicists omit the niceties, and assume that other physicists understand that both maxima and minima are possible.



p / Paths AQB and APB are two conceivable paths that a ray could follow to get from A to B with one reflection, but only AQB is physically possible. We wish to prove that the path AQB, with equal angles of incidence and reflection, is shorter than any other path, such as APB. The trick is to construct a third point, C, lying as far below the surface as B lies above it. Then path AQC is a straight line whose length is the same as AQB's, and path APC has the same length as path APB. Since AQC is straight, it must be shorter than any other path such as APC that connects A and C, and therefore AQB must be shorter than any path such as APB.



q / Light is emitted at the center of an elliptical mirror. There are four physically possible paths by which a ray can be reflected and return to the center.

# Summary

## Selected vocabulary

absorption . . . .	what happens when light hits matter and gives up some of its energy
reflection . . . . .	what happens when light hits matter and bounces off, retaining at least some of its energy
specular reflection . . . . .	reflection from a smooth surface, in which the light ray leaves at the same angle at which it came in
diffuse reflection	reflection from a rough surface, in which a single ray of light is divided up into many weaker reflected rays going in many directions
normal . . . . .	the line perpendicular to a surface at a given point

## Notation

$c$ . . . . .	the speed of light
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## Summary

We can understand many phenomena involving light without having to use sophisticated models such as the wave model or the particle model. Instead, we simply describe light according to the path it takes, which we call a ray. The ray model of light is useful when light is interacting with material objects that are much larger than a wavelength of light. Since a wavelength of visible light is so short compared to the human scale of existence, the ray model is useful in many practical cases.

We see things because light comes from them to our eyes. Objects that glow may send light directly to our eyes, but we see an object that doesn't glow via light from another source that has been reflected by the object.

Many of the interactions of light and matter can be understood by considering what happens when light reaches the boundary between two different substances. In this situation, part of the light is reflected (bounces back) and part passes on into the new medium. This is not surprising — it is typical behavior for a wave, and light is a wave. Light energy can also be absorbed by matter, i.e., converted into heat.

A smooth surface produces specular reflection, in which the reflected ray exits at the same angle with respect to the normal as that of the incoming ray. A rough surface gives diffuse reflection, where a single ray of light is divided up into many weaker reflected rays going in many directions.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** Draw a ray diagram showing why a small light source (a candle, say) produces sharper shadows than a large one (e.g., a long fluorescent bulb).

**2** A Global Positioning System (GPS) receiver is a device that lets you figure out where you are by receiving timed radio signals from satellites. It works by measuring the travel time for the signals, which is related to the distance between you and the satellite. By finding the ranges to several different satellites in this way, it can pin down your location in three dimensions to within a few meters. How accurate does the measurement of the time delay have to be to determine your position to this accuracy?

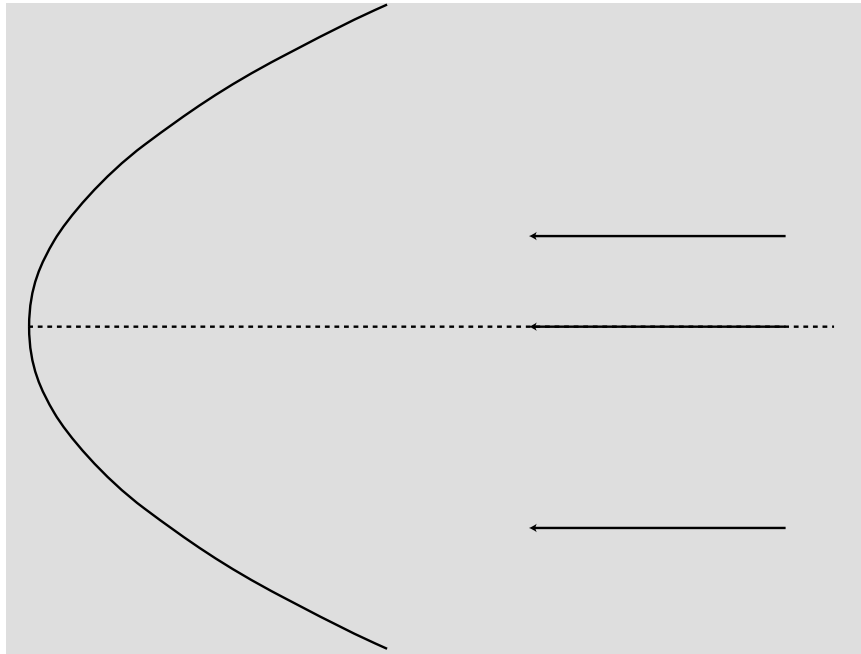
**3** Estimate the frequency of an electromagnetic wave whose wavelength is similar in size to an atom (about a nm). Referring back to figure 24.5.3 on p. 667, in what part of the electromagnetic spectrum would such a wave lie (infrared, gamma-rays, ...)?

**4** The Stealth bomber is designed with flat, smooth surfaces. Why would this make it difficult to detect via radar?

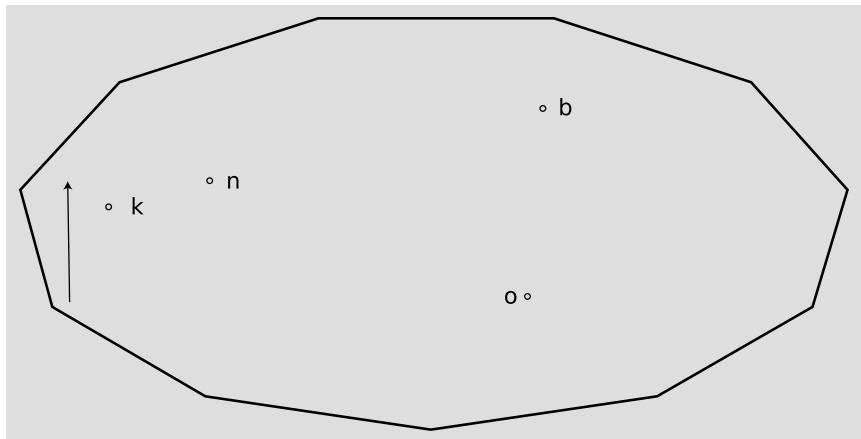
**5** The figure on the next page shows a curved (parabolic) mirror, with three parallel light rays coming toward it. One ray is approaching along the mirror's center line. (a) Trace the drawing accurately, and continue the light rays until they are about to undergo their second reflection. To get good enough accuracy, you'll need to photocopy the page (or download the book and print the page) and draw in the normal at each place where a ray is reflected. What do you notice? (b) Make up an example of a practical use for this device. (c) How could you use this mirror with a small lightbulb to produce a parallel beam of light rays going off to the right?

**6** The natives of planet Wumpus play pool using light rays on an eleven-sided table with mirrors for bumpers, shown in the figure on the next page. Trace this shot accurately with a ruler to reveal the hidden message. To get good enough accuracy, you'll need to photocopy the page (or download the book and print the page) and construct each reflection using a protractor.





Problem 5.



Problem 6.



Narcissus, by Michelangelo Caravaggio, ca. 1598.

## Chapter 29

# Images by reflection

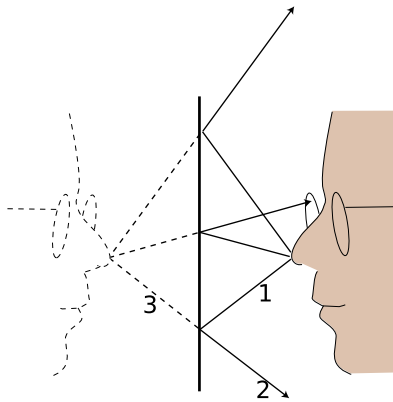
Infants are always fascinated by the antics of the Baby in the Mirror. Now if you want to know something about mirror images that most people don't understand, try this. First bring this page closer and closer to your eyes, until you can no longer focus on it without straining. Then go in the bathroom and see how close you can get your face to the surface of the mirror before you can no longer easily focus on the image of your own eyes. You will find that the shortest comfortable eye-mirror distance is much less than the shortest comfortable eye-paper distance. This demonstrates that the image of your face in the mirror acts as if it had depth and

existed in the space *behind* the mirror. If the image was like a flat picture in a book, then you wouldn't be able to focus on it from such a short distance.

In this chapter we will study the images formed by flat and curved mirrors on a qualitative, conceptual basis. Although this type of image is not as commonly encountered in everyday life as images formed by lenses, images formed by reflection are simpler to understand, so we discuss them first. In chapter 30 we will turn to a more mathematical treatment of images made by reflection. Surprisingly, the same equations can also be applied to lenses, which are the topic of chapter 31.

## 29.1 A virtual image

We can understand a mirror image using a ray diagram. Figure a shows several light rays, 1, that originated by diffuse reflection at the person's nose. They bounce off the mirror, producing new rays, 2. To anyone whose eye is in the right position to get one of these rays, they appear to have come from a behind the mirror, 3, where they would have originated from a single point. This point is where the tip of the image-person's nose appears to be. A similar analysis applies to every other point on the person's face, so it looks as though there was an entire face behind the mirror. The customary way of describing the situation requires some explanation:



a / An image formed by a mirror.

**Customary description in physics:** There is an image of the face behind the mirror.

**Translation:** The pattern of rays coming from the mirror is exactly the same as it would be if there were a face behind the mirror. Nothing is really behind the mirror.

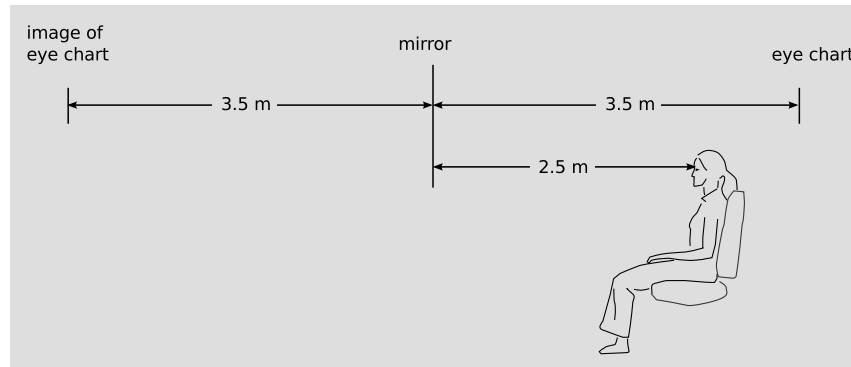
This is referred to as a *virtual* image, because the rays do not actually cross at the point behind the mirror. They only appear to have originated there.

### *self-check A*

Imagine that the person in figure a moves his face down quite a bit — a couple of feet in real life, or a few inches on this scale drawing. The mirror stays where it is. Draw a new ray diagram. Will there still be an image? If so, where is it visible from? ▷ Answer, p. 979

The geometry of specular reflection tells us that rays 1 and 2 are at equal angles to the normal (the imaginary perpendicular line piercing the mirror at the point of reflection). This means that ray 2's imaginary continuation, 3, forms the same angle with the mirror as ray 3. Since each ray of type 3 forms the same angles with

the mirror as its partner of type 1, we see that the distance of the image from the mirror is the same as that of the actual face from the mirror, and it lies directly across from it. The image therefore appears to be the same size as the actual face.



b / Example 1.

#### *An eye exam*

#### *example 1*

Figure b shows a typical setup in an optometrist's examination room. The patient's vision is supposed to be tested at a distance of 6 meters (20 feet in the U.S.), but this distance is larger than the amount of space available in the room. Therefore a mirror is used to create an image of the eye chart behind the wall.

#### *The Praxinoscope*

#### *example 2*

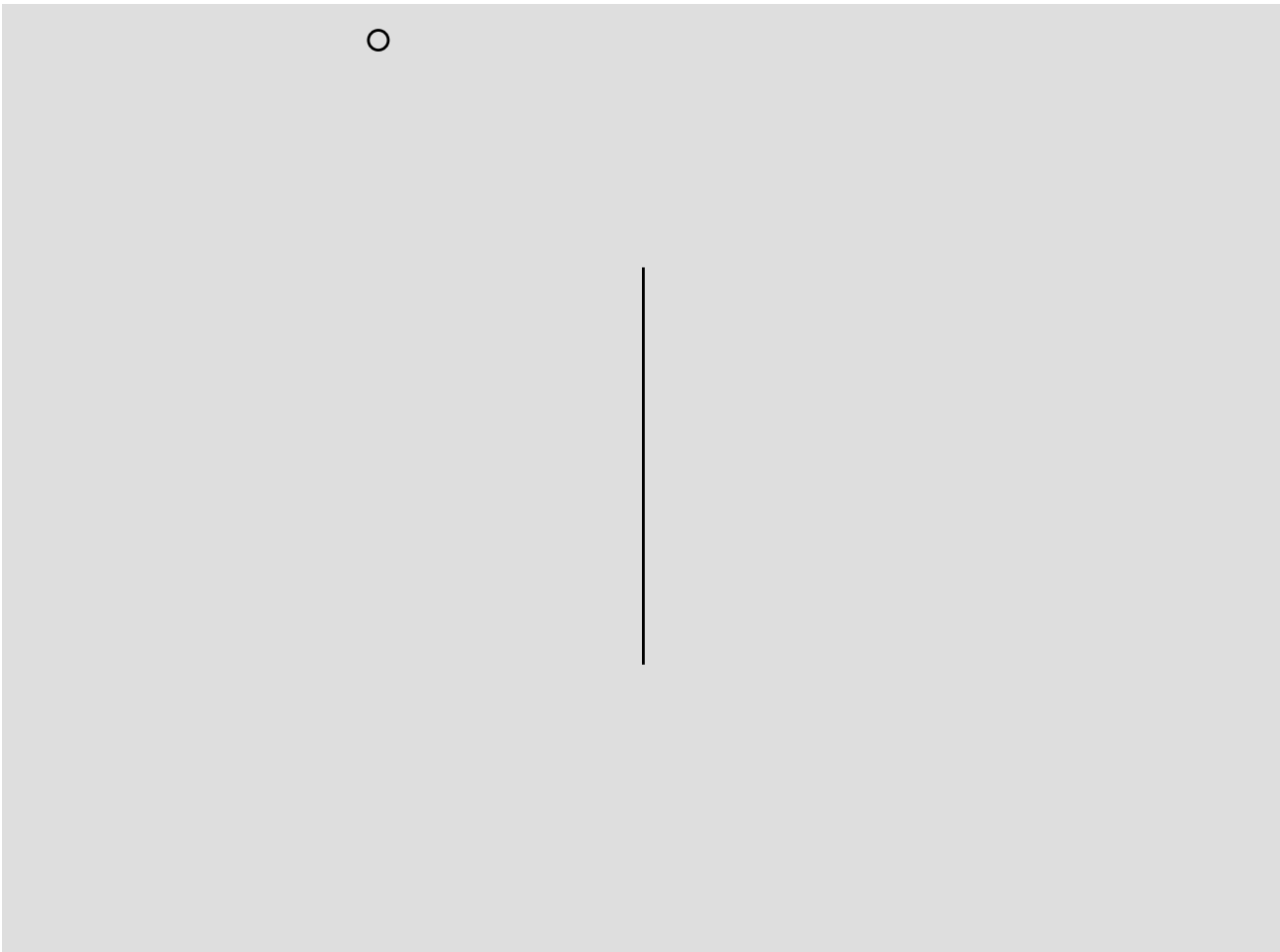
Figure c shows an old-fashioned device called a praxinoscope, which displays an animated picture when spun. The removable strip of paper with the pictures printed on it has twice the radius of the inner circle made of flat mirrors, so each picture's virtual image is at the center. As the wheel spins, each picture's image is replaced by the next.



c / The praxinoscope.

### Discussion question

**A** The figure shows an object that is off to one side of a mirror. Draw a ray diagram. Is an image formed? If so, where is it, and from which directions would it be visible?



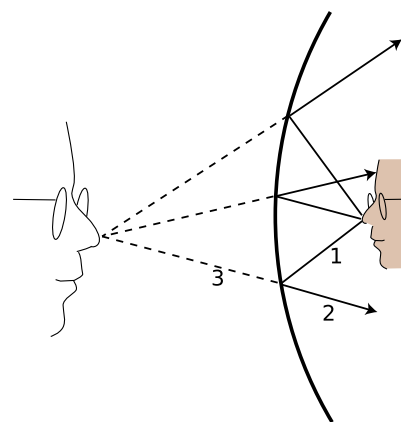
## 29.2 Curved mirrors

An image in a flat mirror is a pretechnological example: even animals can look at their reflections in a calm pond. We now pass to our first nontrivial example of the manipulation of an image by technology: an image in a curved mirror. Before we dive in, let's consider why this is an important example. If it was just a question of memorizing a bunch of facts about curved mirrors, then you would rightly rebel against an effort to spoil the beauty of your liberally educated brain by force-feeding you technological trivia. The reason this is an important example is not that curved mirrors are so important in and of themselves, but that the results we derive for curved bowl-shaped mirrors turn out to be true for a large class of other optical devices, including mirrors that bulge outward rather than inward, and lenses as well. A microscope or a telescope is simply a combination of lenses or mirrors or both. What you're really learning about here is the basic building block of all optical devices from movie projectors to octopus eyes.

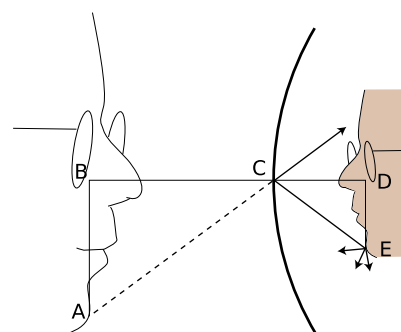
Because the mirror in figure d is curved, it bends the rays back closer together than a flat mirror would: we describe it as *converging*. Note that the term refers to what it does to the light rays, not to the physical shape of the mirror's surface. (The surface itself would be described as *concave*. The term is not all that hard to remember, because the hollowed-out interior of the mirror is like a cave.) It is surprising but true that all the rays like 3 really do converge on a point, forming a good image. We will not prove this fact, but it is true for any mirror whose curvature is gentle enough and that is symmetric with respect to rotation about the perpendicular line passing through its center (not asymmetric like a potato chip). The old-fashioned method of making mirrors and lenses is by grinding them in grit by hand, and this automatically tends to produce an almost perfect spherical surface.

Bending a ray like 2 inward implies bending its imaginary continuation 3 outward, in the same way that raising one end of a seesaw causes the other end to go down. The image therefore forms deeper behind the mirror. This doesn't just show that there is extra distance between the image-nose and the mirror; it also implies that the image itself is bigger from front to back. It has been *magnified* in the front-to-back direction.

It is easy to prove that the same magnification also applies to the image's other dimensions. Consider a point like E in figure e. The trick is that out of all the rays diffusely reflected by E, we pick the one that happens to head for the mirror's center, C. The equal-angle property of specular reflection plus a little straightforward geometry easily leads us to the conclusion that triangles ABC and CDE are the same shape, with ABC being simply a scaled-up version of CDE. The magnification of depth equals the ratio  $BC/CD$ , and the up-



d / An image formed by a curved mirror.



e / The image is magnified by the same factor in depth and in its other dimensions.



f / Increased magnification always comes at the expense of decreased field of view.

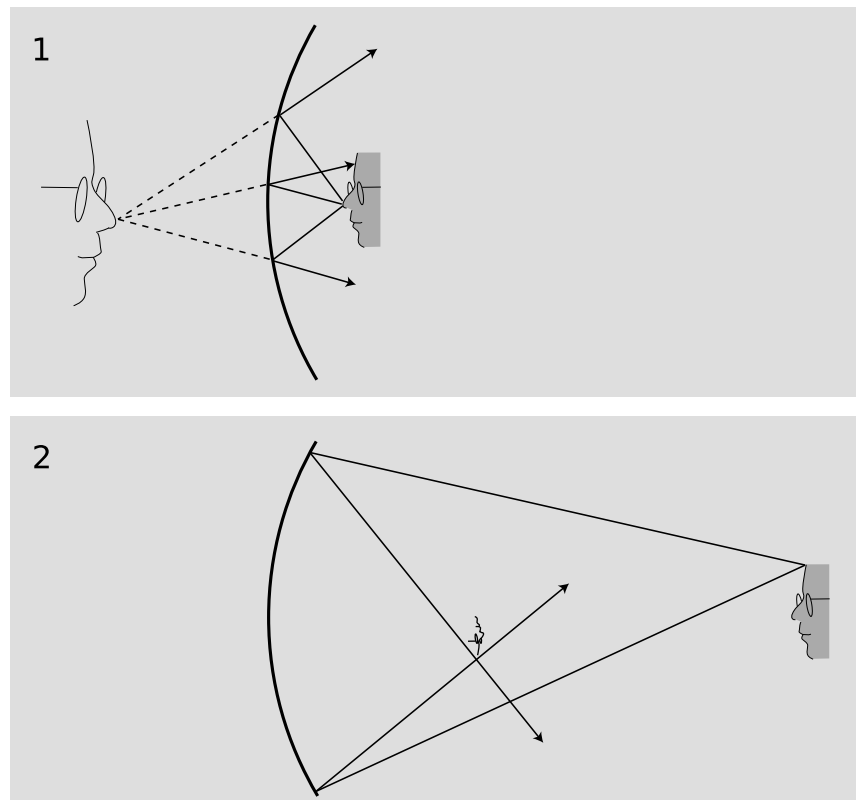
down magnification is  $AB/DE$ . A repetition of the same proof shows that the magnification in the third dimension (out of the page) is also the same. This means that the image-head is simply a larger version of the real one, without any distortion. The scaling factor is called the magnification,  $M$ . The image in the figure is magnified by a factor  $M = 1.9$ .

Note that we did not explicitly specify whether the mirror was a sphere, a paraboloid, or some other shape. However, we assumed that a focused image would be formed, which would not necessarily be true, for instance, for a mirror that was asymmetric or very deeply curved.

### 29.3 A real image

If we start by placing an object very close to the mirror,  $g/1$ , and then move it farther and farther away, the image at first behaves as we would expect from our everyday experience with flat mirrors, receding deeper and deeper behind the mirror. At a certain point, however, a dramatic change occurs. When the object is more than a certain distance from the mirror,  $g/2$ , the image appears upside-down and in *front* of the mirror.

$g/1$ . A virtual image. 2. A real image. As you'll verify in homework problem 6, the image is upside-down



Here's what's happened. The mirror bends light rays inward, but

when the object is very close to it, as in g/1, the rays coming from a given point on the object are too strongly diverging (spreading) for the mirror to bring them back together. On reflection, the rays are still diverging, just not as strongly diverging. But when the object is sufficiently far away, g/2, the mirror is only intercepting the rays that came out in a narrow cone, and it is able to bend these enough so that they will reconverge.

Note that the rays shown in the figure, which both originated at the same point on the object, reunite when they cross. The point where they cross is the image of the point on the original object. This type of image is called a *real image*, in contradistinction to the virtual images we've studied before. The use of the word "real" is perhaps unfortunate. It sounds as though we are saying the image was an actual material object, which of course it is not.

The distinction between a real image and a virtual image is an important one, because a real image can be projected onto a screen or photographic film. If a piece of paper is inserted in figure g/2 at the location of the image, the image will be visible on the paper (provided the object is bright and the room is dark). Your eye uses a lens to make a real image on the retina.

#### *self-check B*

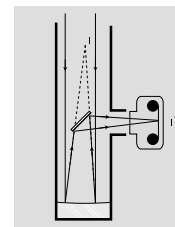
Sketch another copy of the face in figure g/1, even farther from the mirror, and draw a ray diagram. What has happened to the location of the image?

▷ Answer, p. 979

## 29.4 Images of images

If you are wearing glasses right now, then the light rays from the page are being manipulated first by your glasses and then by the lens of your eye. You might think that it would be extremely difficult to analyze this, but in fact it is quite easy. In any series of optical elements (mirrors or lenses or both), each element works on the rays furnished by the previous element in exactly the same manner as if the image formed by the previous element was an actual object.

Figure h shows an example involving only mirrors. The Newtonian telescope, invented by Isaac Newton, consists of a large curved mirror, plus a second, flat mirror that brings the light out of the tube. (In very large telescopes, there may be enough room to put a camera or even a person inside the tube, in which case the second mirror is not needed.) The tube of the telescope is not vital; it is mainly a structural element, although it can also be helpful for blocking out stray light. The lens has been removed from the front of the camera body, and is not needed for this setup. Note that the two sample rays have been drawn parallel, because an astronomical telescope is used for viewing objects that are extremely far away. These two "parallel" lines actually meet at a certain point, say a crater on the



h / A Newtonian telescope being used with a camera.

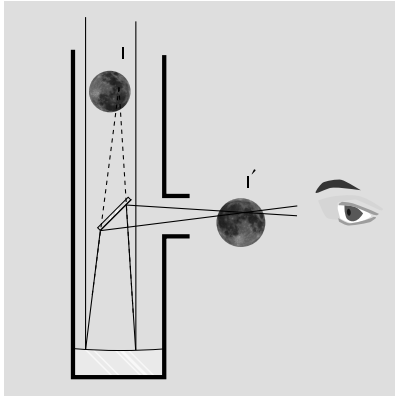


moon, so they can't actually be perfectly parallel, but they are parallel for all practical purposes since we would have to follow them upward for a quarter of a million miles to get to the point where they intersect.

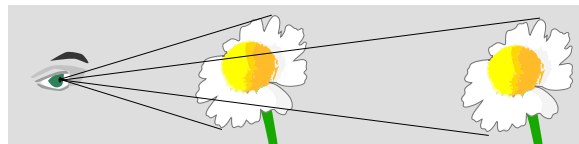
The large curved mirror by itself would form an image  $I$ , but the small flat mirror creates an image of the image,  $I'$ . The relationship between  $I$  and  $I'$  is exactly the same as it would be if  $I$  was an actual object rather than an image:  $I$  and  $I'$  are at equal distances from the plane of the mirror, and the line between them is perpendicular to the plane of the mirror.

One surprising wrinkle is that whereas a flat mirror used by itself forms a virtual image of an object that is real, here the mirror is forming a real image of virtual image  $I$ . This shows how pointless it would be to try to memorize lists of facts about what kinds of images are formed by various optical elements under various circumstances. You are better off simply drawing a ray diagram.

Although the main point here was to give an example of an image of an image, figure i also shows an interesting case where we need to make the distinction between *magnification* and *angular magnification*. If you are looking at the moon through this telescope, then the images  $I$  and  $I'$  are much *smaller* than the actual moon. Otherwise, for example, image  $I$  would not fit inside the telescope! However, these images are very close to your eye compared to the actual moon. The small size of the image has been more than compensated for by the shorter distance. The important thing here is the amount of *angle* within your field of view that the image covers, and it is this angle that has been increased. The factor by which it is increased is called the *angular magnification*,  $M_a$ .



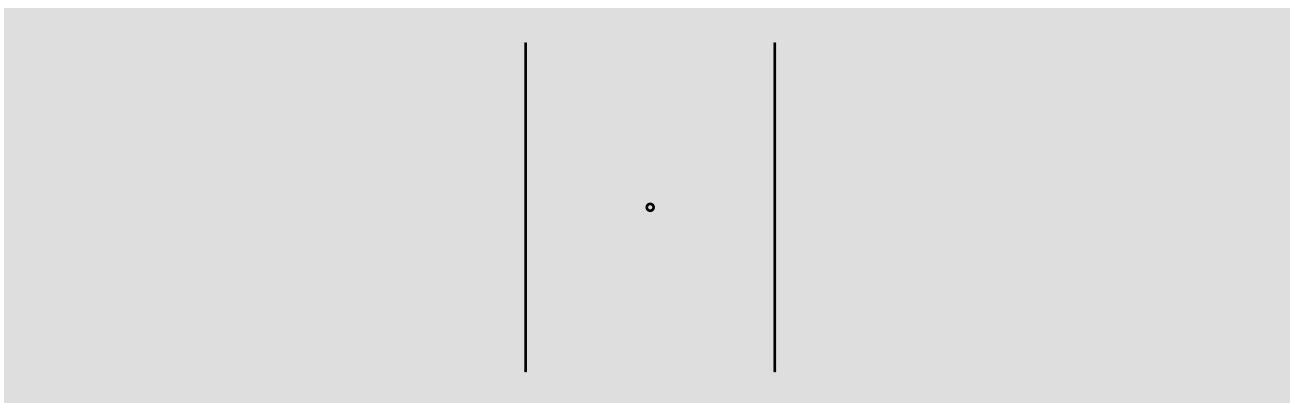
i / A Newtonian telescope being used for visual rather than photographic observing. In real life, an eyepiece lens is normally used for additional magnification, but this simpler setup will also work.



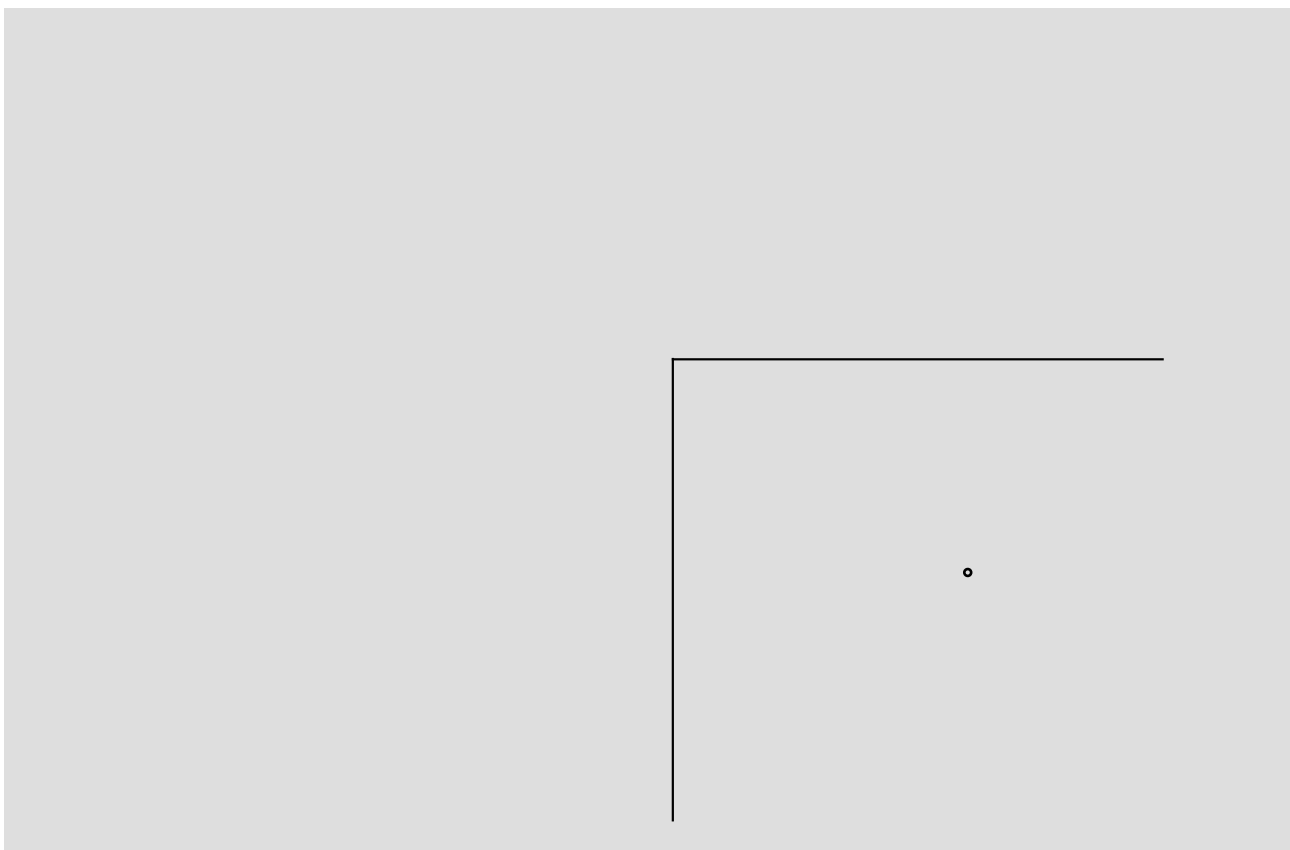
j / The angular size of the flower depends on its distance from the eye.

### Discussion questions

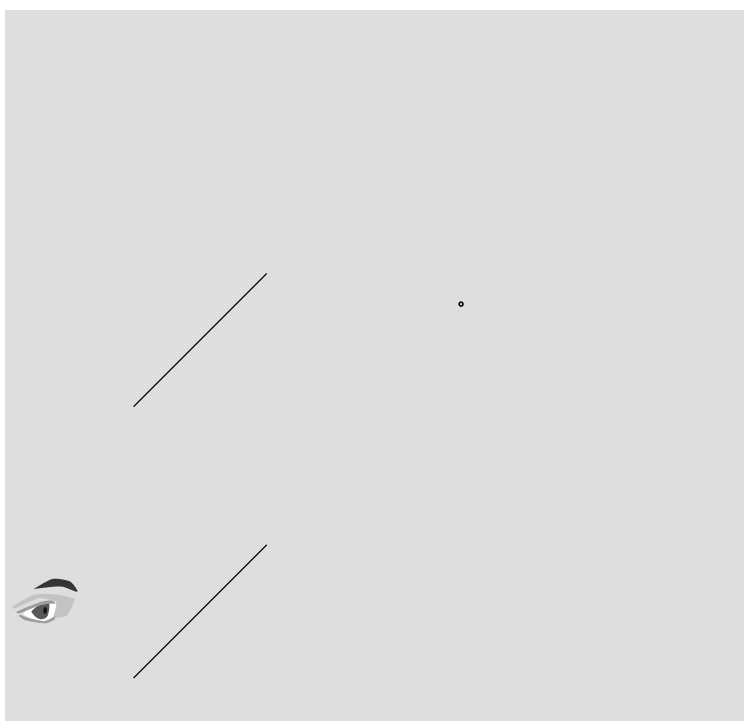
**A** Locate the images of you that will be formed if you stand between two parallel mirrors.



**B** Locate the images formed by two perpendicular mirrors, as in the figure. What happens if the mirrors are not perfectly perpendicular?



**C** Locate the images formed by the periscope.



## Summary

### Selected vocabulary

real image . . . .	a place where an object appears to be, because the rays diffusely reflected from any given point on the object have been bent so that they come back together and then spread out again from the new point
virtual image . .	like a real image, but the rays don't actually cross again; they only appear to have come from the point on the image
converging . . . .	describes an optical device that brings light rays closer to the optical axis
diverging . . . . .	bends light rays farther from the optical axis
magnification . .	the factor by which an image's linear size is increased (or decreased)
angular magnification . . . . .	the factor by which an image's apparent angular size is increased (or decreased)
concave . . . . .	describes a surface that is hollowed out like a cave
convex . . . . .	describes a surface that bulges outward

### Notation

$M$ . . . . .	the magnification of an image
$M_a$ . . . . .	the angular magnification of an image

### Summary

A large class of optical devices, including lenses and flat and curved mirrors, operates by bending light rays to form an image. A real image is one for which the rays actually cross at each point of the image. A virtual image, such as the one formed behind a flat mirror, is one for which the rays only appear to have crossed at a point on the image. A real image can be projected onto a screen; a virtual one cannot.

Mirrors and lenses will generally make an image that is either smaller than or larger than the original object. The scaling factor is called the magnification. In many situations, the angular magnification is more important than the actual magnification.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** A man is walking at 1.0 m/s directly towards a flat mirror. At what speed is his separation from his image decreasing? ✓

**2** If a mirror on a wall is only big enough for you to see yourself from your head down to your waist, can you see your entire body by backing up? Test this experimentally and come up with an explanation for your observations, including a ray diagram.

Note that when you do the experiment, it's easy to confuse yourself if the mirror is even a tiny bit off of vertical. One way to check yourself is to artificially lower the top of the mirror by putting a piece of tape or a post-it note where it blocks your view of the top of your head. You can then check whether you are able to see more of yourself both above *and* below by backing up.

**3** In this chapter we've only done examples of mirrors with hollowed-out shapes (called concave mirrors). Now draw a ray diagram for a curved mirror that has a bulging outward shape (called a convex mirror). (a) How does the image's distance from the mirror compare with the actual object's distance from the mirror? From this comparison, determine whether the magnification is greater than or less than one. (b) Is the image real, or virtual? Could this mirror ever make the other type of image?

**4** As discussed in question 3, there are two types of curved mirrors, concave and convex. Make a list of all the possible combinations of types of images (virtual or real) with types of mirrors (concave and convex). (Not all of the four combinations are physically possible.) Now for each one, use ray diagrams to determine whether increasing the distance of the object from the mirror leads to an increase or a decrease in the distance of the image from the mirror.

Draw BIG ray diagrams! Each diagram should use up about half a page of paper.

Some tips: To draw a ray diagram, you need two rays. For one of these, pick the ray that comes straight along the mirror's axis, since its reflection is easy to draw. After you draw the two rays and locate the image for the original object position, pick a new object position that results in the same type of image, and start a new ray diagram, in a different color of pen, right on top of the first one. For the two new rays, pick the ones that just happen to hit the mirror at the same two places; this makes it much easier to get the result right without depending on extreme accuracy in your ability to draw the

reflected rays.

**5** If the user of an astronomical telescope moves her head closer to or farther away from the image she is looking at, does the magnification change? Does the angular magnification change? Explain. (For simplicity, assume that no eyepiece is being used.)

**6** In figure g/2 in on page 804, only the image of my forehead was located by drawing rays. Either photocopy the figure or download the book and print out the relevant page. On this copy of the figure, make a new set of rays coming from my chin, and locate its image. To make it easier to judge the angles accurately, draw rays from the chin that happen to hit the mirror at the same points where the two rays from the forehead were shown hitting it. By comparing the locations of the chin's image and the forehead's image, verify that the image is actually upside-down, as shown in the original figure.

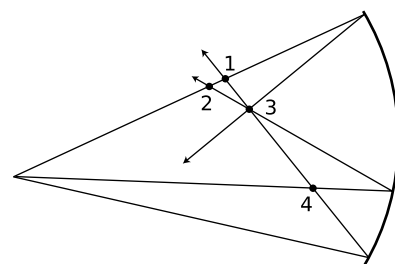
**7** The figure shows four points where rays cross. Of these, which are image points? Explain.

**8** Here's a game my kids like to play. I sit next to a sunny window, and the sun reflects from the glass on my watch, making a disk of light on the wall or floor, which they pretend to chase as I move it around. Is the spot a disk because that's the shape of the sun, or because it's the shape of my watch? In other words, would a square watch make a square spot, or do we just have a circular image of the circular sun, which will be circular no matter what?

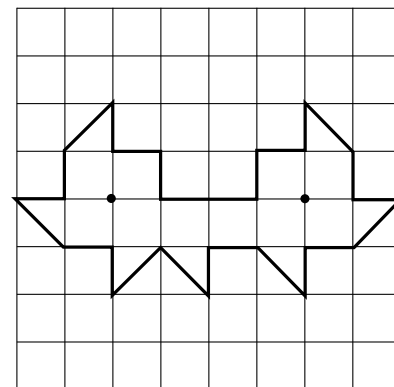
**9** Suppose we have a polygonal room whose walls are mirrors, and there a pointlike light source in the room. In most such examples, every point in the room ends up being illuminated by the light source after some finite number of reflections. A difficult mathematical question, first posed in the middle of the last century, is whether it is ever possible to have an example in which the whole room is *not* illuminated. (Rays are assumed to be absorbed if they strike exactly at a vertex of the polygon, or if they pass exactly through the plane of a mirror.)

The problem was finally solved in 1995 by G.W. Tokarsky, who found an example of a room that was not illuminable from a certain point. Figure 9 shows a slightly simpler example found two years later by D. Castro. If a light source is placed at either of the locations shown with dots, the other dot remains unilluminated, although every other point is lit up. It is not straightforward to prove rigorously that Castro's solution has this property. However, the plausibility of the solution can be demonstrated as follows.

Suppose the light source is placed at the right-hand dot. Locate all the images formed by single reflections. Note that they form a regular pattern. Convince yourself that none of these images illumi-



Problem 7.



Problem 9.

nates the left-hand dot. Because of the regular pattern, it becomes plausible that even if we form images of images, images of images of images, etc., none of them will ever illuminate the other dot.

There are various other versions of the problem, some of which remain unsolved. The book by Klee and Wagon gives a good introduction to the topic, although it predates Tokarsky and Castro's work.

References:

G.W. Tokarsky, "Polygonal Rooms Not Illuminable from Every Point." Amer. Math. Monthly 102, 867-879, 1995.

D. Castro, "Corrections." Quantum 7, 42, Jan. 1997.

V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*. Mathematical Association of America, 1991. ★

## Exercise 29: Exploring images with a curved mirror

Equipment:

- concave mirrors with deep curvature
- concave mirrors with gentle curvature
- convex mirrors

1. Obtain a curved mirror from your instructor. If it is silvered on both sides, make sure you're working with the concave side, which bends light rays inward. Look at your own face in the mirror. Now change the distance between your face and the mirror, and see what happens. Explore the full range of possible distances between your face and the mirror.

In these observations you've been changing two variables at once: the distance between the object (your face) and the mirror, and the distance from the mirror to your eye. In general, scientific experiments become easier to interpret if we practice isolation of variables, i.e., only change one variable while keeping all the others constant. In parts 2 and 3 you'll form an image of an object that's not your face, so that you can have independent control of the object distance and the point of view.

2. With the mirror held far away from you, observe the image of something behind you, over your shoulder. Now bring your eye closer and closer to the mirror. Can you see the image with your eye very close to the mirror? See if you can explain your observation by drawing a ray diagram.

—————> *turn page*



3. Now imagine the following new situation, but *don't actually do it yet*. Suppose you lay the mirror face-up on a piece of tissue paper, put your finger a few cm above the mirror, and look at the image of your finger. As in part 2, you can bring your eye closer and closer to the mirror. Will you be able to see the image with your eye very close to the mirror? Draw a ray diagram to help you predict what you will observe.

Prediction:\_\_\_\_\_

Now test your prediction. If your prediction was incorrect, see if you can figure out what went wrong, or ask your instructor for help.

4. For parts 4 and 5, it's more convenient to use concave mirrors that are more gently curved; obtain one from your instructor. Lay the mirror on the tissue paper, and use it to create an image of the overhead lights on a piece of paper above it and a little off to the side. What do you have to do in order to make the image clear? Can you explain this observation using a ray diagram?

—————> *turn page*

5. Now imagine the following experiment, but *don't do it yet*. What will happen to the image on the paper if you cover half of the mirror with your hand?

Prediction:\_\_\_\_\_

Test your prediction. If your prediction was incorrect, can you explain what happened?

6. Now imagine forming an image with a convex mirror (one that bulges outward), and that therefore bends light rays away from the central axis (i.e., is diverging). Draw a typical ray diagram.

Is the image real, or virtual? Will there be more than one type of image?

Prediction:\_\_\_\_\_

Test your prediction.



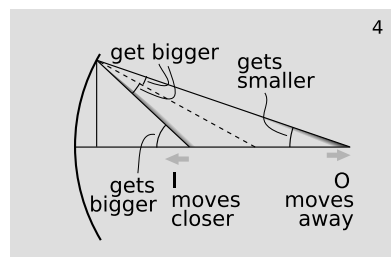
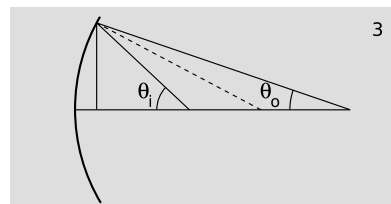
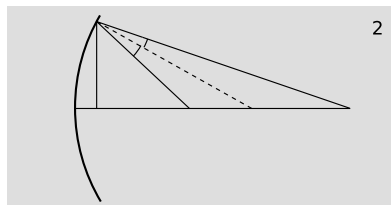
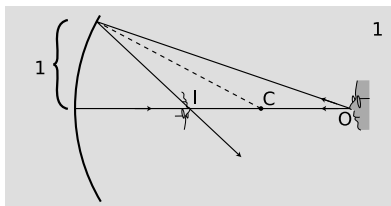


Breakfast Table, by Willem Claesz. de Heda, 17th century. The painting shows a variety of images, some of them distorted, resulting both from reflection and from refraction (ch. 31).

## Chapter 30

# Images, quantitatively

It sounds a bit odd when a scientist refers to a theory as “beautiful,” but to those in the know it makes perfect sense. One mark of a beautiful theory is that it surprises us by being simple. The mathematical theory of lenses and curved mirrors gives us just such a surprise. We expect the subject to be complex because there are so many cases: a converging mirror forming a real image, a diverging lens that makes a virtual image, and so on for a total of six possibilities. If we want to predict the location of the images in all these situations, we might expect to need six different equations, and six more for predicting magnifications. Instead, it turns out that we can use just one equation for the location of the image and one equation for its magnification, and these two equations work in all the different cases with no changes except for plus and minus signs. This is the kind of thing the physicist Eugene Wigner referred to as “the unreasonable effectiveness of mathematics.” Sometimes



a / The relationship between the object's position and the image's can be expressed in terms of the angles  $\theta_o$  and  $\theta_i$ .

we can find a deeper reason for this kind of unexpected simplicity, but sometimes it almost seems as if God went out of Her way to make the secrets of universe susceptible to attack by the human thought-tool called math.

## 30.1 A real image formed by a converging mirror

### Location of the image

We will now derive the equation for the location of a real image formed by a converging mirror. We assume for simplicity that the mirror is spherical, but actually this isn't a restrictive assumption, because any shallow, symmetric curve can be approximated by a sphere. The shape of the mirror can be specified by giving the location of its center,  $C$ . A deeply curved mirror is a sphere with a small radius, so  $C$  is close to it, while a weakly curved mirror has  $C$  farther away. Given the point  $O$  where the object is, we wish to find the point  $I$  where the image will be formed.

To locate an image, we need to track a minimum of two rays coming from the same point. Since we have proved in the previous chapter that this type of image is not distorted, we can use an on-axis point,  $O$ , on the object, as in figure a/1. The results we derive will also hold for off-axis points, since otherwise the image would have to be distorted, which we know is not true. We let one of the rays be the one that is emitted along the axis; this ray is especially easy to trace, because it bounces straight back along the axis again. As our second ray, we choose one that strikes the mirror at a distance of 1 from the axis. "One what?" asks the astute reader. The answer is that it doesn't really matter. When a mirror has shallow curvature, all the reflected rays hit the same point, so 1 could be expressed in any units you like. It could, for instance, be 1 cm, unless your mirror is smaller than 1 cm!

The only way to find out anything mathematical about the rays is to use the sole mathematical fact we possess concerning specular reflection: the incident and reflected rays form equal angles with respect to the normal, which is shown as a dashed line. Therefore the two angles shown in figure a/2 are the same, and skipping some straightforward geometry, this leads to the visually reasonable result that the two angles in figure a/3 are related as follows:

$$\theta_i + \theta_o = \text{constant}$$

(Note that  $\theta_i$  and  $\theta_o$ , which are measured from the image and the object, not from the eye like the angles we referred to in discussing angular magnification on page 806.) For example, move  $O$  farther from the mirror. The top angle in figure a/2 is increased, so the bottom angle must increase by the same amount, causing the image

point, I, to move closer to the mirror. In terms of the angles shown in figure a/3, the more distant object has resulted in a smaller angle  $\theta_o$ , while the closer image corresponds to a larger  $\theta_i$ ; One angle increases by the same amount that the other decreases, so their sum remains constant. These changes are summarized in figure a/4.

The sum  $\theta_i + \theta_o$  is a constant. What does this constant represent? Geometrically, we interpret it as double the angle made by the dashed radius line. Optically, it is a measure of the strength of the mirror, i.e., how strongly the mirror focuses light, and so we call it the focal angle,  $\theta_f$ ,

$$\theta_i + \theta_o = \theta_f \quad .$$

Suppose, for example, that we wish to use a quick and dirty optical test to determine how strong a particular mirror is. We can lay it on the floor as shown in figure c, and use it to make an image of a lamp mounted on the ceiling overhead, which we assume is very far away compared to the radius of curvature of the mirror, so that the mirror intercepts only a very narrow cone of rays from the lamp. This cone is so narrow that its rays are nearly parallel, and  $\theta_o$  is nearly zero. The real image can be observed on a piece of paper. By moving the paper nearer and farther, we can bring the image into focus, at which point we know the paper is located at the image point. Since  $\theta_o \approx 0$ , we have  $\theta_i \approx \theta_f$ , and we can then determine this mirror's focal angle either by measuring  $\theta_i$  directly with a protractor, or indirectly via trigonometry. A strong mirror will bring the rays together to form an image close to the mirror, and these rays will form a blunt-angled cone with a large  $\theta_i$  and  $\theta_f$ .

#### *An alternative optical test*

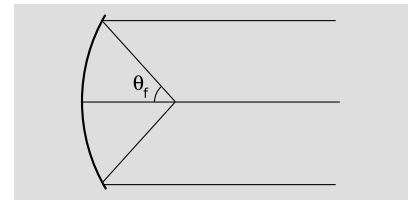
#### *example 1*

▷ Figure c shows an alternative optical test. Rather than placing the object at infinity as in figure b, we adjust it so that the image is right on top of the object. Points O and I coincide, and the rays are reflected right back on top of themselves. If we measure the angle  $\theta$  shown in figure c, how can we find the focal angle?

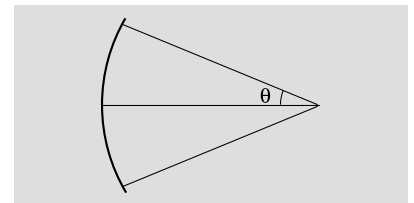
▷ The object and image angles are the same; the angle labeled  $\theta$  in the figure equals both of them. We therefore have  $\theta_i + \theta_o = \theta = \theta_f$ . Comparing figures b and c, it is indeed plausible that the angles are related by a factor of two.

At this point, we could consider our work to be done. Typically, we know the strength of the mirror, and we want to find the image location for a given object location. Given the mirror's focal angle and the object location, we can determine  $\theta_o$  by trigonometry, subtract to find  $\theta_i = \theta_f - \theta_o$ , and then do more trig to find the image location.

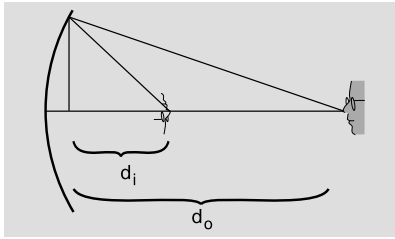
There is, however, a shortcut that can save us from doing so much work. Figure a/3 shows two right triangles whose legs of length 1 coincide and whose acute angles are  $\theta_o$  and  $\theta_i$ . These can be related



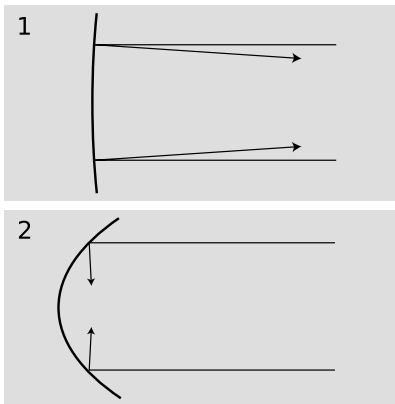
b / The geometrical interpretation of the focal angle.



c / Example 1, an alternative test for finding the focal angle. The mirror is the same as in figure b.



d / The object and image distances



e / Mirror 1 is weaker than mirror 2. It has a shallower curvature, a longer focal length, and a smaller focal angle. It reflects rays at angles not much different than those that would be produced with a flat mirror.

by trigonometry to the object and image distances shown in figure d:

$$\tan \theta_o = 1/d_o \quad \tan \theta_i = 1/d_i$$

Ever since chapter 2, we've been assuming small angles. For small angles, we can use the small-angle approximation  $\tan x \approx x$  (for  $x$  in radians), giving simply

$$\theta_o = 1/d_o \quad \theta_i = 1/d_i \quad .$$

We likewise define a distance called the focal length,  $f$  according to  $\theta_f = 1/f$ . In figure b,  $f$  is the distance from the mirror to the place where the rays cross. We can now reexpress the equation relating the object and image positions as

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o} \quad .$$

Figure e summarizes the interpretation of the focal length and focal angle.<sup>1</sup>

Which form is better,  $\theta_f = \theta_i + \theta_o$  or  $1/f = 1/d_i + 1/d_o$ ? The angular form has in its favor its simplicity and its straightforward visual interpretation, but there are two reasons why we might prefer the second version. First, the numerical values of the angles depend on what we mean by “one unit” for the distance shown as 1 in figure a/1. Second, it is usually easier to measure distances rather than angles, so the distance form is more convenient for number crunching. Neither form is superior overall, and we will often need to use both to solve any given problem.<sup>2</sup>

### A searchlight

### example 2

Suppose we need to create a parallel beam of light, as in a searchlight. Where should we place the lightbulb? A parallel beam has zero angle between its rays, so  $\theta_i = 0$ . To place the lightbulb correctly, however, we need to know a distance, not an angle: the distance  $d_o$  between the bulb and the mirror. The problem involves a mixture of distances and angles, so we need to get everything in terms of one or the other in order to solve it. Since

<sup>1</sup>There is a standard piece of terminology which is that the “focal point” is the point lying on the optical axis at a distance from the mirror equal to the focal length. This term isn't particularly helpful, because it names a location where nothing normally happens. In particular, it is *not* normally the place where the rays come to a focus! — that would be the *image* point. In other words, we don't normally have  $d_i = f$ , unless perhaps  $d_o = \infty$ . A recent online discussion among some physics teachers (<https://carnot.physics.buffalo.edu/archives>, Feb. 2006) showed that many disliked the terminology, felt it was misleading, or didn't know it and would have misinterpreted it if they had come across it. That is, it appears to be what grammarians call a “skunked term” — a word that bothers half the population when it's used incorrectly, and the other half when it's used correctly.

<sup>2</sup>I would like to thank Fouad Ajami for pointing out the pedagogical advantages of using both equations side by side.

the goal is to find a distance, let's figure out the image distance corresponding to the given angle  $\theta_i = 0$ . These are related by  $d_i = 1/\theta_i$ , so we have  $d_i = \infty$ . (Yes, dividing by zero gives infinity. Don't be afraid of infinity. Infinity is a useful problem-solving device.) Solving the distance equation for  $d_o$ , we have

$$\begin{aligned}d_o &= (1/f - 1/d_i)^{-1} \\&= (1/f - 0)^{-1} \\&= f\end{aligned}$$

The bulb has to be placed at a distance from the mirror equal to its focal point.

### *Diopters*

### *example 3*

An equation like  $d_i = 1/\theta_i$  really doesn't make sense in terms of units. Angles are unitless, since radians aren't really units, so the right-hand side is unitless. We can't have a left-hand side with units of distance if the right-hand side of the same equation is unitless. This is an artifact of my cavalier statement that the conical bundles of rays spread out to a distance of 1 from the axis where they strike the mirror, without specifying the units used to measure this 1. In real life, optometrists define the thing we're calling  $\theta_i = 1/d_i$  as the "dioptric strength" of a lens or mirror, and measure it in units of inverse meters ( $\text{m}^{-1}$ ), also known as diopters ( $1 \text{ D} = 1 \text{ m}^{-1}$ ).

## **Magnification**

We have already discussed in the previous chapter how to find the magnification of a virtual image made by a curved mirror. The result is the same for a real image, and we omit the proof, which is very similar. In our new notation, the result is  $M = d_i/d_o$ . A numerical example is given in section 30.2.

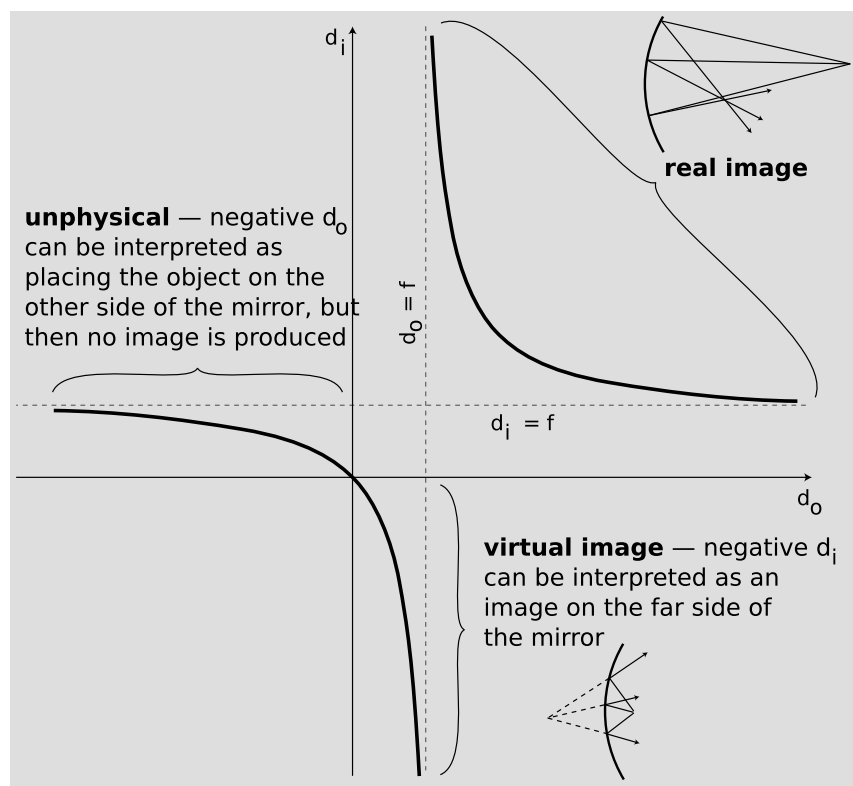
## **30.2 Other cases with curved mirrors**

The equation  $d_i = (1/f - 1/d_o)^{-1}$  can easily produce a negative result, but we have been thinking of  $d_i$  as a distance, and distances can't be negative. A similar problem occurs with  $\theta_i = \theta_f - \theta_o$  for  $\theta_o > \theta_f$ . What's going on here?

The interpretation of the angular equation is straightforward. As we bring the object closer and closer to the image,  $\theta_o$  gets bigger and bigger, and eventually we reach a point where  $\theta_o = \theta_f$  and  $\theta_i = 0$ . This large object angle represents a bundle of rays forming a cone that is very broad, so broad that the mirror can no longer bend them back so that they reconverge on the axis. The image angle  $\theta_i = 0$  represents an outgoing bundle of rays that are parallel. The outgoing rays never cross, so this is not a real image, unless we want to be charitable and say that the rays cross at infinity. If we go on bringing the object even closer, we get a virtual image.



$f$  / A graph of the image distance  $d_i$  as a function of the object distance  $d_o$ .



To analyze the distance equation, let's look at a graph of  $d_i$  as a function of  $d_o$ . The branch on the upper right corresponds to the case of a real image. Strictly speaking, this is the only part of the graph that we've proven corresponds to reality, since we never did any geometry for other cases, such as virtual images. As discussed in the previous section, making  $d_o$  bigger causes  $d_i$  to become smaller, and vice-versa.

Letting  $d_o$  be less than  $f$  is equivalent to  $\theta_o > \theta_f$ : a virtual image is produced on the far side of the mirror. This is the first example of Wigner's "unreasonable effectiveness of mathematics" that we have encountered in optics. Even though our proof depended on the assumption that the image was real, the equation we derived turns out to be applicable to virtual images, provided that we either interpret the positive and negative signs in a certain way, or else modify the equation to have different positive and negative signs.

#### self-check A

Interpret the three places where, in physically realistic parts of the graph, the graph approaches one of the dashed lines. [This will come more naturally if you have learned the concept of limits in a math class.] >

Answer, p. 979

*A flat mirror*

*example 4*

We can even apply the equation to a flat mirror. As a sphere gets

bigger and bigger, its surface is more and more gently curved. The planet Earth is so large, for example, that we cannot even perceive the curvature of its surface. To represent a flat mirror, we let the mirror's radius of curvature, and its focal length, become infinite. Dividing by infinity gives zero, so we have

$$1/d_o = -1/d_i \quad ,$$

or

$$d_o = -d_i \quad .$$

If we interpret the minus sign as indicating a virtual image on the far side of the mirror from the object, this makes sense.

It turns out that for any of the six possible combinations of real or virtual images formed by converging or diverging lenses or mirrors, we can apply equations of the form

$$\theta_f = \theta_i + \theta_o$$

and

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o} \quad ,$$

with only a modification of plus or minus signs. There are two possible approaches here. The approach we have been using so far is the more popular approach in American textbooks: leave the equation the same, but attach interpretations to the resulting negative or positive values of the variables. The trouble with this approach is that one is then forced to memorize tables of sign conventions, e.g., that the value of  $d_i$  should be negative when the image is a virtual image formed by a converging mirror. Positive and negative signs also have to be memorized for focal lengths. Ugh! It's highly unlikely that any student has ever retained these lengthy tables in his or her mind for more than five minutes after handing in the final exam in a physics course. Of course one can always look such things up when they are needed, but the effect is to turn the whole thing into an exercise in blindly plugging numbers into formulas.

As you have gathered by now, there is another method which I think is better, and which I'll use throughout the rest of this book. In this method, all distances and angles are *positive by definition*, and we put in positive and negative signs in the *equations* depending on the situation. (I thought I was the first to invent this method, but I've been told that this is known as the European sign convention, and that it's fairly common in Europe.) Rather than memorizing these signs, we start with the generic equations

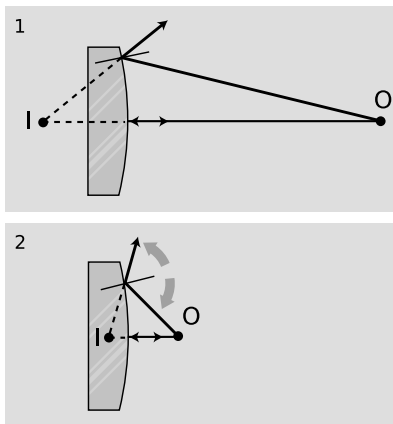
$$\begin{aligned} \theta_f &= \pm\theta_i \pm \theta_o \\ \frac{1}{f} &= \pm\frac{1}{d_i} \pm \frac{1}{d_o} \quad , \end{aligned}$$

and then determine the signs by a two-step method that depends on ray diagrams. There are really only two signs to determine, not four; the signs in the two equations match up in the way you'd expect. The method is as follows:

1. Use ray diagrams to decide whether  $\theta_o$  and  $\theta_i$  vary in the same way or in opposite ways. (In other words, decide whether making  $\theta_o$  greater results in a greater value of  $\theta_i$  or a smaller one.) Based on this, decide whether the two signs in the angle equation are the same or opposite. If the signs are opposite, go on to step 2 to determine which is positive and which is negative.
2. If the signs are opposite, we need to decide which is the positive one and which is the negative. Since the focal angle is never negative, the smaller angle must be the one with a minus sign.

In step 1, many students have trouble drawing the ray diagram correctly. For simplicity, you should always do your diagram for a point on the object that is on the axis of the mirror, and let one of your rays be the one that is emitted along the axis and reflected straight back on itself, as in the figures in section 30.1. As shown in figure a/4 in section 30.1, there are four angles involved: two at the mirror, one at the object ( $\theta_o$ ), and one at the image ( $\theta_i$ ). Make sure to draw in the normal to the mirror so that you can see the two angles at the mirror. These two angles are equal, so as you change the object position, they fan out or fan in, like opening or closing a book. Once you've drawn this effect, you should easily be able to tell whether  $\theta_o$  and  $\theta_i$  change in the same way or in opposite ways.

Although focal lengths are always positive in the method used in this book, you should be aware that diverging mirrors and lenses are assigned negative focal lengths in the other method, so if you see a lens labeled  $f = -30$  cm, you'll know what it means.



g / Example 5.

#### *An anti-shoplifting mirror*

*example 5*

▷ Convenience stores often install a diverging mirror so that the clerk has a view of the whole store and can catch shoplifters. Use a ray diagram to show that the image is reduced, bringing more into the clerk's field of view. If the focal length of the mirror is 3.0 m, and the mirror is 7.0 m from the farthest wall, how deep is the image of the store?

▷ As shown in ray diagram g/1,  $d_i$  is less than  $d_o$ . The magnification,  $M = d_i/d_o$ , will be less than one, i.e., the image is actually reduced rather than magnified.

Apply the method outlined above for determining the plus and minus signs. Step 1: The object is the point on the opposite wall. As an experiment, g/2, move the object closer. I did these drawings using illustration software, but if you were doing them by hand, you'd want to make the scale much larger for greater accuracy. Also, although I split figure g into two separate drawings

in order to make them easier to understand, you're less likely to make a mistake if you do them on top of each other.

The two angles at the mirror fan out from the normal. Increasing  $\theta_o$  has clearly made  $\theta_i$  larger as well. (All four angles got bigger.) There must be a cancellation of the effects of changing the two terms on the right in the same way, and the only way to get such a cancellation is if the two terms in the angle equation have opposite signs:

$$\theta_f = +\theta_i - \theta_o$$

or

$$\theta_f = -\theta_i + \theta_o \quad .$$

Step 2: Now which is the positive term and which is negative? Since the image angle is bigger than the object angle, the angle equation must be

$$\theta_f = \theta_i - \theta_o \quad ,$$

in order to give a positive result for the focal angle. The signs of the distance equation behave the same way:

$$\frac{1}{f} = \frac{1}{d_i} - \frac{1}{d_o} \quad .$$

Solving for  $d_i$ , we find

$$\begin{aligned} d_i &= \left( \frac{1}{f} + \frac{1}{d_o} \right)^{-1} \\ &= 2.1 \text{ m} \quad . \end{aligned}$$

The image of the store is reduced by a factor of  $2.1/7.0 = 0.3$ , i.e., it is smaller by 70%.

#### *A shortcut for real images*

#### *example 6*

In the case of a real image, there is a shortcut for step 1, the determination of the signs. In a real image, the rays cross at both the object and the image. We can therefore time-reverse the ray diagram, so that all the rays are coming from the image and reconverging at the object. Object and image swap roles. Due to this time-reversal symmetry, the object and image cannot be treated differently in any of the equations, and they must therefore have the same signs. They are both positive, since they must add up to a positive result.

### 30.3 ★ Aberrations

An imperfection or distortion in an image is called an aberration. An aberration can be produced by a flaw in a lens or mirror, but even with a perfect optical surface some degree of aberration is unavoidable. To see why, consider the mathematical approximation

h / A diverging mirror in the shape of a sphere. The image is reduced ( $M < 1$ ). This is similar to example 5, but here the image is distorted because the mirror's curve is not shallow.



we've been making, which is that the depth of the mirror's curve is small compared to  $d_o$  and  $d_i$ . Since only a flat mirror can satisfy this shallow-mirror condition perfectly, any curved mirror will deviate somewhat from the mathematical behavior we derived by assuming that condition. There are two main types of aberration in curved mirrors, and these also occur with lenses.

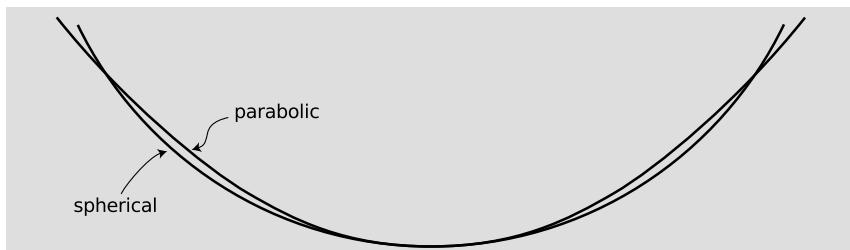
(1) An object on the axis of the lens or mirror may be imaged correctly, but off-axis objects may be out of focus or distorted. In a camera, this type of aberration would show up as a fuzziness or warping near the sides of the picture when the center was perfectly focused. An example of this is shown in figure i, and in that particular example, the aberration is not a sign that the equipment was of low quality or wasn't right for the job but rather an inevitable result of trying to flatten a panoramic view; in the limit of a 360-degree panorama, the problem would be similar to the problem of representing the Earth's surface on a flat map, which can't be accomplished without distortion.

(2) The image may be sharp when the object is at certain distances and blurry when it is at other distances. The blurriness occurs because the rays do not all cross at exactly the same point. If we know in advance the distance of the objects with which the mirror or lens will be used, then we can optimize the shape of the optical surface to make in-focus images in that situation. For instance, a spherical mirror will produce a perfect image of an object that is at the center of the sphere, because each ray is reflected directly onto the radius along which it was emitted. For objects at greater distances, however, the focus will be somewhat blurry. In astronomy the objects being used are always at infinity, so a spherical mirror



i / This photo was taken using a “fish-eye lens,” which gives an extremely large field of view.

is a poor choice for a telescope. A different shape (a parabola) is better specialized for astronomy.



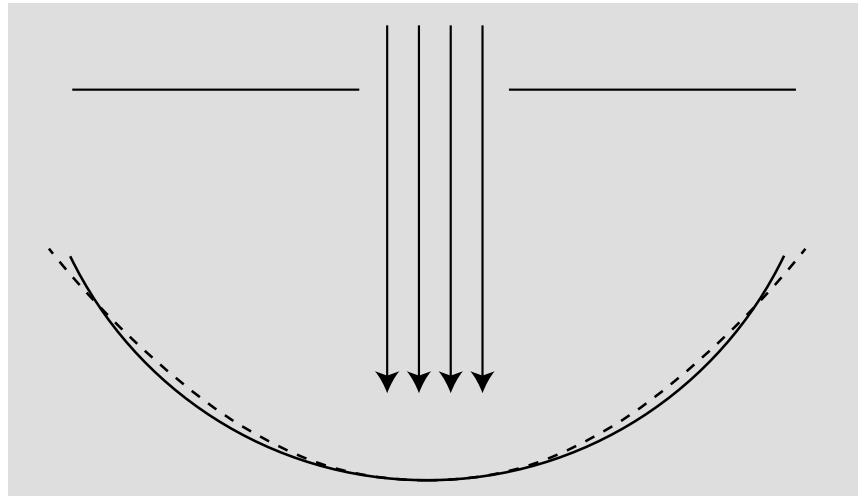
j / Spherical mirrors are the cheapest to make, but parabolic mirrors are better for making images of objects at infinity. A sphere has equal curvature everywhere, but a parabola has tighter curvature at its center and gentler curvature at the sides.

One way of decreasing aberration is to use a small-diameter mirror or lens, or block most of the light with an opaque screen with a hole in it, so that only light that comes in close to the axis can get through. Either way, we are using a smaller portion of the lens or mirror whose curvature will be more shallow, thereby making the shallow-mirror (or thin-lens) approximation more accurate. Your eye does this by narrowing down the pupil to a smaller hole. In a camera, there is either an automatic or manual adjustment, and narrowing the opening is called “stopping down.” The disadvantage of stopping down is that light is wasted, so the image will be dimmer



or a longer exposure must be used.

k / Even though the spherical mirror (solid line) is not well adapted for viewing an object at infinity, we can improve its performance greatly by stopping it down. Now the only part of the mirror being used is the central portion, where its shape is virtually indistinguishable from a parabola (dashed line).



What I would suggest you take away from this discussion for the sake of your general scientific education is simply an understanding of what an aberration is, why it occurs, and how it can be reduced, not detailed facts about specific types of aberrations.

l / The Hubble Space Telescope was placed into orbit with faulty optics in 1990. Its main mirror was supposed to have been nearly parabolic, since it is an astronomical telescope, meant for producing images of objects at infinity. However, contractor Perkin Elmer had delivered a faulty mirror, which produced aberrations. The large photo shows astronauts putting correcting mirrors in place in 1993. The two small photos show images produced by the telescope before and after the fix.



## Summary

### Selected vocabulary

focal length . . . a property of a lens or mirror, equal to the distance from the lens or mirror to the image it forms of an object that is infinitely far away

### Notation

$f$  . . . . . the focal length  
 $d_o$  . . . . . the distance of the object from the mirror  
 $d_i$  . . . . . the distance of the image from the mirror  
 $\theta_f$  . . . . . the focal angle, defined as  $1/f$   
 $\theta_o$  . . . . . the object angle, defined as  $1/d_o$   
 $\theta_i$  . . . . . the image angle, defined as  $1/d_i$

### Other terminology and notation

$f > 0$  . . . . . describes a converging lens or mirror; in this book, all focal lengths are positive, so there is no such implication  
 $f < 0$  . . . . . describes a diverging lens or mirror; in this book, all focal lengths are positive  
 $M < 0$  . . . . . indicates an inverted image; in this book, all magnifications are positive

## Summary

Every lens or mirror has a property called the focal length, which is defined as the distance from the lens or mirror to the image it forms of an object that is infinitely far away. A stronger lens or mirror has a shorter focal length.

The relationship between the locations of an object and its image formed by a lens or mirror can always be expressed by equations of the form

$$\theta_f = \pm\theta_i \pm \theta_o$$

$$\frac{1}{f} = \pm\frac{1}{d_i} \pm \frac{1}{d_o} \quad .$$

The choice of plus and minus signs depends on whether we are dealing with a lens or a mirror, whether the lens or mirror is converging or diverging, and whether the image is real or virtual. A method for determining the plus and minus signs is as follows:

1. Use ray diagrams to decide whether  $\theta_i$  and  $\theta_o$  vary in the same way or in opposite ways. Based on this, decide whether the two signs in the equation are the same or opposite. If the signs are opposite, go on to step 2 to determine which is positive and which is negative.
2. If the signs are opposite, we need to decide which is the positive one and which is the negative. Since the focal angle is never negative, the smaller angle must be the one with a minus sign.



Once the correct form of the equation has been determined, the magnification can be found via the equation

$$M = \frac{d_i}{d_o} \quad .$$

This equation expresses the idea that the entire image-world is shrunk consistently in all three dimensions.

## Problems

### Key

✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

1 Apply the equation  $M = d_i/d_o$  to the case of a flat mirror.

2 Use the method described in the text to derive the equation relating object distance to image distance for the case of a virtual image produced by a converging mirror. ▷ Solution, p. 976

3 (a) Make up a numerical example of a virtual image formed by a converging mirror with a certain focal length, and determine the magnification. (You will need the result of problem 2.) Make sure to choose values of  $d_o$  and  $f$  that would actually produce a virtual image, not a real one. Now change the location of the object *a little bit* and redetermine the magnification, showing that it changes. At my local department store, the cosmetics department sells mirrors advertised as giving a magnification of 5 times. How would you interpret this?

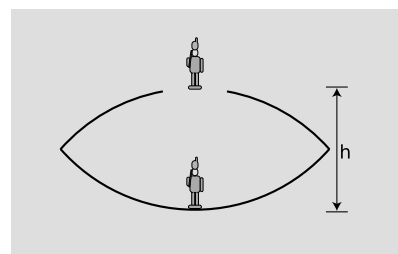
(b) Suppose a Newtonian telescope is being used for astronomical observing. Assume for simplicity that no eyepiece is used, and assume a value for the focal length of the mirror that would be reasonable for an amateur instrument that is to fit in a closet. Is the angular magnification different for objects at different distances? For example, you could consider two planets, one of which is twice as far as the other.

4 (a) Find a case where the magnification of a curved mirror is infinite. Is the *angular* magnification infinite from any realistic viewing position? (b) Explain why an arbitrarily large magnification can't be achieved by having a sufficiently small value of  $d_o$ .

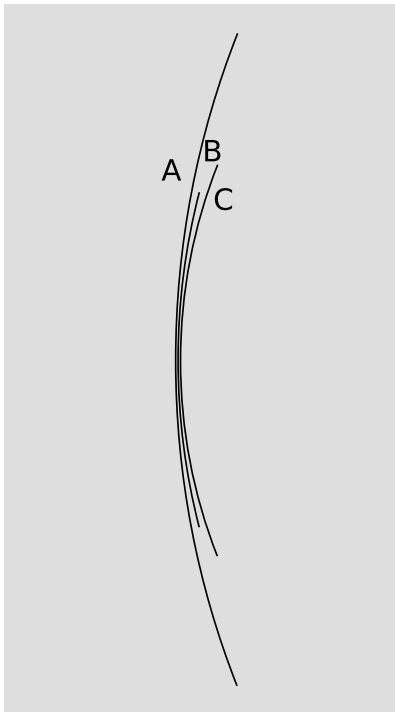
5 The figure shows a device for constructing a realistic optical illusion. Two mirrors of equal focal length are put against each other with their silvered surfaces facing inward. A small object placed in the bottom of the cavity will have its image projected in the air above. The way it works is that the top mirror produces a virtual image, and the bottom mirror then creates a real image of the virtual image. (a) Show that if the image is to be positioned as shown, at the mouth of the cavity, then the focal length of the mirrors is related to the dimension  $h$  via the equation

$$\frac{1}{f} = \frac{1}{h} + \frac{1}{h + \left(\frac{1}{h} - \frac{1}{f}\right)^{-1}}.$$

(b) Restate the equation in terms of a single variable  $x = h/f$ , and show that there are two solutions for  $x$ . Which solution is physically consistent with the assumptions of the calculation? ★



Problem 5.



Problem 8.

**6** A concave surface that reflects sound waves can act just like a converging mirror. Suppose that, standing near such a surface, you are able to find a point where you can place your head so that your own whispers are focused back on your head, so that they sound loud to you. Given your distance to the surface, what is the surface's focal length?

**7** Find the focal length of the mirror in problem 5 in chapter 28. ✓

**8** Rank the focal lengths of the mirrors, from shortest to longest.

**9** (a) A converging mirror is being used to create a virtual image. What is the range of possible magnifications? (b) Do the same for the other types of images that can be formed by curved mirrors (both converging and diverging).

**10** (a) A converging mirror with a focal length of 20 cm is used to create an image, using an object at a distance of 10 cm. Is the image real, or is it virtual? (b) How about  $f = 20$  cm and  $d_o = 30$  cm? (c) What if it was a *diverging* mirror with  $f = 20$  cm and  $d_o = 10$  cm? (d) A diverging mirror with  $f = 20$  cm and  $d_o = 30$  cm?  
▷ Solution, p. 976

**11** A diverging mirror of focal length  $f$  is fixed, and faces down. An object is dropped from the surface of the mirror, and falls away from it with acceleration  $g$ . The goal of the problem is to find the maximum velocity of the image.

(a) Describe the motion of the image verbally, and explain why we should expect there to be a maximum velocity.

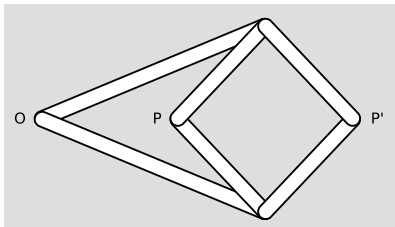
(b) Use arguments based on units to determine the form of the solution, up to an unknown unitless multiplicative constant.

(c) Complete the solution by determining the unitless constant.

∫

**12** A mechanical linkage is a device that changes one type of motion into another. The most familiar example occurs in a gasoline car's engine, where a connecting rod changes the linear motion of the piston into circular motion of the crankshaft. The top panel of the figure shows a mechanical linkage invented by Peaucellier in 1864, and independently by Lipkin around the same time. It consists of six rods joined by hinges, the four short ones forming a rhombus. Point O is fixed in space, but the apparatus is free to rotate about O. Motion at P is transformed into a different motion at P' (or vice versa).

Geometrically, the linkage is a mechanical implementation of the ancient problem of inversion in a circle. Considering the case in which the rhombus is folded flat, let the  $k$  be the distance from O to the point where P and P' coincide. Form the circle of radius  $k$  with its center at O. As P and P' move in and out, points on the



Problem 12.

inside of the circle are always mapped to points on its outside, such that  $rr' = k^2$ . That is, the linkage is a type of analog computer that exactly solves the problem of finding the inverse of a number  $r$ . Inversion in a circle has many remarkable geometrical properties, discussed in H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1961. If a pen is inserted through a hole at P, and P' is traced over a geometrical figure, the Peaucellier linkage can be used to draw a kind of image of the figure.

A related problem is the construction of pictures, like the one in the bottom panel of the figure, called anamorphs. The drawing of the column on the paper is highly distorted, but when the reflecting cylinder is placed in the correct spot on top of the page, an undistorted image is produced inside the cylinder. (Wide-format movie technologies such as Cinemascope are based on similar principles.)

Show that the Peaucellier linkage does *not* convert correctly between an image and its anamorph, and design a modified version of the linkage that does. Some knowledge of analytic geometry will be helpful. ★

## Exercise 30: Object and image distances

Equipment:

optical benches

converging mirrors

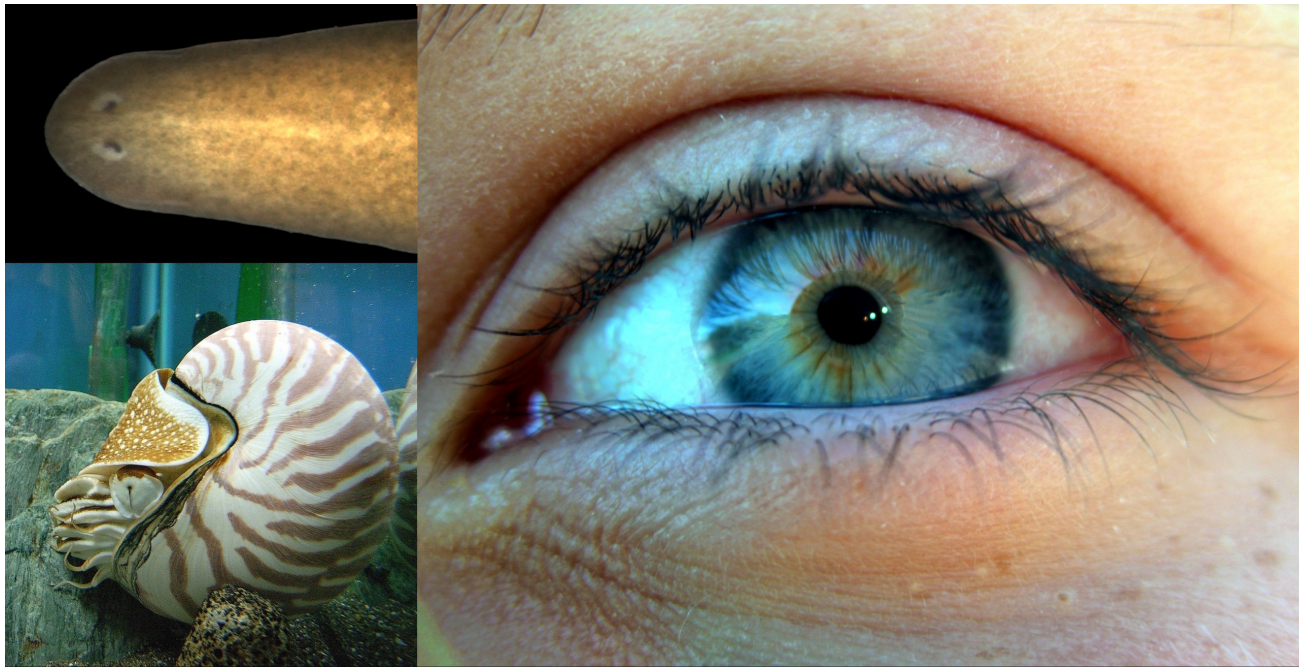
illuminated objects

1. Set up the optical bench with the mirror at zero on the centimeter scale. Set up the illuminated object on the bench as well.
2. Each group will locate the image for their own value of the object distance, by finding where a piece of paper has to be placed in order to see the image on it. (The instructor will do one point as well.) Note that you will have to tilt the mirror a little so that the paper on which you project the image doesn't block the light from the illuminated object.

Is the image real or virtual? How do you know? Is it inverted, or uninverted?

Draw a ray diagram.

3. Measure the image distance and write your result in the table on the board. Do the same for the magnification.
4. What do you notice about the trend of the data on the board? Draw a second ray diagram with a different object distance, and show why this makes sense. Some tips for doing this correctly: (1) For simplicity, use the point on the object that is on the mirror's axis. (2) You need to trace two rays to locate the image. To save work, don't just do two rays at random angles. You can either use the on-axis ray as one ray, or do two rays that come off at the same angle, one above and one below the axis. (3) Where each ray hits the mirror, draw the normal line, and make sure the ray is at equal angles on both sides of the normal.
5. We will find the mirror's focal length from the instructor's data-point. Then, using this focal length, calculate a theoretical prediction of the image distance, and write it on the board next to the experimentally determined image distance.



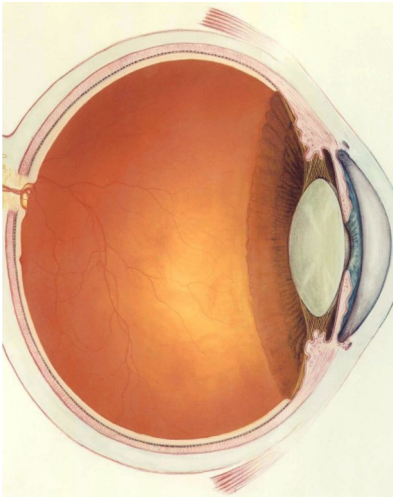
Three stages in the evolution of the eye. The flatworm has two eye pits. The nautilus's eyes are pinhole cameras. The human eye incorporates a lens.

## Chapter 31

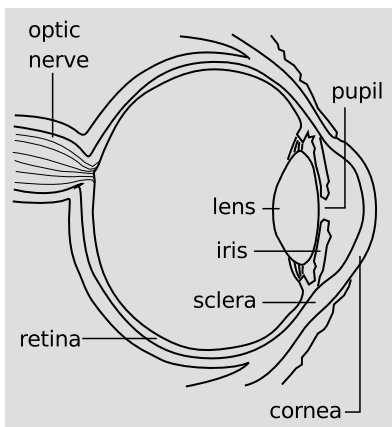
# Refraction

Economists normally consider free markets to be the natural way of judging the monetary value of something, but social scientists also use questionnaires to gauge the relative value of privileges, disadvantages, or possessions that cannot be bought or sold. They ask people to *imagine* that they could trade one thing for another and ask which they would choose. One interesting result is that the average light-skinned person in the U.S. would rather lose an arm than suffer the racist treatment routinely endured by African-Americans. Even more impressive is the value of sight. Many prospective parents can imagine without too much fear having a deaf child, but would have a far more difficult time coping with raising a blind one.

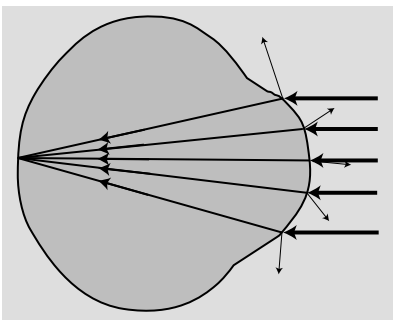
So great is the value attached to sight that some have imbued it with mystical aspects. Moses “had vision,” George Bush did not. Christian fundamentalists who perceive a conflict between evolution and their religion have claimed that the eye is such a perfect device that it could never have arisen through a process as helter-skelter as evolution, or that it could not have evolved because half of an eye would be useless. In fact, the structure of an eye is fundamentally dictated by physics, and it has arisen separately by evolution some-



a / A human eye.



b / The anatomy of the eye.



c / A simplified optical diagram of the eye. Light rays are bent when they cross from the air into the eye. (A little of the incident rays' energy goes into the reflected rays rather than the ones transmitted into the eye.)

where between eight and 40 times, depending on which biologist you ask. We humans have a version of the eye that can be traced back to the evolution of a light-sensitive “eye spot” on the head of an ancient invertebrate. A sunken pit then developed so that the eye would only receive light from one direction, allowing the organism to tell where the light was coming from. (Modern flatworms have this type of eye.) The top of the pit then became partially covered, leaving a hole, for even greater directionality (as in the nautilus). At some point the cavity became filled with jelly, and this jelly finally became a lens, resulting in the general type of eye that we share with the bony fishes and other vertebrates. Far from being a perfect device, the vertebrate eye is marred by a serious design flaw due to the lack of planning or intelligent design in evolution: the nerve cells of the retina and the blood vessels that serve them are all in front of the light-sensitive cells, blocking part of the light. Squids and other molluscs, whose eyes evolved on a separate branch of the evolutionary tree, have a more sensible arrangement, with the light-sensitive cells out in front.

## 31.1 Refraction

### Refraction

The fundamental physical phenomenon at work in the eye is that when light crosses a boundary between two media (such as air and the eye's jelly), part of its energy is reflected, but part passes into the new medium. In the ray model of light, we describe the original ray as splitting into a reflected ray and a transmitted one (the one that gets through the boundary). Of course the reflected ray goes in a direction that is different from that of the original one, according to the rules of reflection we have already studied. More surprisingly — and this is the crucial point for making your eye focus light — the transmitted ray is bent somewhat as well. This bending phenomenon is called *refraction*. The origin of the word is the same as that of the word “fracture,” i.e., the ray is bent or “broken.” (Keep in mind, however, that light rays are not physical objects that can really be “broken.”) Refraction occurs with all waves, not just light waves.

The actual anatomy of the eye, b, is quite complex, but in essence it is very much like every other optical device based on refraction. The rays are bent when they pass through the front surface of the eye, c. Rays that enter farther from the central axis are bent more, with the result that an image is formed on the retina. There is only one slightly novel aspect of the situation. In most human-built optical devices, such as a movie projector, the light is bent as it passes into a lens, bent again as it reemerges, and then reaches a focus beyond the lens. In the eye, however, the “screen” is inside the eye, so the rays are only refracted once, on entering the jelly, and never emerge

again.

A common misconception is that the “lens” of the eye is what does the focusing. All the transparent parts of the eye are made of fairly similar stuff, so the dramatic change in medium is when a ray crosses from the air into the eye (at the outside surface of the cornea). This is where nearly all the refraction takes place. The lens medium differs only slightly in its optical properties from the rest of the eye, so very little refraction occurs as light enters and exits the lens. The lens, whose shape is adjusted by muscles attached to it, is only meant for fine-tuning the focus to form images of near or far objects.

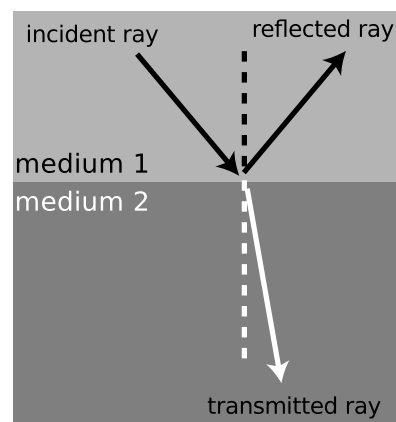
### Refractive properties of media

What are the rules governing refraction? The first thing to observe is that just as with reflection, the new, bent part of the ray lies in the same plane as the normal (perpendicular) and the incident ray, d.

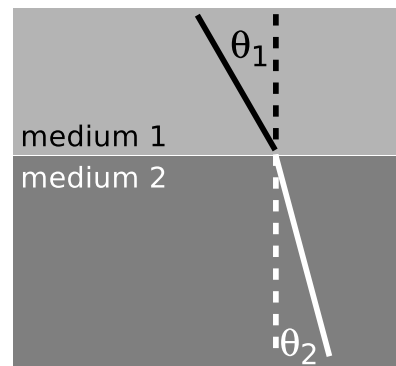
If you try shooting a beam of light at the boundary between two substances, say water and air, you’ll find that regardless of the angle at which you send in the beam, the part of the beam in the water is always closer to the normal line, e. It doesn’t matter if the ray is entering the water or leaving, so refraction is symmetric with respect to time-reversal, f.

If, instead of water and air, you try another combination of substances, say plastic and gasoline, again you’ll find that the ray’s angle with respect to the normal is consistently smaller in one and larger in the other. Also, we find that if substance A has rays closer to normal than in B, and B has rays closer to normal than in C, then A has rays closer to normal than C. This means that we can rank-order all materials according to their refractive properties. Isaac Newton did so, including in his list many amusing substances, such as “Danzig vitriol” and “a pseudo-topazius, being a natural, pellucid, brittle, hairy stone, of a yellow color.” Several general rules can be inferred from such a list:

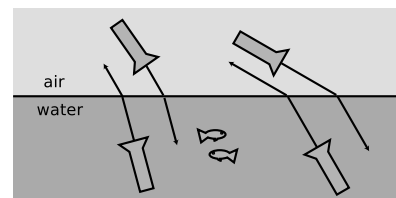
- Vacuum lies at one end of the list. In refraction across the interface between vacuum and any other medium, the other medium has rays closer to the normal.
- Among gases, the ray gets closer to the normal if you increase the density of the gas by pressurizing it more.
- The refractive properties of liquid mixtures and solutions vary in a smooth and systematic manner as the proportions of the mixture are changed.
- Denser substances usually, but not always, have rays closer to the normal.



d / The incident, reflected, and transmitted (refracted) rays all lie in a plane that includes the normal (dashed line).



e / The angles  $\theta_1$  and  $\theta_2$  are related to each other, and also depend on the properties of the two media. Because refraction is time-reversal symmetric, there is no need to label the rays with arrowheads.



f / Refraction has time-reversal symmetry. Regardless of whether the light is going into or out of the water, the relationship between the two angles is the same, and the ray is closer to the normal while in the water.



The second and third rules provide us with a method for measuring the density of an unknown sample of gas, or the concentration of a solution. The latter technique is very commonly used, and the CRC Handbook of Physics and Chemistry, for instance, contains extensive tables of the refractive properties of sugar solutions, cat urine, and so on.

### Snell's law

The numerical rule governing refraction was discovered by Snell, who must have collected experimental data something like what is shown on this graph and then attempted by trial and error to find the right equation. The equation he came up with was

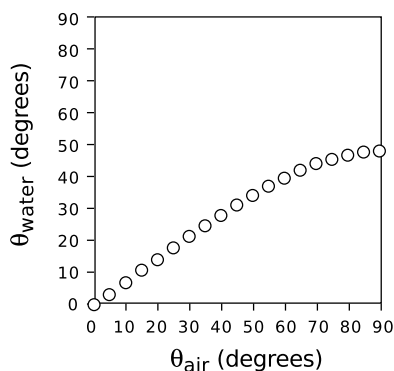
$$\frac{\sin \theta_1}{\sin \theta_2} = \text{constant} \quad .$$

The value of the constant would depend on the combination of media used. For instance, any one of the data points in the graph would have sufficed to show that the constant was 1.3 for an air-water interface (taking air to be substance 1 and water to be substance 2).

Snell further found that if media A and B gave a constant  $K_{AB}$  and media B and C gave a constant  $K_{BC}$ , then refraction at an interface between A and C would be described by a constant equal to the product,  $K_{AC} = K_{AB}K_{BC}$ . This is exactly what one would expect if the constant depended on the ratio of some number characterizing one medium to the number characteristic of the second medium. This number is called the *index of refraction* of the medium, written as  $n$  in equations. Since measuring the angles would only allow him to determine the *ratio* of the indices of refraction of two media, Snell had to pick some medium and define it as having  $n = 1$ . He chose to define vacuum as having  $n = 1$ . (The index of refraction of air at normal atmospheric pressure is 1.0003, so for most purposes it is a good approximation to assume that air has  $n = 1$ .) He also had to decide which way to define the ratio, and he chose to define it so that media with their rays closer to the normal would have larger indices of refraction. This had the advantage that denser media would typically have higher indices of refraction, and for this reason the index of refraction is also referred to as the optical density. Written in terms of indices of refraction, Snell's equation becomes

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \quad ,$$

but rewriting it in the form



g / The relationship between the angles in refraction.

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

[relationship between angles of rays at the interface between media with indices of refraction  $n_1$  and  $n_2$ ; angles are defined with respect to the normal]

makes us less likely to get the 1's and 2's mixed up, so this the way most people remember Snell's law. A few indices of refraction are given in the back of the book.

#### self-check A

(1) What would the graph look like for two substances with the same index of refraction?

(2) Based on the graph, when does refraction at an air-water interface change the direction of a ray most strongly? ▷ Answer, p. 979

#### Finding an angle using Snell's law

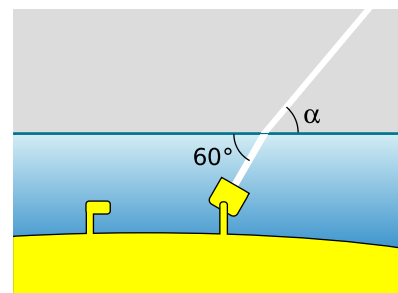
#### example 1

▷ A submarine shines its searchlight up toward the surface of the water. What is the angle  $\alpha$  shown in the figure?

▷ The tricky part is that Snell's law refers to the angles with respect to the normal. Forgetting this is a very common mistake. The beam is at an angle of  $30^\circ$  with respect to the normal in the water. Let's refer to the air as medium 1 and the water as 2. Solving Snell's law for  $\theta_1$ , we find

$$\theta_1 = \sin^{-1} \left( \frac{n_2}{n_1} \sin \theta_2 \right) .$$

As mentioned above, air has an index of refraction very close to 1, and water's is about 1.3, so we find  $\theta_1 = 40^\circ$ . The angle  $\alpha$  is therefore  $50^\circ$ .



h / Example 1.

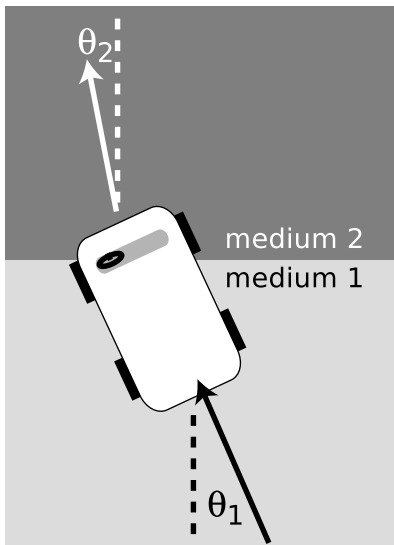
### The index of refraction is related to the speed of light.

What neither Snell nor Newton knew was that there is a very simple interpretation of the index of refraction. This may come as a relief to the reader who is taken aback by the complex reasoning involving proportionalities that led to its definition. Later experiments showed that the index of refraction of a medium was inversely proportional to the speed of light in that medium. Since  $c$  is defined as the speed of light in vacuum, and  $n = 1$  is defined as the index of refraction of vacuum, we have

$$n = \frac{c}{v} .$$

[ $n$  = medium's index of refraction,  $v$  = speed of light in that medium,  $c$  = speed of light in a vacuum]

Many textbooks start with this as the definition of the index of refraction, although that approach makes the quantity's name somewhat of a mystery, and leaves students wondering why  $c/v$  was used rather than  $v/c$ . It should also be noted that measuring angles of refraction is a far more practical method for determining  $n$  than direct measurement of the speed of light in the substance of interest.

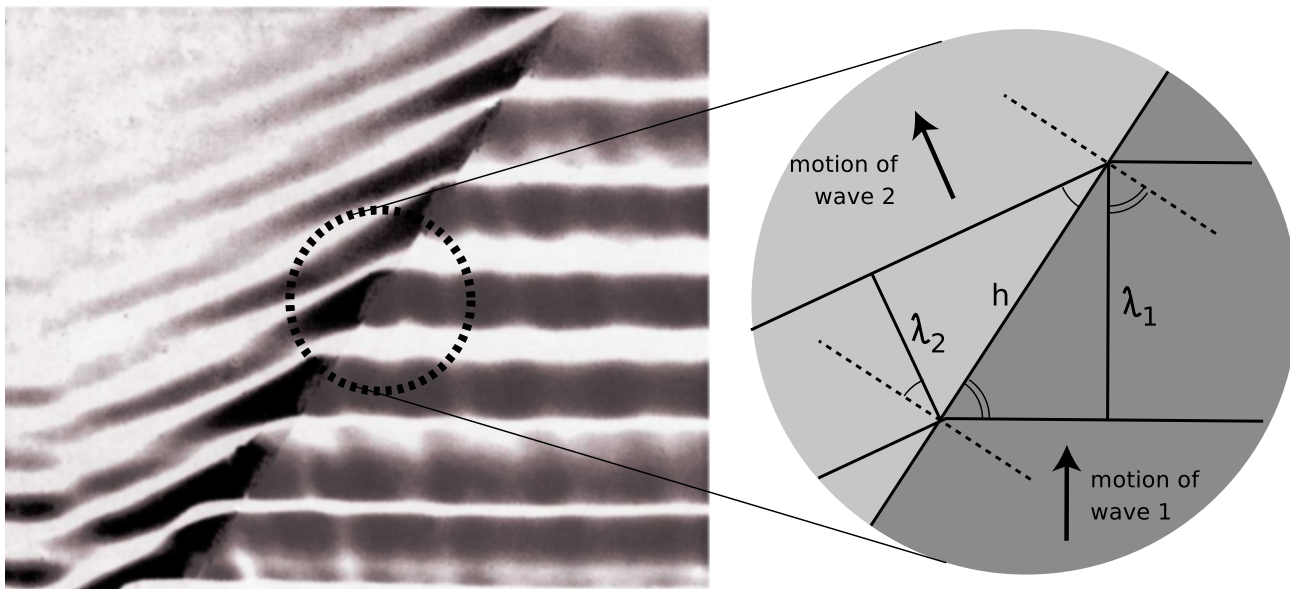


i / A mechanical model of refraction.

### A mechanical model of Snell's law

Why should refraction be related to the speed of light? The mechanical model shown in the figure may help to make this more plausible. Suppose medium 2 is thick, sticky mud, which slows down the car. The car's right wheel hits the mud first, causing the right side of the car to slow down. This will cause the car to turn to the right until it moves far enough forward for the left wheel to cross into the mud. After that, the two sides of the car will once again be moving at the same speed, and the car will go straight.

Of course, light isn't a car. Why should a beam of light have anything resembling a "left wheel" and "right wheel?" After all, the mechanical model would predict that a motorcycle would go straight, and a motorcycle seems like a better approximation to a ray of light than a car. The whole thing is just a model, not a description of physical reality.



j / A derivation of Snell's law.

### A derivation of Snell's law

However intuitively appealing the mechanical model may be, light is a wave, and we should be using wave models to describe refraction. In fact Snell's law can be derived quite simply from wave concepts. Figure j shows the refraction of a water wave. The water in the upper left part of the tank is shallower, so the speed of the waves is slower there, and their wavelengths are shorter. The reflected part of the wave is also very faintly visible.

In the close-up view on the right, the dashed lines are normals to the interface. The two marked angles on the right side are both equal to  $\theta_1$ , and the two on the left to  $\theta_2$ .

Trigonometry gives

$$\begin{aligned}\sin \theta_1 &= \lambda_1/h & \text{and} \\ \sin \theta_2 &= \lambda_2/h & .\end{aligned}$$

Eliminating  $h$  by dividing the equations, we find

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{\lambda_1}{\lambda_2} .$$

The frequencies of the two waves must be equal or else they would get out of step, so by  $v = f\lambda$  we know that their wavelengths are proportional to their velocities. Combining  $\lambda \propto v$  with  $v \propto 1/n$  gives  $\lambda \propto 1/n$ , so we find

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} ,$$

which is one form of Snell's law.

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*Ocean waves near and far from shore* *example 2*

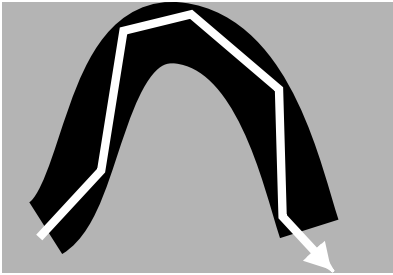
Ocean waves are formed by winds, typically on the open sea, and the wavefronts are perpendicular to the direction of the wind that formed them. At the beach, however, you have undoubtedly observed that waves tend come in with their wavefronts very nearly (but not exactly) parallel to the shoreline. This is because the speed of water waves in shallow water depends on depth: the shallower the water, the slower the wave. Although the change from the fast-wave region to the slow-wave region is gradual rather than abrupt, there is still refraction, and the wave motion is nearly perpendicular to the normal in the slow region.

### Color and refraction

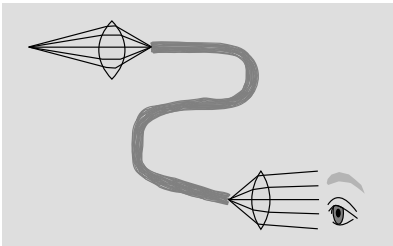
In general, the speed of light in a medium depends both on the medium and on the wavelength of the light. Another way of saying it is that a medium's index of refraction varies with wavelength. This is why a prism can be used to split up a beam of white light into a rainbow. Each wavelength of light is refracted through a different angle.

### How much light is reflected, and how much is transmitted?

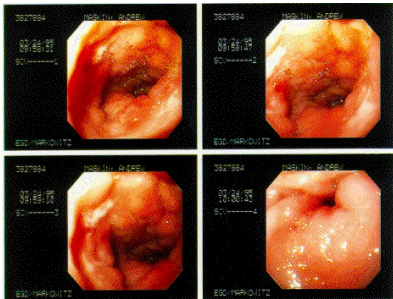
In chapter 20 we developed an equation for the percentage of the wave energy that is transmitted and the percentage reflected at a boundary between media. This was only done in the case of waves in one dimension, however, and rather than discuss the full three dimensional generalization it will be more useful to go into some qualitative observations about what happens. First, reflection happens



k / Total internal reflection in a fiber-optic cable.



l / A simplified drawing of a surgical endoscope. The first lens forms a real image at one end of a bundle of optical fibers. The light is transmitted through the bundle, and is finally magnified by the eyepiece.



m / Endoscopic images of a duodenal ulcer.

only at the interface between two media, and two media with the same index of refraction act as if they were a single medium. Thus, at the interface between media with the same index of refraction, there is no reflection, and the ray keeps going straight. Continuing this line of thought, it is not surprising that we observe very little reflection at an interface between media with similar indices of refraction.

The next thing to note is that it is possible to have situations where no possible angle for the refracted ray can satisfy Snell’s law. Solving Snell’s law for  $\theta_2$ , we find

$$\theta_2 = \sin^{-1} \left( \frac{n_1}{n_2} \sin \theta_1 \right) \quad ,$$

and if  $n_1$  is greater than  $n_2$ , then there will be large values of  $\theta_1$  for which the quantity  $(n_1/n_2) \sin \theta$  is greater than one, meaning that your calculator will flash an error message at you when you try to take the inverse sine. What can happen physically in such a situation? The answer is that all the light is reflected, so there is no refracted ray. This phenomenon is known as *total internal reflection*, and is used in the fiber-optic cables that nowadays carry almost all long-distance telephone calls. The electrical signals from your phone travel to a switching center, where they are converted from electricity into light. From there, the light is sent across the country in a thin transparent fiber. The light is aimed straight into the end of the fiber, and as long as the fiber never goes through any turns that are too sharp, the light will always encounter the edge of the fiber at an angle sufficiently oblique to give total internal reflection. If the fiber-optic cable is thick enough, one can see an image at one end of whatever the other end is pointed at.

Alternatively, a bundle of cables can be used, since a single thick cable is too hard to bend. This technique for seeing around corners is useful for making surgery less traumatic. Instead of cutting a person wide open, a surgeon can make a small “keyhole” incision and insert a bundle of fiber-optic cable (known as an endoscope) into the body.

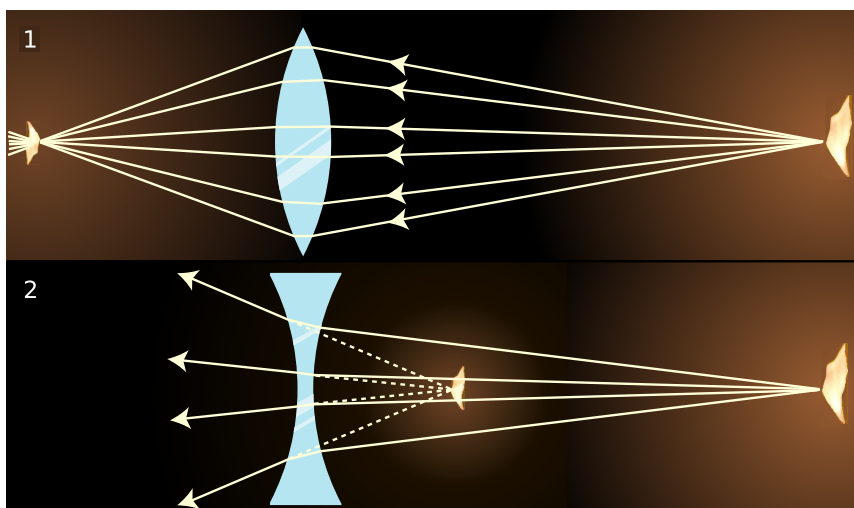
Since rays at sufficiently large angles with respect to the normal may be completely reflected, it is not surprising that the relative amount of reflection changes depending on the angle of incidence, and is greatest for large angles of incidence.

## Discussion questions

- A** What index of refraction should a fish have in order to be invisible to other fish?
- B** Does a surgeon using an endoscope need a source of light inside the body cavity? If so, how could this be done without inserting a light bulb through the incision?
- C** A denser sample of a gas has a higher index of refraction than a less dense sample (i.e., a sample under lower pressure), but why would it not make sense for the index of refraction of a gas to be proportional to density?
- D** The earth's atmosphere gets thinner and thinner as you go higher in altitude. If a ray of light comes from a star that is below the zenith, what will happen to it as it comes into the earth's atmosphere?
- E** Does total internal reflection occur when light in a denser medium encounters a less dense medium, or the other way around? Or can it occur in either case?

## 31.2 Lenses

Figures n/1 and n/2 show examples of lenses forming images. There is essentially nothing for you to learn about imaging with lenses that is truly new. You already know how to construct and use ray diagrams, and you know about real and virtual images. The concept of the focal length of a lens is the same as for a curved mirror. The equations for locating images and determining magnifications are of the same form. It's really just a question of flexing your mental muscles on a few examples. The following self-checks and discussion questions will get you started.



n / 1. A converging lens forms an image of a candle flame. 2. A diverging lens.

### self-check B

- (1) In figures n/1 and n/2, classify the images as real or virtual.

(2) Glass has an index of refraction that is greater than that of air. Consider the topmost ray in figure n/1. Explain why the ray makes a slight left turn upon entering the lens, and another left turn when it exits.

(3) If the flame in figure n/2 was moved closer to the lens, what would happen to the location of the image? ▷ Answer, p. 979

### Discussion questions

**A** In figures n/1 and n/2, the front and back surfaces are parallel to each other at the center of the lens. What will happen to a ray that enters near the center, but not necessarily along the axis of the lens? Draw a BIG ray diagram, and show a ray that comes from off axis.

*In discussion questions B-F, don't draw ultra-detailed ray diagrams as in A.*

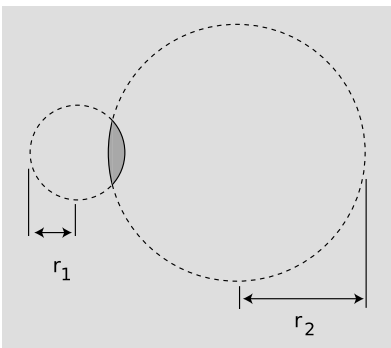
**B** Suppose you wanted to change the setup in figure n/1 so that the location of the actual flame in the figure would instead be occupied by an image of a flame. Where would you have to move the candle to achieve this? What about in n/2?

**C** There are three qualitatively different types of image formation that can occur with lenses, of which figures n/1 and n/2 exhaust only two. Figure out what the third possibility is. Which of the three possibilities can result in a magnification greater than one?

**D** Classify the examples shown in figure o according to the types of images delineated in discussion question C.

**E** In figures n/1 and n/2, the only rays drawn were those that happened to enter the lenses. Discuss this in relation to figure o.

**F** In the right-hand side of figure o, the image viewed through the lens is in focus, but the side of the rose that sticks out from behind the lens is not. Why?



p / The radii of curvature appearing in the lensmaker's equation.

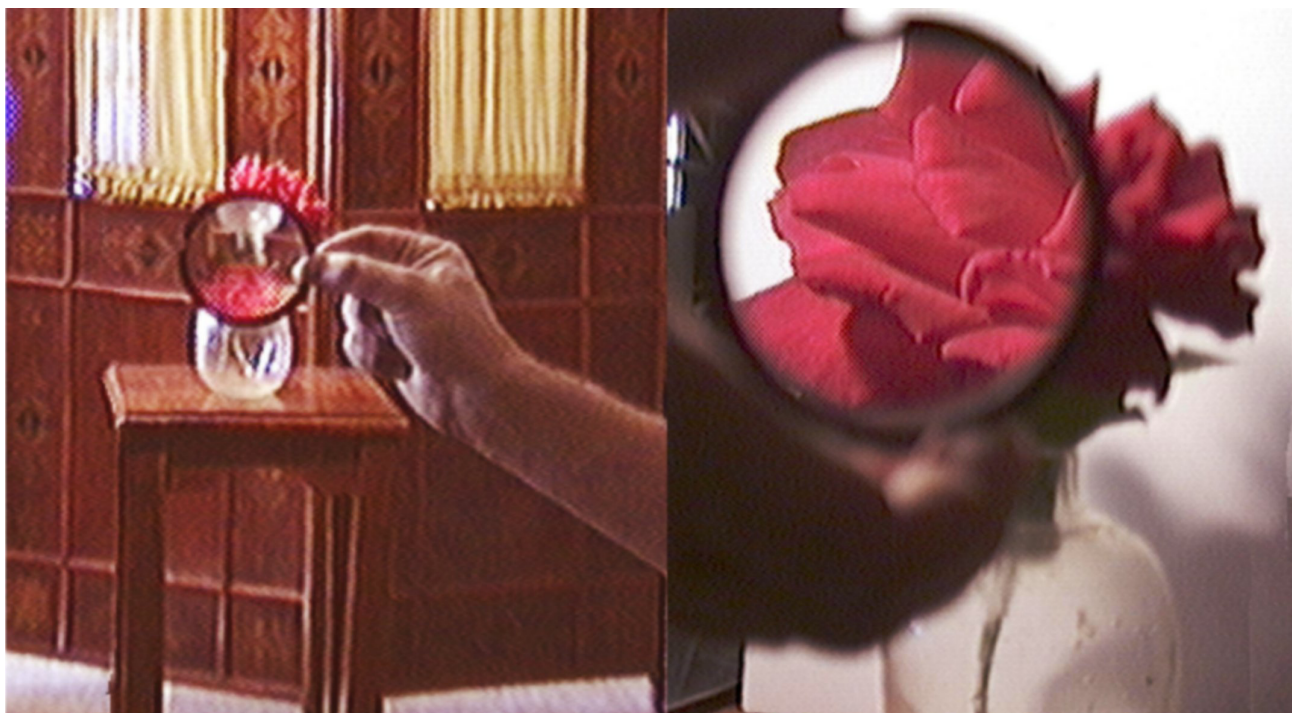
## 31.3 ★ The lensmaker's equation

The focal length of a spherical mirror is simply  $r/2$ , but we cannot expect the focal length of a lens to be given by pure geometry, since it also depends on the index of refraction of the lens. Suppose we have a lens whose front and back surfaces are both spherical. (This is no great loss of generality, since any surface with a sufficiently shallow curvature can be approximated with a sphere.) Then if the lens is immersed in a medium with an index of refraction of 1, its focal length is given approximately by

$$f = \left[ (n - 1) \left| \frac{1}{r_1} \pm \frac{1}{r_2} \right| \right]^{-1},$$

where  $n$  is the index of refraction and  $r_1$  and  $r_2$  are the radii of curvature of the two surfaces of the lens. This is known as the lensmaker's equation. In my opinion it is not particularly worthy of memorization. The positive sign is used when both surfaces are



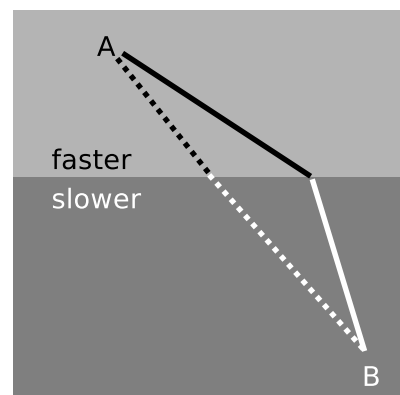


o / Two images of a rose created by the same lens and recorded with the same camera.

curved outward or both are curved inward; otherwise a negative sign applies. The proof of this equation is left as an exercise to those readers who are sufficiently brave and motivated.

### 31.4 ★ The principle of least time for refraction

We have seen previously how the rules governing straight-line motion of light and reflection of light can be derived from the principle of least time. What about refraction? In the figure, it is indeed plausible that the bending of the ray serves to minimize the time required to get from a point A to point B. If the ray followed the unbent path shown with a dashed line, it would have to travel a longer distance in the medium in which its speed is slower. By bending the correct amount, it can reduce the distance it has to cover in the slower medium without going too far out of its way. It is true that Snell's law gives exactly the set of angles that minimizes the time required for light to get from one point to another. The proof of this fact is left as an exercise (problem 9, p. 850).



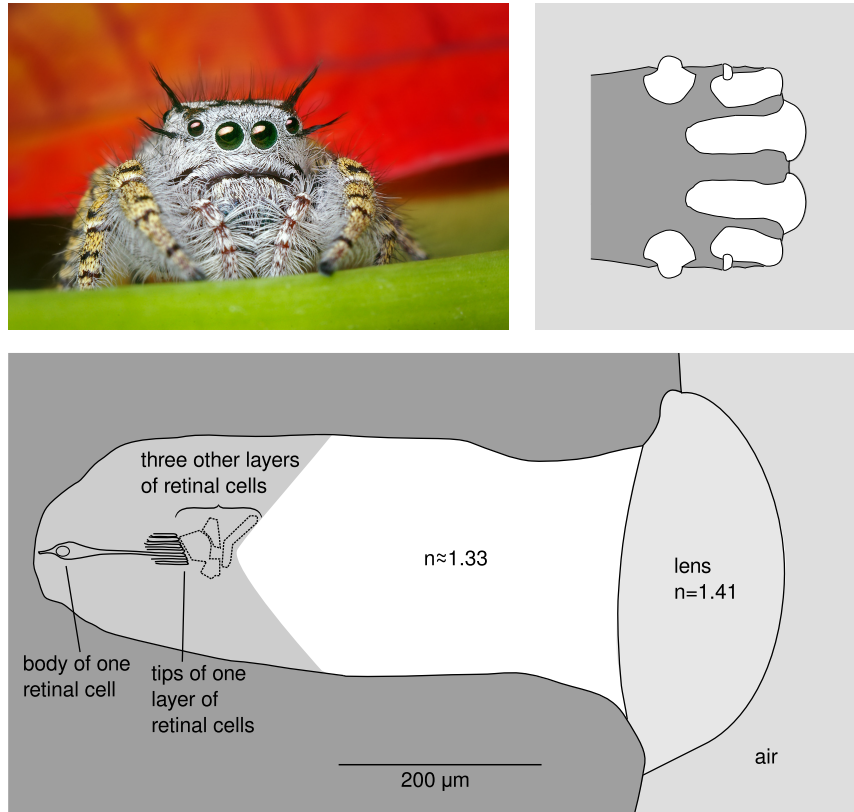
q / The principle of least time applied to refraction.



### 31.5 ★ Case study: the eye of the jumping spider

Figure r shows an exceptionally cute jumping spider. The jumping spider does not build a web. It stalks its prey like a cat, so it needs excellent eyesight. In some ways, its visual system is more sophisticated and more functional than that of a human, illustrating how evolution does not progress systematically toward “higher” forms of life.

r / Top left: A female jumping spider, *Phidippus mystaceus*. Top right: Cross-section in a horizontal plane, viewed from above, of the jumping spider *Metaphidippus aeneolus*. The eight eyes are shown in white. Bottom: Close-up of one of the large principal eyes.



One way in which the spider outdoes us is that it has eight eyes to our two. (Each eye is simple, not compound like that of a fly.) The reason this works well has to do with the trade-off between magnification and field of view. The elongated principal eyes at the front of the head have a large value of  $d_i$ , resulting in a large magnification  $M = d_i/d_o$ . This high magnification is used for sophisticated visual tasks like distinguishing prey from a potential mate. (The pretty stripes on the legs in the photo are probably evolved to aid in making this distinction, which is a crucial one on a Saturday night.) As always with a high magnification, this results in a reduction in the field of view: making the image bigger means reducing the amount of the potential image that can actually fit on the retina. The animal has tunnel vision in these forward eyes. To allow it to glimpse prey from other angles, it has the additional eyes on the sides of its

head. These are not elongated, and the smaller  $d_i$  gives a smaller magnification but a larger field of view. When the spider sees something moving in these eyes, it turns its body so that it can take a look with the front eyes. The tiniest pair of eyes are too small to be useful. These vestigial organs, like the maladaptive human appendix, are an example of the tendency of evolution to produce unfortunate accidents due to the lack of intelligent design. The use of multiple eyes for these multiple purposes is far superior to the two-eye arrangement found in humans, octopi, etc., especially because of its compactness. If the spider had only two spherical eyes, they would have to have the same front-to-back dimension in order to produce the same acuity, but then the eyes would take up nearly all of the front of the head.

Another beautiful feature of these eyes is that they will never need bifocals. A human eye uses muscles to adjust for seeing near and far, varying  $f$  in order to achieve a fixed  $d_i$  for differing values of  $d_o$ . On older models of *H. sap.*, this poorly engineered feature is usually one of the first things to break down. The spider's front eyes have muscles, like a human's, that rotate the tube, but none that vary  $f$ , which is fixed. However, the retina consists of four separate layers at slightly different values of  $d_i$ . The figure only shows the detailed cellular structure of the rearmost layer, which is the most acute. Depending on  $d_o$ , the image may lie closest to any one of the four layers, and the spider can then use that layer to get a well-focused view. The layering is also believed to help eliminate problems caused by the variation of the index of refraction with wavelength (cf. problem 2, p. 849).

Although the spider's eye is different in many ways from a human's or an octopus's, it shares the same fundamental construction, being essentially a lens that forms a real image on a screen inside a darkened chamber. From this perspective, the main difference is simply the scale, which is miniaturized by about a factor of  $10^2$  in the linear dimensions. How far down can this scaling go? Does an amoeba or a white blood cell lack an eye merely because it doesn't have a nervous system that could make sense of the signals? In fact there is an optical limit on the miniaturization of any eye or camera. The spider's eye is already so small that on the scale of the bottom panel in figure r, one wavelength of visible light would be easily distinguishable — about the length of the comma in this sentence. Chapter 32 is about optical effects that occur when the wave nature of light is important, and problem 14 on p. 876 specifically addresses the effect on this spider's vision.

# Summary

## Selected vocabulary

refraction . . . .	the change in direction that occurs when a wave encounters the interface between two media
index of refraction . . . . .	an optical property of matter; the speed of light in a vacuum divided by the speed of light in the substance in question

## Notation

$n$ . . . . .	the index of refraction
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## Summary

Refraction is a change in direction that occurs when a wave encounters the interface between two media. Together, refraction and reflection account for the basic principles behind nearly all optical devices.

Snell discovered the equation for refraction,

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad ,$$

[angles measured with respect to the normal]

through experiments with light rays, long before light was proven to be a wave. Snell's law can be proven based on the geometrical behavior of waves. Here  $n$  is the index of refraction. Snell invented this quantity to describe the refractive properties of various substances, but it was later found to be related to the speed of light in the substance,

$$n = \frac{c}{v} \quad ,$$

where  $c$  is the speed of light in a vacuum. In general a material's index of refraction is different for different wavelengths of light.

As discussed in chapter 20, any wave is partially transmitted and partially reflected at the boundary between two media in which its speeds are different. It is not particularly important to know the equation that tells what fraction is transmitted (and thus refracted), but important technologies such as fiber optics are based on the fact that this fraction becomes *zero* for sufficiently oblique angles. This phenomenon is referred to as total internal reflection. It occurs when there is no angle that satisfies Snell's law.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** Suppose a converging lens is constructed of a type of plastic whose index of refraction is less than that of water. How will the lens's behavior be different if it is placed underwater?

**2** There are two main types of telescopes, refracting (using lenses) and reflecting (using mirrors). (Some telescopes use a mixture of the two types of elements: the light first encounters a large curved mirror, and then goes through an eyepiece that is a lens.) What implications would the color-dependence of focal length have for the relative merits of the two types of telescopes? Describe the case where an image is formed of a white star. You may find it helpful to draw a ray diagram.

**3** Based on Snell's law, explain why rays of light passing through the edges of a converging lens are bent more than rays passing through parts closer to the center. It might seem like it should be the other way around, since the rays at the edge pass through less glass — shouldn't they be affected less? In your answer:

- Include a ray diagram showing a huge close-up view of the relevant part of the lens.
- Make use of the fact that the front and back surfaces aren't always parallel; a lens in which the front and back surfaces *are* always parallel doesn't focus light at all, so if your explanation doesn't make use of this fact, your argument must be incorrect.
- Make sure your argument still works even if the rays don't come in parallel to the axis.

**4** When you take pictures with a camera, the distance between the lens and the film has to be adjusted, depending on the distance at which you want to focus. This is done by moving the lens. If you want to change your focus so that you can take a picture of something farther away, which way do you have to move the lens? Explain using ray diagrams. [Based on a problem by Eric Mazur.]

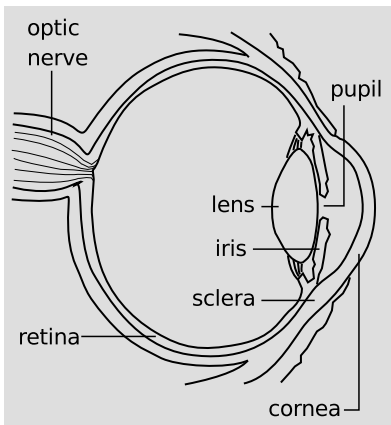
**5** (a) Light is being reflected diffusely from an object 1.000 m underwater. The light that comes up to the surface is refracted at the water-air interface. If the refracted rays all appear to come from the same point, then there will be a virtual image of the object in the water, above the object's actual position, which will be visible

to an observer above the water. Consider three rays, A, B and C, whose angles in the water with respect to the normal are  $\theta_i = 0.000^\circ$ ,  $1.000^\circ$  and  $20.000^\circ$  respectively. Find the depth of the point at which the refracted parts of A and B appear to have intersected, and do the same for A and C. Show that the intersections are at nearly the same depth, but not quite. [Check: The difference in depth should be about 4 cm.]

(b) Since all the refracted rays do not quite appear to have come from the same point, this is technically not a virtual image. In practical terms, what effect would this have on what you see?

(c) In the case where the angles are all small, use algebra and trig to show that the refracted rays do appear to come from the same point, and find an equation for the depth of the virtual image. Do not put in any numerical values for the angles or for the indices of refraction — just keep them as symbols. You will need the approximation  $\sin \theta \approx \tan \theta \approx \theta$ , which is valid for small angles measured in radians.

★



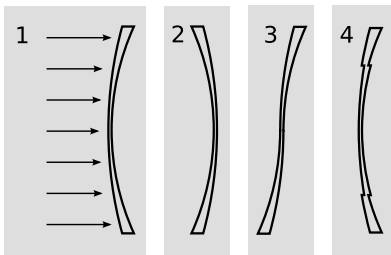
Problem 6.

**6** The drawing shows the anatomy of the human eye, at twice life size. Find the radius of curvature of the outer surface of the cornea by measurements on the figure, and then derive the focal length of the air-cornea interface, where almost all the focusing of light occurs. You will need to use physical reasoning to modify the lensmaker's equation for the case where there is only a single refracting surface. Assume that the index of refraction of the cornea is essentially that of water.

★

**7** When swimming underwater, why is your vision made much clearer by wearing goggles with flat pieces of glass that trap air behind them? [Hint: You can simplify your reasoning by considering the special case where you are looking at an object far away, and along the optic axis of the eye.]

**8** The figure shows four lenses. Lens 1 has two spherical surfaces. Lens 2 is the same as lens 1 but turned around. Lens 3 is made by cutting through lens 1 and turning the bottom around. Lens 4 is made by cutting a central circle out of lens 1 and recessing it.



Problem 8.

(a) A parallel beam of light enters lens 1 from the left, parallel to its axis. Reasoning based on Snell's law, will the beam emerging from the lens be bent inward, or outward, or will it remain parallel to the axis? Explain your reasoning. As part of your answer, make a huge drawing of one small part of the lens, and apply Snell's law at both interfaces. Recall that rays are bent more if they come to the interface at a larger angle with respect to the normal.

(b) What will happen with lenses 2, 3, and 4? Explain. Drawings are not necessary.

**9** Prove that the principle of least time leads to Snell's law. ★

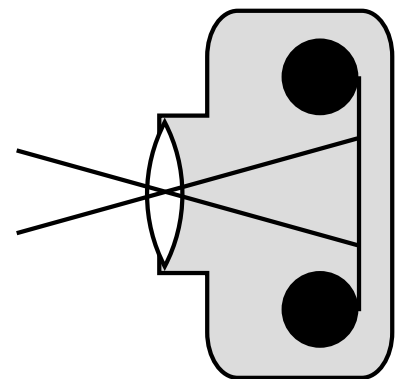
**10** An object is more than one focal length from a converging lens. (a) Draw a ray diagram. (b) Using reasoning like that developed in chapter 30, determine the positive and negative signs in the equation  $1/f = \pm 1/d_i \pm 1/d_o$ . (c) The images of the rose in section 4.2 were made using a lens with a focal length of 23 cm. If the lens is placed 80 cm from the rose, locate the image. ✓

**11** An object is less than one focal length from a converging lens. (a) Draw a ray diagram. (b) Using reasoning like that developed in chapter 30, determine the positive and negative signs in the equation  $1/f = \pm 1/d_i \pm 1/d_o$ . (c) The images of the rose in section 4.2 were made using a lens with a focal length of 23 cm. If the lens is placed 10 cm from the rose, locate the image. ✓

**12** Nearsighted people wear glasses whose lenses are diverging. (a) Draw a ray diagram. For simplicity pretend that there is no eye behind the glasses. (b) Using reasoning like that developed in chapter 30, determine the positive and negative signs in the equation  $1/f = \pm 1/d_i \pm 1/d_o$ . (c) If the focal length of the lens is 50.0 cm, and the person is looking at an object at a distance of 80.0 cm, locate the image. ✓

**13** Two standard focal lengths for camera lenses are 50 mm (standard) and 28 mm (wide-angle). To see how the focal lengths relate to the angular size of the field of view, it is helpful to visualize things as represented in the figure. Instead of showing many rays coming from the same point on the same object, as we normally do, the figure shows two rays from two different objects. Although the lens will intercept infinitely many rays from each of these points, we have shown only the ones that pass through the center of the lens, so that they suffer no angular deflection. (Any angular deflection at the front surface of the lens is canceled by an opposite deflection at the back, since the front and back surfaces are parallel at the lens's center.) What is special about these two rays is that they are aimed at the edges of one 35-mm-wide frame of film; that is, they show the limits of the field of view. Throughout this problem, we assume that  $d_o$  is much greater than  $d_i$ . (a) Compute the angular width of the camera's field of view when these two lenses are used. (b) Use small-angle approximations to find a simplified equation for the angular width of the field of view,  $\theta$ , in terms of the focal length,  $f$ , and the width of the film,  $w$ . Your equation should not have any trig functions in it. Compare the results of this approximation with your answers from part a. (c) Suppose that we are holding constant the aperture (amount of surface area of the lens being used to collect light). When switching from a 50-mm lens to a 28-mm lens, how many times longer or shorter must the exposure be in order to make a properly developed picture, i.e., one that is not under- or overexposed? [Based on a problem by Arnold Arons.]

▷ Solution, p. 976



Problem 13.

**14** A nearsighted person is one whose eyes focus light too strongly, and who is therefore unable to relax the lens inside her eye sufficiently to form an image on her retina of an object that is too far away.

(a) Draw a ray diagram showing what happens when the person tries, with uncorrected vision, to focus at infinity.

(b) What type of lenses do her glasses have? Explain.

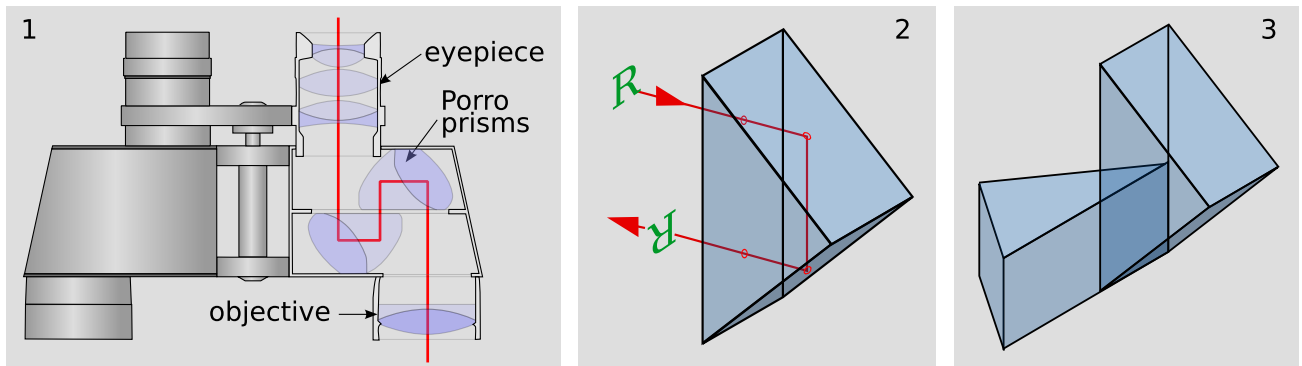
(c) Draw a ray diagram showing what happens when she wears glasses. Locate both the image formed by the glasses and the final image.

(d) Suppose she sometimes uses contact lenses instead of her glasses. Does the focal length of her contacts have to be less than, equal to, or greater than that of her glasses? Explain.

**15** Diamond has an index of refraction of 2.42, and part of the reason diamonds sparkle is that this encourages a light ray to undergo many total internal reflections before it emerges. (a) Calculate the critical angle at which total internal reflection occurs in diamond. (b) Explain the interpretation of your result: Is it measured from the normal, or from the surface? Is it a minimum, or a maximum? How would the critical angle have been different for a substance such as glass or plastic, with a lower index of refraction?

✓

**16** Fred's eyes are able to focus on things as close as 5.0 cm. Fred holds a magnifying glass with a focal length of 3.0 cm at a height of 2.0 cm above a flatworm. (a) Locate the image, and find the magnification. (b) Without the magnifying glass, from what distance would Fred want to view the flatworm to see its details as well as possible? With the magnifying glass? (c) Compute the angular magnification.



**Problem 17.**

**17** Panel 1 of the figure shows the optics inside a pair of binoculars. They are essentially a pair of telescopes, one for each eye. But to make them more compact, and allow the eyepieces to be the right distance apart for a human face, they incorporate a set of eight prisms, which fold the light path. In addition, the prisms make the image upright. Panel 2 shows one of these prisms, known as a Porro prism. The light enters along a normal, undergoes two total internal reflections at angles of 45 degrees with respect to the back surfaces, and exits along a normal. The image of the letter R has been flipped across the horizontal. Panel 3 shows a pair of these prisms glued together. The image will be flipped across both the horizontal and the vertical, which makes it oriented the right way for the user of the binoculars.

(a) Find the minimum possible index of refraction for the glass used in the prisms.

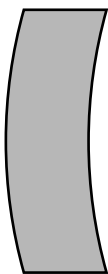
(b) For a material of this minimal index of refraction, find the fraction of the incoming light that will be lost to reflection in the four Porro prisms on a each side of a pair of binoculars. (See ch. 20.) In real, high-quality binoculars, the optical surfaces of the prisms have antireflective coatings, but carry out your calculation for the case where there is no such coating.

(c) Discuss the reasons why a designer of binoculars might or might not want to use a material with exactly the index of refraction found in part a. ★

**18** It would be annoying if your eyeglasses produced a magnified or reduced image. Prove that when the eye is very close to a lens, and the lens produces a virtual image, the angular magnification is always approximately equal to 1 (regardless of whether the lens is diverging or converging).

**19** A typical mirror consists of a pane of glass of thickness  $t$  and index of refraction  $n$ , “silvered” on the back with a reflective coating. Let  $d_o$  and  $d_i$  be measured from the back of the mirror.





Show that  $d_i = d_o - 2(1 - 1/n)t$ . Use the result of, and make the approximation employed in, problem 5c. As a check on your result, consider separately the special values of  $n$  and  $t$  that would recover the case without any glass.

Problem 20.

**20** The figure shows a lens with surfaces that are curved, but whose thickness is constant. Use the lensmaker's equation to prove that this "lens" is not really a lens at all.   ▷ Solution, p. 977

**21** Estimate the radii of curvature of the two optical surfaces in the eye of the jumping spider in figure r on p. 846. Use physical reasoning to modify the lensmaker's equation for a case like this one, in which there are three indices of refraction  $n_1$  (air),  $n_2$  (lens), and  $n_3$  (the material behind the lens), and  $n_1 \neq n_3$ . Show that the interface between  $n_2$  and  $n_3$  contributes negligibly to focusing, and verify that the image is produced at approximately the right place in the eye when the object is far away. As a check on your result, direct optical measurements by M.F. Land in 1969 gave  $f = 512 \mu\text{m}$ .  
✓

## Exercise 31: How strong are your glasses?

This exercise was created by Dan MacIsaac.

Equipment:

eyeglasses

diverging lenses for students who don't wear glasses, or who use converging glasses

rulers and metersticks

scratch paper

marking pens

Most people who wear glasses have glasses whose lenses are diverging, which allows them to focus on objects far away. Such a lens cannot form a real image, so its focal length cannot be measured as easily as that of an converging lens. In this exercise you will determine the focal length of your own glasses by taking them off, holding them at a distance from your face, and looking through them at a set of parallel lines on a piece of paper. The lines will be reduced (the lens's magnification is less than one), and by adjusting the distance between the lens and the paper, you can make the magnification equal  $1/2$  exactly, so that two spaces between lines as seen through the lens fit into one space as seen simultaneously to the side of the lens. This object distance can be used in order to find the focal length of the lens.

1. Does this technique really measure magnification or does it measure angular magnification? What can you do in your experiment in order to make these two quantities nearly the same, so the math is simpler?
2. Before taking any numerical data, use algebra to find the focal length of the lens in terms of  $d_o$ , the object distance that results in a magnification of  $1/2$ .
3. Use a marker to draw three evenly spaced parallel lines on the paper. (A spacing of a few cm works well.) Measure the object distance that results in a magnification of  $1/2$ , and determine the focal length of your lens.



This image of the Pleiades star cluster shows haloes around the stars due to the wave nature of light.

## Chapter 32

# Wave optics

Electron microscopes can make images of individual atoms, but why will a visible-light microscope never be able to? Stereo speakers create the illusion of music that comes from a band arranged in your living room, but why doesn't the stereo illusion work with bass notes? Why are computer chip manufacturers investing billions of dollars in equipment to etch chips with x-rays instead of visible light?

The answers to all of these questions have to do with the subject of wave optics. So far this book has discussed the interaction of light waves with matter, and its practical applications to optical devices like mirrors, but we have used the ray model of light almost exclusively. Hardly ever have we explicitly made use of the fact that light is an electromagnetic wave. We were able to get away with the

simple ray model because the chunks of matter we were discussing, such as lenses and mirrors, were thousands of times larger than a wavelength of light. We now turn to phenomena and devices that can only be understood using the wave model of light.

## 32.1 Diffraction

Figure a shows a typical problem in wave optics, enacted with water waves. It may seem surprising that we don't get a simple pattern like figure b, but the pattern would only be that simple if the wavelength was hundreds of times shorter than the distance between the gaps in the barrier and the widths of the gaps.

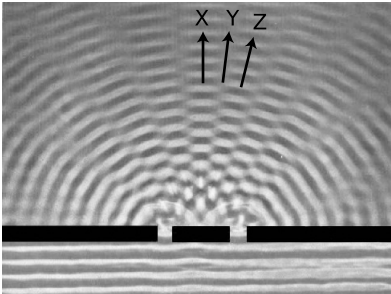
Wave optics is a broad subject, but this example will help us to pick out a reasonable set of restrictions to make things more manageable:

- (1) We restrict ourselves to cases in which a wave travels through a uniform medium, encounters a certain area in which the medium has different properties, and then emerges on the other side into a second uniform region.
- (2) We assume that the incoming wave is a nice tidy sine-wave pattern with wavefronts that are lines (or, in three dimensions, planes).
- (3) In figure a we can see that the wave pattern immediately beyond the barrier is rather complex, but farther on it sorts itself out into a set of wedges separated by gaps in which the water is still. We will restrict ourselves to studying the simpler wave patterns that occur farther away, so that the main question of interest is how intense the outgoing wave is at a given angle.

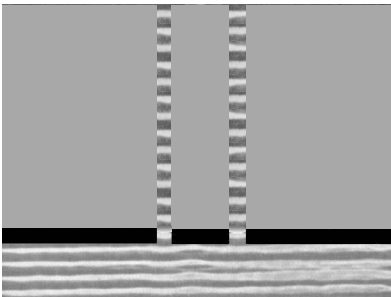
The kind of phenomenon described by restriction (1) is called *diffraction*. Diffraction can be defined as the behavior of a wave when it encounters an obstacle or a nonuniformity in its medium. In general, diffraction causes a wave to bend around obstacles and make patterns of strong and weak waves radiating out beyond the obstacle. Understanding diffraction is the central problem of wave optics. If you understand diffraction, even the subset of diffraction problems that fall within restrictions (2) and (3), the rest of wave optics is icing on the cake.

Diffraction can be used to find the structure of an unknown diffracting object: even if the object is too small to study with ordinary imaging, it may be possible to work backward from the diffraction pattern to learn about the object. The structure of a crystal, for example, can be determined from its x-ray diffraction pattern.

Diffraction can also be a bad thing. In a telescope, for example, light waves are diffracted by all the parts of the instrument. This will cause the image of a star to appear fuzzy even when the focus has been adjusted correctly. By understanding diffraction, one can learn how a telescope must be designed in order to reduce this problem



a / In this view from overhead, a straight, sinusoidal water wave encounters a barrier with two gaps in it. Strong wave vibration occurs at angles X and Z, but there is none at all at angle Y. (The figure has been retouched from a real photo of water waves. In reality, the waves beyond the barrier would be much weaker than the ones before it, and they would therefore be difficult to see.)



b / This doesn't happen.

— essentially, it should have the biggest possible diameter.

There are two ways in which restriction (2) might commonly be violated. First, the light might be a mixture of wavelengths. If we simply want to observe a diffraction pattern or to use diffraction as a technique for studying the object doing the diffracting (e.g., if the object is too small to see with a microscope), then we can pass the light through a colored filter before diffracting it.

A second issue is that light from sources such as the sun or a light-bulb does not consist of a nice neat plane wave, except over very small regions of space. Different parts of the wave are out of step with each other, and the wave is referred to as *incoherent*. One way of dealing with this is shown in figure c. After filtering to select a certain wavelength of red light, we pass the light through a small pinhole. The region of the light that is intercepted by the pinhole is so small that one part of it is not out of step with another. Beyond the pinhole, light spreads out in a spherical wave; this is analogous to what happens when you speak into one end of a paper towel roll and the sound waves spread out in all directions from the other end. By the time the spherical wave gets to the double slit it has spread out and reduced its curvature, so that we can now think of it as a simple plane wave.

If this seems laborious, you may be relieved to know that modern technology gives us an easier way to produce a single-wavelength, coherent beam of light: the laser.

The parts of the final image on the screen in c are called diffraction fringes. The center of each fringe is a point of maximum brightness, and halfway between two fringes is a minimum.

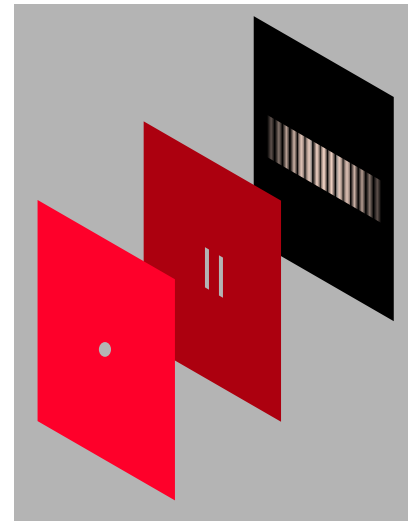
### Discussion question

**A** Why would x-rays rather than visible light be used to find the structure of a crystal? Sound waves are used to make images of fetuses in the womb. What would influence the choice of wavelength?

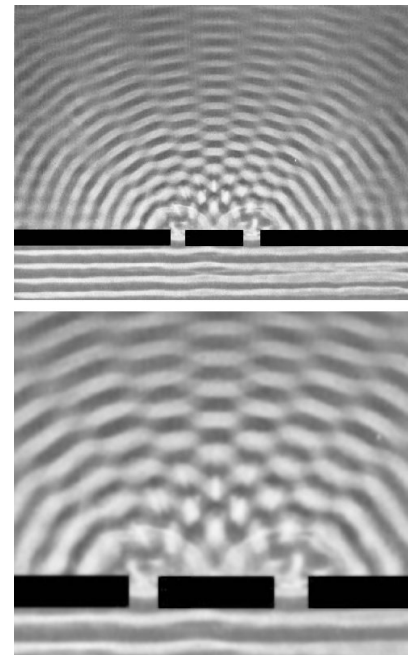
## 32.2 Scaling of diffraction

This chapter has “optics” in its title, so it is nominally about light, but we started out with an example involving water waves. Water waves are certainly easier to visualize, but is this a legitimate comparison? In fact the analogy works quite well, despite the fact that a light wave has a wavelength about a million times shorter. This is because diffraction effects scale uniformly. That is, if we enlarge or reduce the whole diffraction situation by the same factor, including both the wavelengths and the sizes of the obstacles the wave encounters, the result is still a valid solution.

This is unusually simple behavior! In section 1.2 we saw many examples of more complex scaling, such as the impossibility of bacteria



c / A practical, low-tech setup for observing diffraction of light.



d / The bottom figure is simply a copy of the middle portion of the top one, scaled up by a factor of two. All the angles are the same. Physically, the angular pattern of the diffraction fringes can't be any different if we scale both  $\lambda$  and  $d$  by the same factor, leaving  $\lambda/d$  unchanged.



the size of dogs, or the need for an elephant to eliminate heat through its ears because of its small surface-to-volume ratio, whereas a tiny shrew's life-style centers around conserving its body heat.

Of course water waves and light waves differ in many ways, not just in scale, but the general facts you will learn about diffraction are applicable to all waves. In some ways it might have been more appropriate to insert this chapter after chapter 20 on bounded waves, but many of the important applications are to light waves, and you would probably have found these much more difficult without any background in optics.

Another way of stating the simple scaling behavior of diffraction is that the diffraction angles we get depend only on the unitless ratio  $\lambda/d$ , where  $\lambda$  is the wavelength of the wave and  $d$  is some dimension of the diffracting objects, e.g., the center-to-center spacing between the slits in figure a. If, for instance, we scale up both  $\lambda$  and  $d$  by a factor of 37, the ratio  $\lambda/d$  will be unchanged.

### 32.3 The correspondence principle

The only reason we don't usually notice diffraction of light in everyday life is that we don't normally deal with objects that are comparable in size to a wavelength of visible light, which is about a millionth of a meter. Does this mean that wave optics contradicts ray optics, or that wave optics sometimes gives wrong results? No. If you hold three fingers out in the sunlight and cast a shadow with them, *either* wave optics or ray optics can be used to predict the straightforward result: a shadow pattern with two bright lines where the light has gone through the gaps between your fingers. Wave optics is a more general theory than ray optics, so in any case where ray optics is valid, the two theories will agree. This is an example of a general idea enunciated by the physicist Niels Bohr, called the *correspondence principle*: when flaws in a physical theory lead to the creation of a new and more general theory, the new theory must still agree with the old theory within its more restricted area of applicability. After all, a theory is only created as a way of describing experimental observations. If the original theory had not worked in any cases at all, it would never have become accepted.

In the case of optics, the correspondence principle tells us that when  $\lambda/d$  is small, both the ray and the wave model of light must give approximately the same result. Suppose you spread your fingers and cast a shadow with them using a coherent light source. The quantity  $\lambda/d$  is about  $10^{-4}$ , so the two models will agree very closely. (To be specific, the shadows of your fingers will be outlined by a series of light and dark fringes, but the angle subtended by a fringe will be on the order of  $10^{-4}$  radians, so they will be invisible and washed out by the natural fuzziness of the edges of sun-shadows, caused by the finite size of the sun.)



e / Christiaan Huygens (1629-1695).

### self-check A

What kind of wavelength would an electromagnetic wave have to have in order to diffract dramatically around your body? Does this contradict the correspondence principle? ▷ Answer, p. 980

## 32.4 Huygens' principle

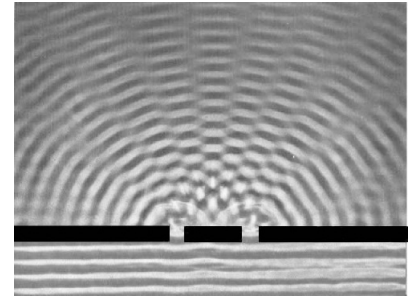
Returning to the example of double-slit diffraction, *f*, note the strong visual impression of two overlapping sets of concentric semicircles. This is an example of *Huygens' principle*, named after a Dutch physicist and astronomer. (The first syllable rhymes with “boy.”) Huygens' principle states that any wavefront can be broken down into many small side-by-side wave peaks, *g*, which then spread out as circular ripples, *h*, and by the principle of superposition, the result of adding up these sets of ripples must give the same result as allowing the wave to propagate forward, *i*. In the case of sound or light waves, which propagate in three dimensions, the “ripples” are actually spherical rather than circular, but we can often imagine things in two dimensions for simplicity.

In double-slit diffraction the application of Huygens' principle is visually convincing: it is as though all the sets of ripples have been blocked except for two. It is a rather surprising mathematical fact, however, that Huygens' principle gives the right result in the case of an unobstructed linear wave, *h* and *i*. A theoretically infinite number of circular wave patterns somehow conspire to add together and produce the simple linear wave motion with which we are familiar.

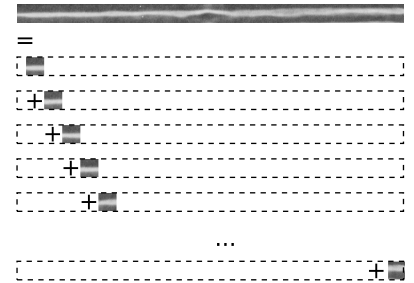
Since Huygens' principle is equivalent to the principle of superposition, and superposition is a property of waves, what Huygens had created was essentially the first wave theory of light. However, he imagined light as a series of pulses, like hand claps, rather than as a sinusoidal wave.

The history is interesting. Isaac Newton loved the atomic theory of matter so much that he searched enthusiastically for evidence that light was also made of tiny particles. The paths of his light particles would correspond to rays in our description; the only significant difference between a ray model and a particle model of light would occur if one could isolate individual particles and show that light had a “graininess” to it. Newton never did this, so although he thought of his model as a particle model, it is more accurate to say he was one of the builders of the ray model.

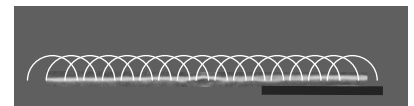
Almost all that was known about reflection and refraction of light could be interpreted equally well in terms of a particle model or a wave model, but Newton had one reason for strongly opposing Huygens' wave theory. Newton knew that waves exhibited diffraction, but diffraction of light is difficult to observe, so Newton believed that light did not exhibit diffraction, and therefore must not be



*f* / Double-slit diffraction.



*g* / A wavefront can be analyzed by the principle of superposition, breaking it down into many small parts.



*h* / If it was by itself, each of the parts would spread out as a circular ripple.

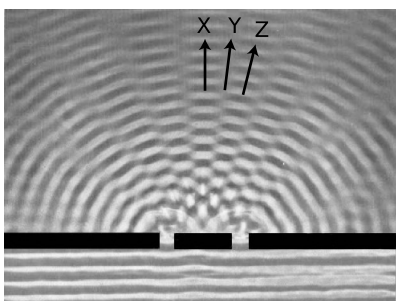


*i* / Adding up the ripples produces a new wavefront.

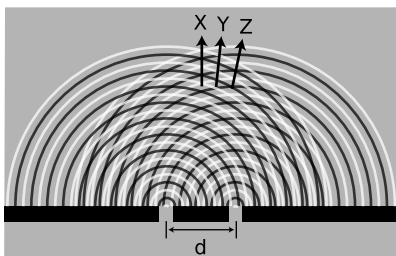




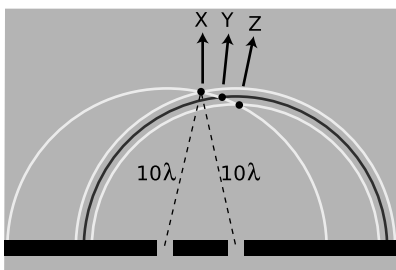
j / Thomas Young



k / Double-slit diffraction.



l / Use of Huygens' principle.



m / Constructive interference along the center-line.

a wave. Although Newton's criticisms were fair enough, the debate also took on the overtones of a nationalistic dispute between England and continental Europe, fueled by English resentment over Leibniz's supposed plagiarism of Newton's calculus. Newton wrote a book on optics, and his prestige and political prominence tended to discourage questioning of his model.

Thomas Young (1773-1829) was the person who finally, a hundred years later, did a careful search for wave interference effects with light and analyzed the results correctly. He observed double-slit diffraction of light as well as a variety of other diffraction effects, all of which showed that light exhibited wave interference effects, and that the wavelengths of visible light waves were extremely short. The crowning achievement was the demonstration by the experimentalist Heinrich Hertz and the theorist James Clerk Maxwell that light was an *electromagnetic* wave. Maxwell is said to have related his discovery to his wife one starry evening and told her that she was the only other person in the world who knew what starlight was.

## 32.5 Double-slit diffraction

Let's now analyze double-slit diffraction, k, using Huygens' principle. The most interesting question is how to compute the angles such as X and Z where the wave intensity is at a maximum, and the in-between angles like Y where it is minimized. Let's measure all our angles with respect to the vertical center line of the figure, which was the original direction of propagation of the wave.

If we assume that the width of the slits is small (on the order of the wavelength of the wave or less), then we can imagine only a single set of Huygens ripples spreading out from each one, l. White lines represent peaks, black ones troughs. The only dimension of the diffracting slits that has any effect on the geometric pattern of the overlapping ripples is then the center-to-center distance,  $d$ , between the slits.

We know from our discussion of the scaling of diffraction that there must be some equation that relates an angle like  $\theta_Z$  to the ratio  $\lambda/d$ ,

$$\frac{\lambda}{d} \leftrightarrow \theta_Z \quad .$$

If the equation for  $\theta_Z$  depended on some other expression such as  $\lambda + d$  or  $\lambda^2/d$ , then it would change when we scaled  $\lambda$  and  $d$  by the same factor, which would violate what we know about the scaling of diffraction.

Along the central maximum line, X, we always have positive waves coinciding with positive ones and negative waves coinciding with negative ones. (I have arbitrarily chosen to take a snapshot of the pattern at a moment when the waves emerging from the slit are experiencing a positive peak.) The superposition of the two sets of

ripples therefore results in a doubling of the wave amplitude along this line. There is constructive interference. This is easy to explain, because by symmetry, each wave has had to travel an equal number of wavelengths to get from its slit to the center line,  $m$ : Because both sets of ripples have ten wavelengths to cover in order to reach the point along direction X, they will be in step when they get there.

At the point along direction Y shown in the same figure, one wave has traveled ten wavelengths, and is therefore at a positive extreme, but the other has traveled only nine and a half wavelengths, so it is at a negative extreme. There is perfect cancellation, so points along this line experience no wave motion.

But the distance traveled does not have to be equal in order to get constructive interference. At the point along direction Z, one wave has gone nine wavelengths and the other ten. They are both at a positive extreme.

#### self-check B

At a point half a wavelength below the point marked along direction X, carry out a similar analysis. ▷ Answer, p. 980

To summarize, we will have perfect constructive interference at any point where the distance to one slit differs from the distance to the other slit by an integer number of wavelengths. Perfect destructive interference will occur when the number of wavelengths of path length difference equals an integer plus a half.

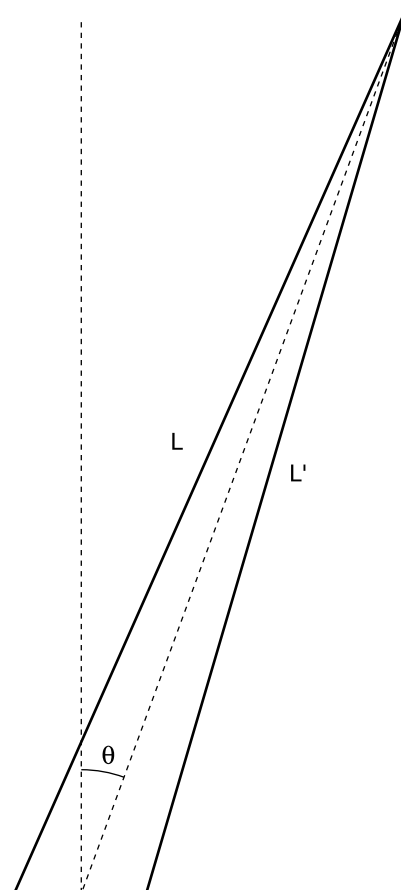
Now we are ready to find the equation that predicts the angles of the maxima and minima. The waves travel different distances to get to the same point in space,  $n$ . We need to find whether the waves are in phase (in step) or out of phase at this point in order to predict whether there will be constructive interference, destructive interference, or something in between.

One of our basic assumptions in this chapter is that we will only be dealing with the diffracted wave in regions very far away from the object that diffracts it, so the triangle is long and skinny. Most real-world examples with diffraction of light, in fact, would have triangles with even skinner proportions than this one. The two long sides are therefore very nearly parallel, and we are justified in drawing the right triangle shown in figure o, labeling one leg of the right triangle as the difference in path length,  $L - L'$ , and labeling the acute angle as  $\theta$ . (In reality this angle is a tiny bit greater than the one labeled  $\theta$  in figure n.)

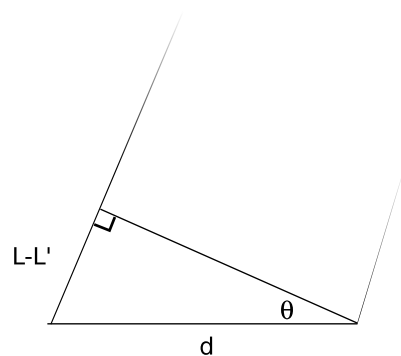
The difference in path length is related to  $d$  and  $\theta$  by the equation

$$\frac{L - L'}{d} = \sin \theta \quad .$$

Constructive interference will result in a maximum at angles for



n / The waves travel distances  $L$  and  $L'$  from the two slits to get to the same point in space, at an angle  $\theta$  from the center line.



o / A close-up view of figure n, showing how the path length difference  $L - L'$  is related to  $d$  and to the angle  $\theta$ .

which  $L - L'$  is an integer number of wavelengths,

$$L - L' = m\lambda \quad . \quad \begin{array}{l} \text{[condition for a maximum;} \\ m \text{ is an integer]} \end{array}$$

Here  $m$  equals 0 for the central maximum,  $-1$  for the first maximum to its left,  $+2$  for the second maximum on the right, etc. Putting all the ingredients together, we find  $m\lambda/d = \sin \theta$ , or

$$\frac{\lambda}{d} = \frac{\sin \theta}{m} \quad . \quad \begin{array}{l} \text{[condition for a maximum;} \\ m \text{ is an integer]} \end{array}$$

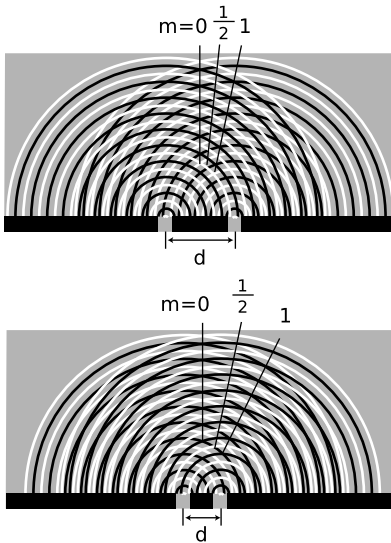
Similarly, the condition for a minimum is

$$\frac{\lambda}{d} = \frac{\sin \theta}{m} \quad . \quad \begin{array}{l} \text{[condition for a minimum;} \\ m \text{ is an integer plus } 1/2] \end{array}$$

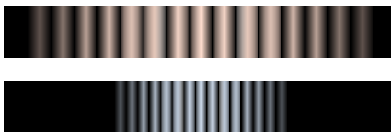
That is, the minima are about halfway between the maxima.

As expected based on scaling, this equation relates angles to the unitless ratio  $\lambda/d$ . Alternatively, we could say that we have proven the scaling property in the special case of double-slit diffraction. It was inevitable that the result would have these scaling properties, since the whole proof was geometric, and would have been equally valid when enlarged or reduced on a photocopying machine!

Counterintuitively, this means that a diffracting object with smaller dimensions produces a bigger diffraction pattern, p.



p / Cutting  $d$  in half doubles the angles of the diffraction fringes.



q / Double-slit diffraction patterns of long-wavelength red light (top) and short-wavelength blue light (bottom).

---

*Double-slit diffraction of blue and red light* *example 1*  
 Blue light has a shorter wavelength than red. For a given double-slit spacing  $d$ , the smaller value of  $\lambda/d$  leads to smaller values of  $\sin \theta$ , and therefore to a more closely spaced set of diffraction fringes, (g)

---

*The correspondence principle* *example 2*  
 Let's also consider how the equations for double-slit diffraction relate to the correspondence principle. When the ratio  $\lambda/d$  is very small, we should recover the case of simple ray optics. Now if  $\lambda/d$  is small,  $\sin \theta$  must be small as well, and the spacing between the diffraction fringes will be small as well. Although we have not proven it, the central fringe is always the brightest, and the fringes get dimmer and dimmer as we go farther from it. For small values

of  $\lambda/d$ , the part of the diffraction pattern that is bright enough to be detectable covers only a small range of angles. This is exactly what we would expect from ray optics: the rays passing through the two slits would remain parallel, and would continue moving in the  $\theta = 0$  direction. (In fact there would be images of the two separate slits on the screen, but our analysis was all in terms of angles, so we should not expect it to address the issue of whether there is structure within a set of rays that are all traveling in the  $\theta = 0$  direction.)

---

*Spacing of the fringes at small angles* *example 3*

At small angles, we can use the approximation  $\sin \theta \approx \theta$ , which is valid if  $\theta$  is measured in radians. The equation for double-slit diffraction becomes simply

$$\frac{\lambda}{d} = \frac{\theta}{m} \quad ,$$

which can be solved for  $\theta$  to give

$$\theta = \frac{m\lambda}{d} \quad .$$

The difference in angle between successive fringes is the change in  $\theta$  that results from changing  $m$  by plus or minus one,

$$\Delta\theta = \frac{\lambda}{d} \quad .$$

For example, if we write  $\theta_7$  for the angle of the seventh bright fringe on one side of the central maximum and  $\theta_8$  for the neighboring one, we have

$$\begin{aligned} \theta_8 - \theta_7 &= \frac{8\lambda}{d} - \frac{7\lambda}{d} \\ &= \frac{\lambda}{d} \quad , \end{aligned}$$

and similarly for any other neighboring pair of fringes.

Although the equation  $\lambda/d = \sin \theta/m$  is only valid for a double slit, it can still be a guide to our thinking even if we are observing diffraction of light by a virus or a flea's leg: it is always true that

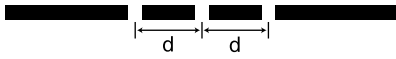
- (1) large values of  $\lambda/d$  lead to a broad diffraction pattern, and
- (2) diffraction patterns are repetitive.

In many cases the equation looks just like  $\lambda/d = \sin \theta/m$  but with an extra numerical factor thrown in, and with  $d$  interpreted as some other dimension of the object, e.g., the diameter of a piece of wire.

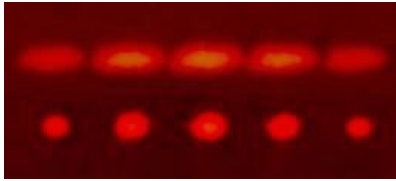
## 32.6 Repetition

Suppose we replace a double slit with a triple slit, *r*. We can think of this as a third repetition of the structures that were present in the double slit. Will this device be an improvement over the double slit for any practical reasons?

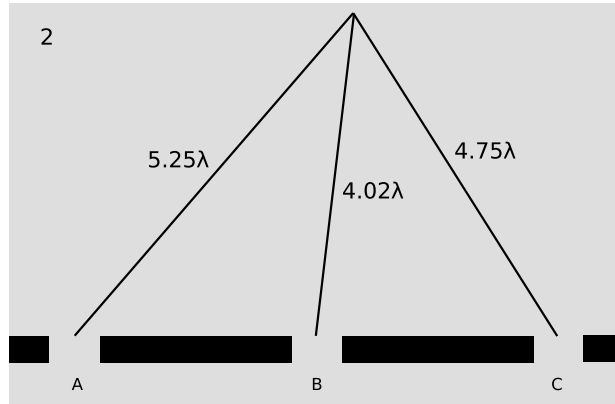
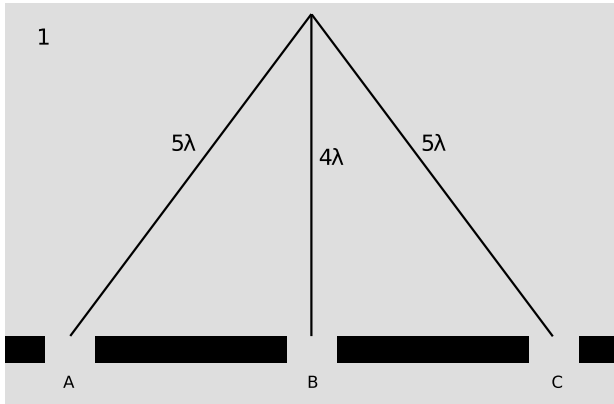
The answer is yes, as can be shown using figure *t*. For ease of visualization, I have violated our usual rule of only considering points very far from the diffracting object. The scale of the drawing is such that a wavelength is one cm. In *t/1*, all three waves travel an integer number of wavelengths to reach the same point, so there is a bright central spot, as we would expect from our experience with the double slit. In figure *t/2*, we show the path lengths to a new point. This point is farther from slit A by a quarter of a wavelength, and correspondingly closer to slit C. The distance from slit B has hardly changed at all. Because the paths lengths traveled from slits A and C differ by half a wavelength, there will be perfect destructive interference between these two waves. There is still some uncanceled wave intensity because of slit B, but the amplitude will be three times less than in figure *t/1*, resulting in a factor of 9 decrease in brightness. Thus, by moving off to the right a little, we have gone from the bright central maximum to a point that is quite dark.



*r* / A triple slit.



*s* / A double-slit diffraction pattern (top), and a pattern made by five slits (bottom).



*t* / 1. There is a bright central maximum. 2. At this point just off the central maximum, the path lengths traveled by the three waves have changed.

Now let's compare with what would have happened if slit C had been covered, creating a plain old double slit. The waves coming from slits A and B would have been out of phase by 0.23 wavelengths, but this would not have caused very severe interference. The point in figure *t/2* would have been quite brightly lit up.

To summarize, we have found that adding a third slit narrows down the central fringe dramatically. The same is true for all the other fringes as well, and since the same amount of energy is concentrated in narrower diffraction fringes, each fringe is brighter and easier to

see, s.

This is an example of a more general fact about diffraction: if some feature of the diffracting object is repeated, the locations of the maxima and minima are unchanged, but they become narrower.

Taking this reasoning to its logical conclusion, a diffracting object with thousands of slits would produce extremely narrow fringes. Such an object is called a diffraction grating.

## 32.7 Single-slit diffraction

If we use only a single slit, is there diffraction? If the slit is not wide compared to a wavelength of light, then we can approximate its behavior by using only a single set of Huygens ripples. There are no other sets of ripples to add to it, so there are no constructive or destructive interference effects, and no maxima or minima. The result will be a uniform spherical wave of light spreading out in all directions, like what we would expect from a tiny lightbulb. We could call this a diffraction pattern, but it is a completely featureless one, and it could not be used, for instance, to determine the wavelength of the light, as other diffraction patterns could.

All of this, however, assumes that the slit is narrow compared to a wavelength of light. If, on the other hand, the slit is broader, there will indeed be interference among the sets of ripples spreading out from various points along the opening. Figure u shows an example with water waves, and figure v with light.

### *self-check C*

How does the wavelength of the waves compare with the width of the slit in figure u?

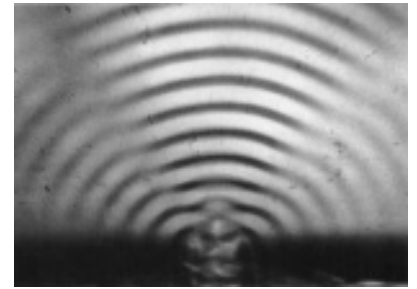
▷ Answer, p. 980

We will not go into the details of the analysis of single-slit diffraction, but let us see how its properties can be related to the general things we've learned about diffraction. We know based on scaling arguments that the angular sizes of features in the diffraction pattern must be related to the wavelength and the width,  $a$ , of the slit by some relationship of the form

$$\frac{\lambda}{a} \leftrightarrow \theta \quad .$$

This is indeed true, and for instance the angle between the maximum of the central fringe and the maximum of the next fringe on one side equals  $1.5\lambda/a$ . Scaling arguments will never produce factors such as the 1.5, but they tell us that the answer must involve  $\lambda/a$ , so all the familiar qualitative facts are true. For instance, shorter-wavelength light will produce a more closely spaced diffraction pattern.

An important scientific example of single-slit diffraction is in telescopes. Images of individual stars, as in figure x, are a good way to



u / Single-slit diffraction of water waves.



v / Single-slit diffraction of red light. Note the double width of the central maximum.



w / A pretty good simulation of the single-slit pattern of figure u, made by using three motors to produce overlapping ripples from three neighboring points in the water.



x / An image of the Pleiades star cluster. The circular rings around the bright stars are due to single-slit diffraction at the mouth of the telescope's tube.



y / A radio telescope.

examine diffraction effects, because all stars except the sun are so far away that no telescope, even at the highest magnification, can image their disks or surface features. Thus any features of a star's image must be due purely to optical effects such as diffraction. A prominent cross appears around the brightest star, and dimmer ones surround the dimmer stars. Something like this is seen in most telescope photos, and indicates that inside the tube of the telescope there were two perpendicular struts or supports. Light diffracted around these struts. You might think that diffraction could be eliminated entirely by getting rid of all obstructions in the tube, but the circles around the stars are diffraction effects arising from single-slit diffraction at the mouth of the telescope's tube! (Actually we have not even talked about diffraction through a circular opening, but the idea is the same.) Since the angular sizes of the diffracted images depend on  $\lambda/a$ , the only way to improve the resolution of the images is to increase the diameter,  $a$ , of the tube. This is one of the main reasons (in addition to light-gathering power) why the best telescopes must be very large in diameter.

#### self-check D

What would this imply about radio telescopes as compared with visible-light telescopes?

▷ Answer, p.

980

Double-slit diffraction is easier to understand conceptually than single-slit diffraction, but if you do a double-slit diffraction experiment in real life, you are likely to encounter a complicated pattern like figure z/1, rather than the simpler one, 2, you were expecting. This is because the slits are fairly big compared to the wavelength of the light being used. We really have two different distances in our pair of slits:  $d$ , the distance between the slits, and  $w$ , the width of each slit. Remember that smaller distances on the object the light diffracts around correspond to larger features of the diffraction pattern. The pattern 1 thus has two spacings in it: a short spacing corresponding to the large distance  $d$ , and a long spacing that relates to the small dimension  $w$ .

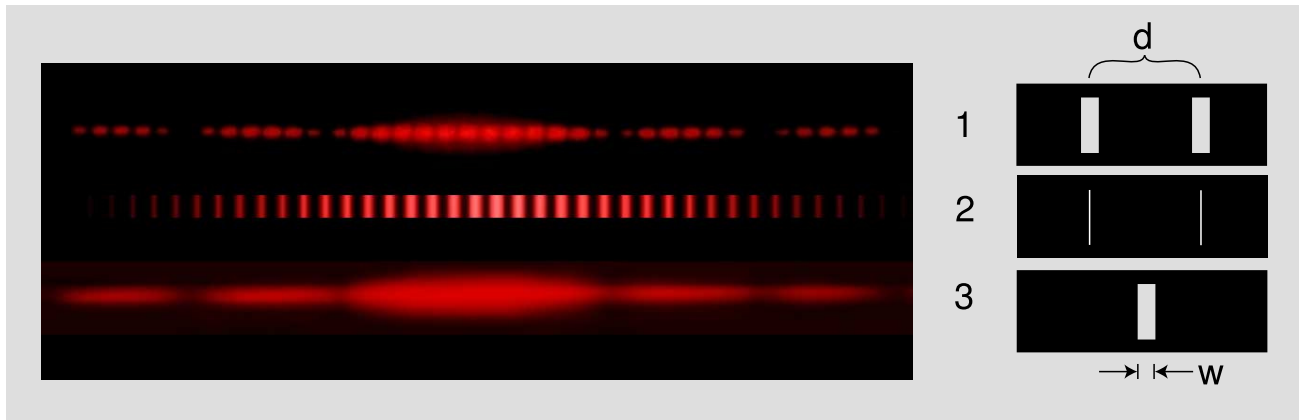
### Discussion question

**A** Why is it optically impossible for bacteria to evolve eyes that use visible light to form images?

## 32.8 $\int$ ★ The principle of least time

In section 28.5 and 31.4, we saw how in the ray model of light, both refraction and reflection can be described in an elegant and beautiful way by a single principle, the principle of least time. We can now justify the principle of least time based on the wave model of light. Consider an example involving reflection, aa. Starting at





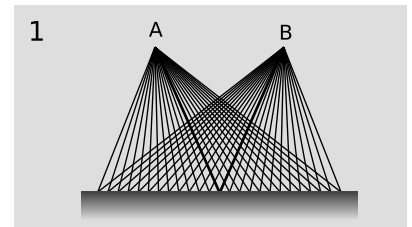
z / 1. A diffraction pattern formed by a real double slit. The width of each slit is fairly big compared to the wavelength of the light. This is a real photo. 2. This idealized pattern is not likely to occur in real life. To get it, you would need each slit to be so narrow that its width was comparable to the wavelength of the light, but that's not usually possible. This is not a real photo. 3. A real photo of a single-slit diffraction pattern caused by a slit whose width is the same as the widths of the slits used to make the top pattern.

point A, Huygens' principle for waves tells us that we can think of the wave as spreading out in all directions. Suppose we imagine all the possible ways that a ray could travel from A to B. We show this by drawing 25 possible paths, of which the central one is the shortest. Since the principle of least time connects the wave model to the ray model, we should expect to get the most accurate results when the wavelength is much shorter than the distances involved — for the sake of this numerical example, let's say that a wavelength is 1/10 of the shortest reflected path from A to B. The table, 2, shows the distances traveled by the 25 rays.

Note how similar are the distances traveled by the group of 7 rays, indicated with a bracket, that come closest to obeying the principle of least time. If we think of each one as a wave, then all 7 are again nearly in phase at point B. However, the rays that are farther from satisfying the principle of least time show more rapidly changing distances; on reuniting at point B, their phases are a random jumble, and they will very nearly cancel each other out. Thus, almost none of the wave energy delivered to point B goes by these longer paths. Physically we find, for instance, that a wave pulse emitted at A is observed at B after a time interval corresponding very nearly to the shortest possible path, and the pulse is not very "smeared out" when it gets there. The shorter the wavelength compared to the dimensions of the figure, the more accurate these approximate statements become.

Instead of drawing a finite number of rays, such 25, what happens if we think of the angle,  $\theta$ , of emission of the ray as a continuously varying variable? Minimizing the distance  $L$  requires

$$\frac{dL}{d\theta} = 0 \quad .$$



1	
2	12.25
	11.91
	11.59
	11.30
	11.03
	10.80
	10.59
	10.41
	10.26
	10.15
	10.07
	10.02
	10.00
	10.02
	10.07
	10.15
	10.26
	10.41
	10.59
	10.80
	11.03
	11.30
	11.59
	11.91
	12.25

aa / Light could take many different paths from A to B.



Because  $L$  is changing slowly in the vicinity of the angle that satisfies the principle of least time, all the rays that come out close to this angle have very nearly the same  $L$ , and remain very nearly in phase when they reach B. This is the basic reason why the discrete table, aa/2, turned out to have a group of rays that all traveled nearly the same distance.

As discussed in section 28.5, the principle of least time is really a principle of least *or greatest* time. This makes perfect sense, since  $dL/d\theta = 0$  can in general describe either a minimum or a maximum

The principle of least time is very general. It does not apply just to refraction and reflection — it can even be used to prove that light rays travel in a straight line through empty space, without taking detours! This general approach to wave motion was used by Richard Feynman, one of the pioneers who in the 1950's reconciled quantum mechanics with relativity. A very readable explanation is given in a book Feynman wrote for laypeople, QED: The Strange Theory of Light and Matter.

## Summary

### Selected vocabulary

- diffraction . . . . . the behavior of a wave when it encounters an obstacle or a nonuniformity in its medium; in general, diffraction causes a wave to bend around obstacles and make patterns of strong and weak waves radiating out beyond the obstacle.
- coherent . . . . . a light wave whose parts are all in phase with each other

### Other terminology and notation

- wavelets . . . . . the ripples in Huygens' principle

### Summary

Wave optics is a more general theory of light than ray optics. When light interacts with material objects that are much larger than one wavelength of the light, the ray model of light is approximately correct, but in other cases the wave model is required.

Huygens' principle states that, given a wavefront at one moment in time, the future behavior of the wave can be found by breaking the wavefront up into a large number of small, side-by-side wave peaks, each of which then creates a pattern of circular or spherical ripples. As these sets of ripples add together, the wave evolves and moves through space. Since Huygens' principle is a purely geometrical construction, diffraction effects obey a simple scaling rule: the behavior is unchanged if the wavelength and the dimensions of the diffracting objects are both scaled up or down by the same factor. If we wish to predict the angles at which various features of the diffraction pattern radiate out, scaling requires that these angles depend only on the unitless ratio  $\lambda/d$ , where  $d$  is the size of some feature of the diffracting object.

Double-slit diffraction is easily analyzed using Huygens' principle if the slits are narrower than one wavelength. We need only construct two sets of ripples, one spreading out from each slit. The angles of the maxima (brightest points in the bright fringes) and minima (darkest points in the dark fringes) are given by the equation

$$\frac{\lambda}{d} = \frac{\sin \theta}{m} \quad ,$$

where  $d$  is the center-to-center spacing of the slits, and  $m$  is an integer at a maximum or an integer plus 1/2 at a minimum.

If some feature of a diffracting object is repeated, the diffraction fringes remain in the same places, but become narrower with each repetition. By repeating a double-slit pattern hundreds or thousands of times, we obtain a diffraction grating.

A single slit can produce diffraction fringes if it is larger than one

wavelength. Many practical instances of diffraction can be interpreted as single-slit diffraction, e.g., diffraction in telescopes. The main thing to realize about single-slit diffraction is that it exhibits the same kind of relationship between  $\lambda$ ,  $d$ , and angles of fringes as in any other type of diffraction.

## Problems

### Key

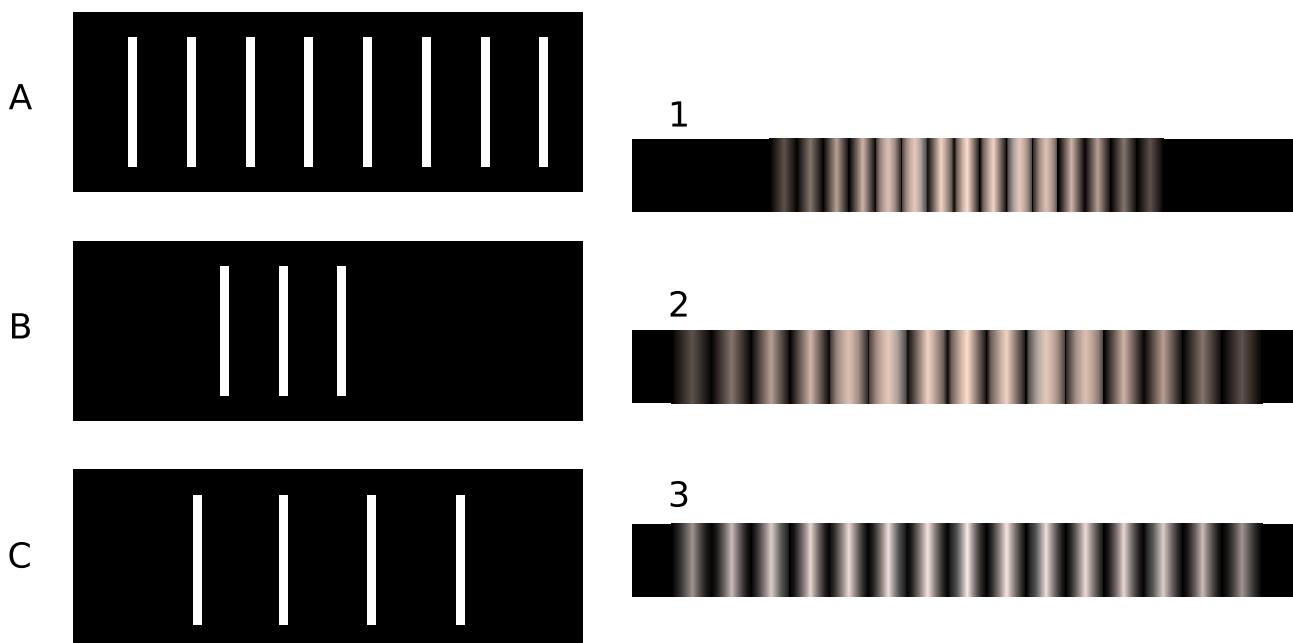
✓ A computerized answer check is available online.

∫ A problem that requires calculus.

★ A difficult problem.

1 Why would blue or violet light be the best for microscopy?

2 Match gratings A-C with the diffraction patterns 1-3 that they produce. Explain.



3 The beam of a laser passes through a diffraction grating, fans out, and illuminates a wall that is perpendicular to the original beam, lying at a distance of 2.0 m from the grating. The beam is produced by a helium-neon laser, and has a wavelength of 694.3 nm. The grating has 2000 lines per centimeter. (a) What is the distance on the wall between the central maximum and the maxima immediately to its right and left? (b) How much does your answer change when you use the small-angle approximations  $\theta \approx \sin \theta \approx \tan \theta$ ? ✓

4 When white light passes through a diffraction grating, what is the smallest value of  $m$  for which the visible spectrum of order  $m$  overlaps the next one, of order  $m + 1$ ? (The visible spectrum runs from about 400 nm to about 700 nm.)

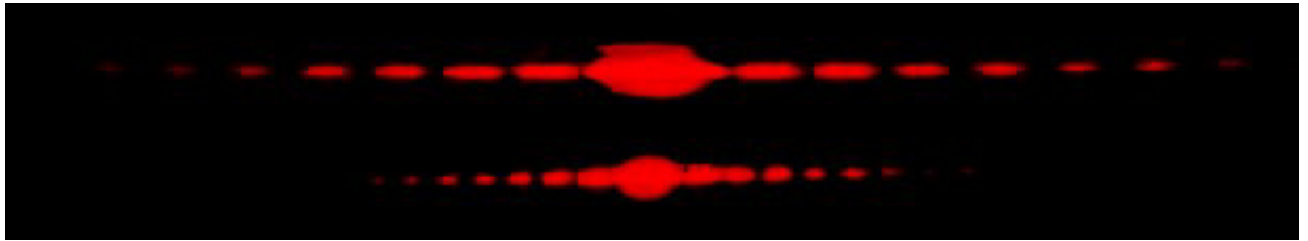
**5**     Ultrasound, i.e., sound waves with frequencies too high to be audible, can be used for imaging fetuses in the womb or for breaking up kidney stones so that they can be eliminated by the body. Consider the latter application. Lenses can be built to focus sound waves, but because the wavelength of the sound is not all that small compared to the diameter of the lens, the sound will not be concentrated exactly at the geometrical focal point. Instead, a diffraction pattern will be created with an intense central spot surrounded by fainter rings. About 85% of the power is concentrated within the central spot. The angle of the first minimum (surrounding the central spot) is given by  $\sin \theta = \lambda/b$ , where  $b$  is the diameter of the lens. This is similar to the corresponding equation for a single slit, but with a factor of 1.22 in front which arises from the circular shape of the aperture. Let the distance from the lens to the patient's kidney stone be  $L = 20$  cm. You will want  $f > 20$  kHz, so that the sound is inaudible. Find values of  $b$  and  $f$  that would result in a usable design, where the central spot is small enough to lie within a kidney stone 1 cm in diameter.

**6**     For star images such as the ones in figure x, estimate the angular width of the diffraction spot due to diffraction at the mouth of the telescope. Assume a telescope with a diameter of 10 meters (the largest currently in existence), and light with a wavelength in the middle of the visible range. Compare with the actual angular size of a star of diameter  $10^9$  m seen from a distance of  $10^{17}$  m. What does this tell you?

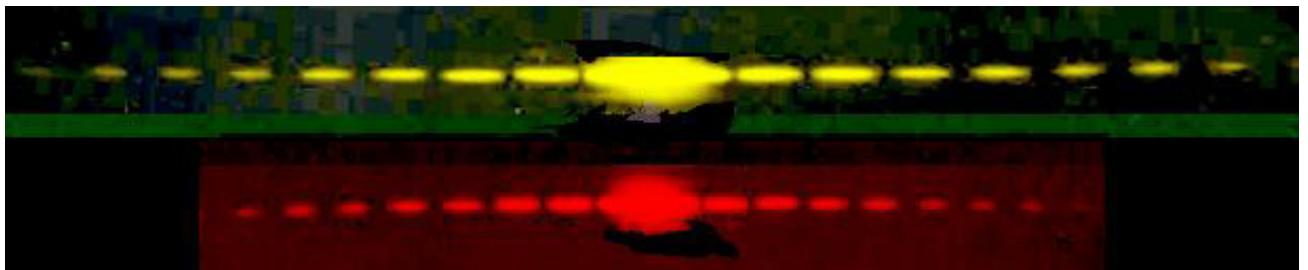
**7**     Under what circumstances could one get a mathematically undefined result by solving the double-slit diffraction equation for  $\theta$ ? Give a physical interpretation of what would actually be observed.

**8**     When ultrasound is used for medical imaging, the frequency may be as high as 5-20 MHz. Another medical application of ultrasound is for therapeutic heating of tissues inside the body; here, the frequency is typically 1-3 MHz. What fundamental physical reasons could you suggest for the use of higher frequencies for imaging?

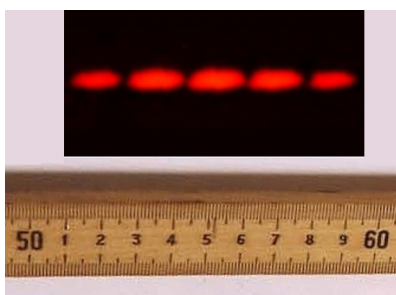
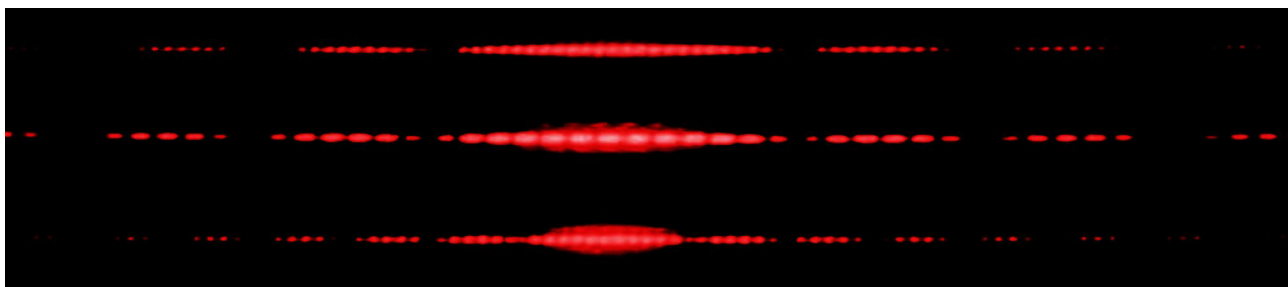
**9** The figure below shows two diffraction patterns, both made with the same wavelength of red light. (a) What type of slits made the patterns? Is it a single slit, double slits, or something else? Explain. (b) Compare the dimensions of the slits used to make the top and bottom pattern. Give a numerical ratio, and state which way the ratio is, i.e., which slit pattern was the larger one. Explain.



**10** The figure below shows two diffraction patterns. The top one was made with yellow light, and the bottom one with red. Could the slits used to make the two patterns have been the same?



**11** The figure below shows three diffraction patterns. All were made under identical conditions, except that a different set of double slits was used for each one. The slits used to make the top pattern had a center-to-center separation  $d = 0.50$  mm, and each slit was  $w = 0.04$  mm wide. (a) Determine  $d$  and  $w$  for the slits used to make the pattern in the middle. (b) Do the same for the slits used to make the bottom pattern.



Problems 12 and 13.

**12** The figure shows a diffraction pattern made by a double slit, along with an image of a meter stick to show the scale. The slits were 146 cm away from the screen on which the diffraction pattern was projected. The spacing of the slits was 0.050 mm. What was the wavelength of the light?

**13** The figure shows a diffraction pattern made by a double slit, along with an image of a meter stick to show the scale. Sketch the diffraction pattern from the figure on your paper. Now consider the four variables in the equation  $\lambda/d = \sin \theta/m$ . Which of these are the same for all five fringes, and which are different for each fringe? Which variable would you naturally use in order to label which fringe was which? Label the fringes on your sketch using the values of that variable.

**14** Figure r on p. 846 shows the anatomy of a jumping spider's principal eye. The smallest feature the spider can distinguish is limited by the size of the receptor cells in its retina. (a) By making measurements on the diagram, estimate this limiting angular size in units of minutes of arc (60 minutes = 1 degree). (b) Show that this is greater than, but roughly in the same ballpark as, the limit imposed by diffraction for visible light.

*Remark:* Evolution is a scientific theory that makes testable predictions, and if observations contradict its predictions, the theory can be disproved. It would be maladaptive for the spider to have retinal receptor cells with sizes much less than the limit imposed by diffraction, since it would increase complexity without giving any improvement in visual acuity. The results of this problem confirm that, as predicted by Darwinian evolution, this is not the case. Work by M.F. Land in 1969 shows that in this spider's eye, aberration is a somewhat bigger effect than diffraction, so that the size of the receptors is very nearly at

an evolutionary optimum.

✓ ✓



## Exercise 32A: Double-source interference

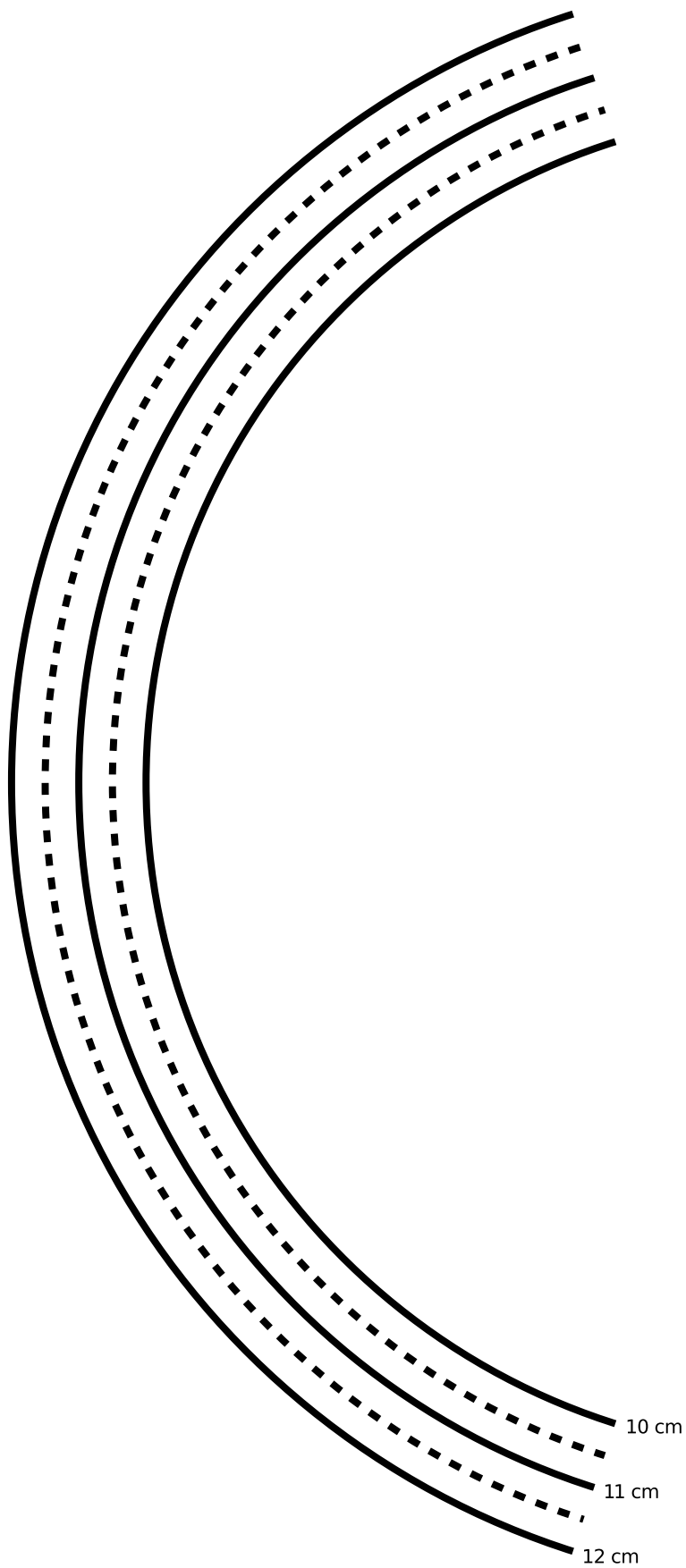
1. Two sources separated by a distance  $d = 2$  cm make circular ripples with a wavelength of  $\lambda = 1$  cm. On a piece of paper, make a life-size drawing of the two sources in the default setup, and locate the following points:

- A. The point that is 10 wavelengths from source #1 and 10 wavelengths from source #2.
- B. The point that is 10.5 wavelengths from #1 and 10.5 from #2.
- C. The point that is 11 wavelengths from #1 and 11 from #2.
- D. The point that is 10 wavelengths from #1 and 10.5 from #2.
- E. The point that is 11 wavelengths from #1 and 11.5 from #2.
- F. The point that is 10 wavelengths from #1 and 11 from #2.
- G. The point that is 11 wavelengths from #1 and 12 from #2.

You can do this either using a compass or by putting the next page under your paper and tracing. It is not necessary to trace all the arcs completely, and doing so is unnecessarily time-consuming; you can fairly easily estimate where these points would lie, and just trace arcs long enough to find the relevant intersections.

What do these points correspond to in the real wave pattern?

- 2. Make a fresh copy of your drawing, showing only point F and the two sources, which form a long, skinny triangle. Now suppose you were to change the setup by doubling  $d$ , while leaving  $\lambda$  the same. It's easiest to understand what's happening on the drawing if you move both sources outward, keeping the center fixed. Based on your drawing, what will happen to the position of point F when you double  $d$ ? Measure its angle with a protractor.
- 3. What would happen if you doubled *both*  $\lambda$  and  $d$  compared to the standard setup?-----
- 4. Combining the ideas from parts 2 and 3, what do you think would happen to your angles if, starting from the standard setup, you doubled  $\lambda$  while leaving  $d$  the same?-----
- 5. Suppose  $\lambda$  was a millionth of a centimeter, while  $d$  was still as in the standard setup. What would happen to the angles? What does this tell you about observing diffraction of light?



## Exercise 32B: Single-slit diffraction

Equipment:

rulers

computer with web browser

The following page is a diagram of a single slit and a screen onto which its diffraction pattern is projected. The class will make a numerical prediction of the intensity of the pattern at the different points on the screen. Each group will be responsible for calculating the intensity at one of the points. (Either 11 groups or six will work nicely – in the latter case, only points a, c, e, g, i, and k are used.) The idea is to break up the wavefront in the mouth of the slit into nine parts, each of which is assumed to radiate semicircular ripples as in Huygens' principle. The wavelength of the wave is 1 cm, and we assume for simplicity that each set of ripples has an amplitude of 1 unit when it reaches the screen.

1. For simplicity, let's imagine that we were only to use two sets of ripples rather than nine. You could measure the distance from each of the two points inside the slit to your point on the screen. Suppose the distances were both 25.0 cm. What would be the amplitude of the superimposed waves at this point on the screen?

Suppose one distance was 24.0 cm and the other was 25.0 cm. What would happen?

What if one was 24.0 cm and the other was 26.0 cm?

What if one was 24.5 cm and the other was 25.0 cm?

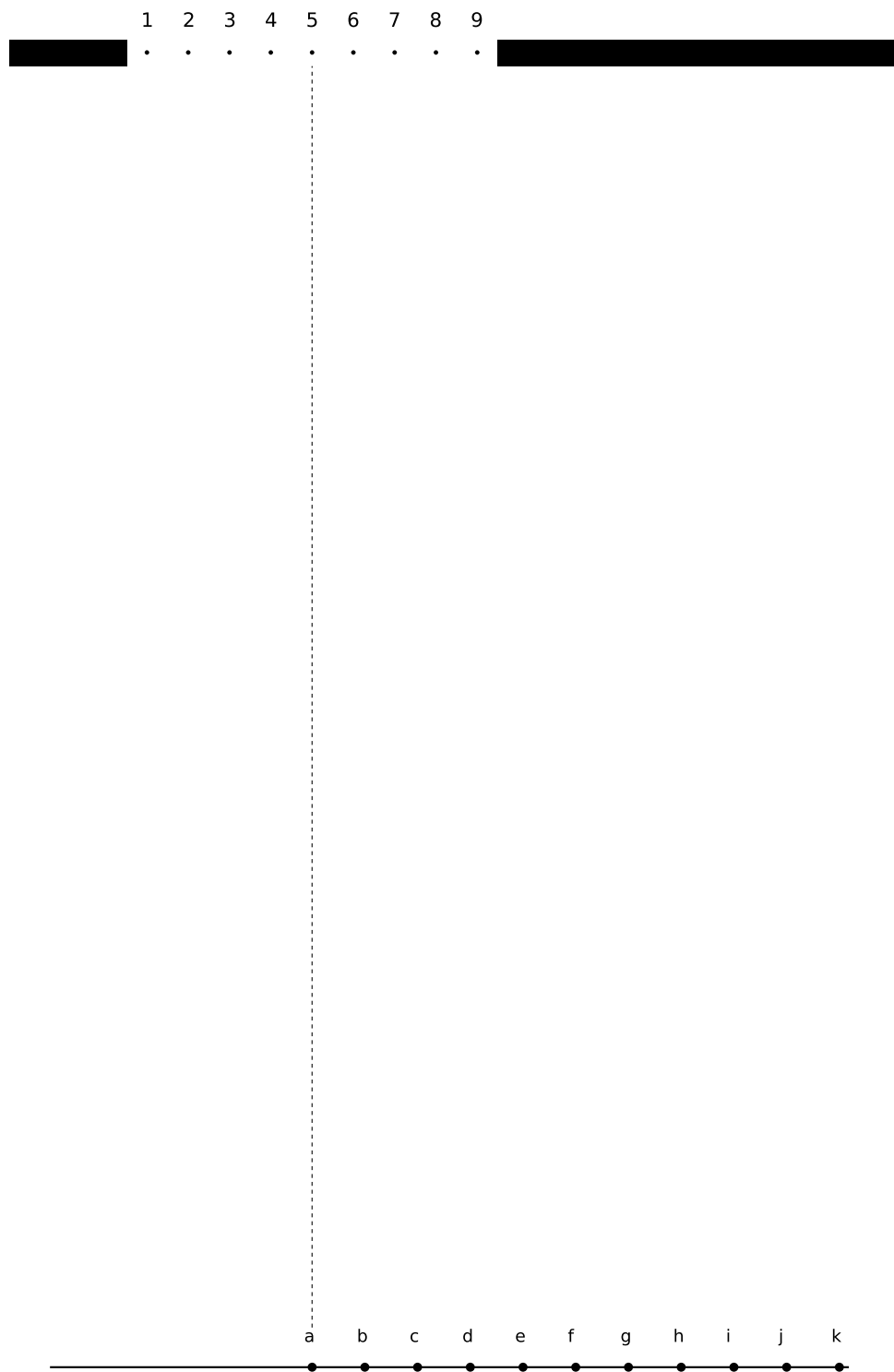
In general, what combinations of distances will lead to completely destructive and completely constructive interference?

Can you estimate the answer in the case where the distances are 24.7 and 25.0 cm?

2. Although it is possible to calculate mathematically the amplitude of the sine wave that results from superimposing two sine waves with an arbitrary phase difference between them, the algebra is rather laborious, and it becomes even more tedious when we have more than two waves to superimpose. Instead, one can simply use a computer spreadsheet or some other computer program to add up the sine waves numerically at a series of points covering one complete cycle. This is what we will actually do. You just need to enter the relevant data into the computer, then examine the results and pick off the amplitude from the resulting list of numbers. You can run the software through a web interface at <http://lightandmatter.com/cgi-bin/diffraction1.cgi>.

3. Measure all nine distances to your group's point on the screen, and write them on the board - that way everyone can see everyone else's data, and the class can try to make sense of why the results came out the way they did. Determine the amplitude of the combined wave, and write it on the board as well.

The class will discuss why the results came out the way they did.



## Exercise 32C: Diffraction of light

Equipment:

slit patterns, lasers, straight-filament bulbs

### station 1

You have a mask with a bunch of different double slits cut out of it. The values of  $w$  and  $d$  are as follows:

pattern A	$w=0.04$ mm	$d=.250$ mm
pattern B	$w=0.04$ mm	$d=.500$ mm
pattern C	$w=0.08$ mm	$d=.250$ mm
pattern D	$w=0.08$ mm	$d=.500$ mm

Predict how the patterns will look different, and test your prediction. The easiest way to get the laser to point at different sets of slits is to stick folded up pieces of paper in one side or the other of the holders.

### station 2

This is just like station 1, but with single slits:

pattern A	$w=0.02$ mm
pattern B	$w=0.04$ mm
pattern C	$w=0.08$ mm
pattern D	$w=0.16$ mm

Predict what will happen, and test your predictions. If you have time, check the actual numerical ratios of the  $w$  values against the ratios of the sizes of the diffraction patterns

### station 3

This is like station 1, but the only difference among the sets of slits is how many slits there are:

pattern A	double slit
pattern B	3 slits
pattern C	4 slits
pattern D	5 slits

### station 4

Hold the diffraction grating up to your eye, and look through it at the straight-filament light bulb. If you orient the grating correctly, you should be able to see the  $m = 1$  and  $m = -1$  diffraction patterns off the left and right. If you have it oriented the wrong way, they'll be above and below the bulb instead, which is inconvenient because the bulb's filament is vertical. Where is the  $m = 0$  fringe? Can you see  $m = 2$ , etc.?

*Station 5 has the same equipment as station 4. If you're assigned to station 5 first, you should actually do activity 4 first, because it's easier.*

### station 5

Use the transformer to increase and decrease the voltage across the bulb. This allows you to control the filament's temperature. Sketch graphs of intensity as a function of wavelength for various temperatures. The inability of the wave model of light to explain the mathematical shapes of these curves was historically one of the reasons for creating a new model, in which light is both a particle and a wave.

# **The modern revolution in physics**





The continental U.S. got its first taste of volcanism in recent memory with the eruption of Mount St. Helens in 1980.

## Chapter 33

# Rules of randomness

Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the things which compose it...nothing would be uncertain, and the future as the past would be laid out before its eyes.

*Pierre Simon de Laplace, 1776*

The Quantum Mechanics is very imposing. But an inner voice tells me that it is still not the final truth. The theory yields much, but it hardly brings us nearer to the secret of the Old



One. In any case, I am convinced that He does not play dice.  
*Albert Einstein*

However radical Newton's clockwork universe seemed to his contemporaries, by the early twentieth century it had become a sort of smugly accepted dogma. Luckily for us, this deterministic picture of the universe breaks down at the atomic level. The clearest demonstration that the laws of physics contain elements of randomness is in the behavior of radioactive atoms. Pick two identical atoms of a radioactive isotope, say the naturally occurring uranium 238, and watch them carefully. They will decay at different times, even though there was no difference in their initial behavior.

We would be in big trouble if these atoms' behavior was as predictable as expected in the Newtonian world-view, because radioactivity is an important source of heat for our planet. In reality, each atom chooses a random moment at which to release its energy, resulting in a nice steady heating effect. The earth would be a much colder planet if only sunlight heated it and not radioactivity. Probably there would be no volcanoes, and the oceans would never have been liquid. The deep-sea geothermal vents in which life first evolved would never have existed. But there would be an even worse consequence if radioactivity was deterministic: after a few billion years of peace, all the uranium 238 atoms in our planet would presumably pick the same moment to decay. The huge amount of stored nuclear energy, instead of being spread out over eons, would all be released at one instant, blowing our whole planet to Kingdom Come.<sup>1</sup>

The new version of physics, incorporating certain kinds of randomness, is called quantum physics (for reasons that will become clear later). It represented such a dramatic break with the previous, deterministic tradition that everything that came before is considered "classical," even the theory of relativity. The remainder of this book is a basic introduction to quantum physics.

### Discussion question

**A** I said "Pick two identical atoms of a radioactive isotope." Are two atoms really identical? If their electrons are orbiting the nucleus, can we distinguish each atom by the particular arrangement of its electrons at some instant in time?

---

<sup>1</sup>This is under the assumption that all the radioactive heating comes from uranium atoms, and that all the atoms were created at the same time. In reality, both uranium and thorium atoms contribute, and they may not have all been created at the same time. We have only a general idea of the processes that created these heavy elements in the gas cloud from which our solar system condensed. Some portion of them may have come from nuclear reactions in supernova explosions in that particular nebula, but some may have come from previous supernova explosions throughout our galaxy, or from exotic events like collisions of white dwarf stars.

## 33.1 Randomness isn't random

Einstein's distaste for randomness, and his association of determinism with divinity, goes back to the Enlightenment conception of the universe as a gigantic piece of clockwork that only had to be set in motion initially by the Builder. Many of the founders of quantum mechanics were interested in possible links between physics and Eastern and Western religious and philosophical thought, but every educated person has a different concept of religion and philosophy. Bertrand Russell remarked, "Sir Arthur Eddington deduces religion from the fact that atoms do not obey the laws of mathematics. Sir James Jeans deduces it from the fact that they do."

Russell's witticism, which implies incorrectly that mathematics cannot describe randomness, reminds us how important it is not to oversimplify this question of randomness. You should not simply surmise, "Well, it's all random, anything can happen." For one thing, certain things simply cannot happen, either in classical physics or quantum physics. The conservation laws of mass, energy, momentum, and angular momentum are still valid, so for instance processes that create energy out of nothing are not just unlikely according to quantum physics, they are impossible.

A useful analogy can be made with the role of randomness in evolution. Darwin was not the first biologist to suggest that species changed over long periods of time. His two new fundamental ideas were that (1) the changes arose through random genetic variation, and (2) changes that enhanced the organism's ability to survive and reproduce would be preserved, while maladaptive changes would be eliminated by natural selection. Doubters of evolution often consider only the first point, about the randomness of natural variation, but not the second point, about the systematic action of natural selection. They make statements such as, "the development of a complex organism like *Homo sapiens* via random chance would be like a whirlwind blowing through a junkyard and spontaneously assembling a jumbo jet out of the scrap metal." The flaw in this type of reasoning is that it ignores the deterministic constraints on the results of random processes. For an atom to violate conservation of energy is no more likely than the conquest of the world by chimpanzees next year.

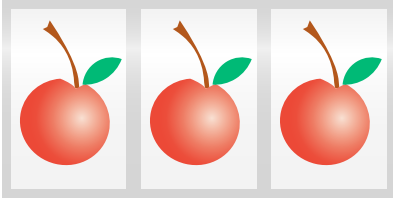
### Discussion question

**A** Economists often behave like wannabe physicists, probably because it seems prestigious to make numerical calculations instead of talking about human relationships and organizations like other social scientists. Their striving to make economics work like Newtonian physics extends to a parallel use of mechanical metaphors, as in the concept of a market's supply and demand acting like a self-adjusting machine, and the idealization of people as economic automatons who consistently strive to maximize their own wealth. What evidence is there for randomness rather than mechanical determinism in economics?

## 33.2 Calculating randomness

You should also realize that even if something is random, we can still understand it, and we can still calculate probabilities numerically. In other words, physicists are good bookmakers. A good bookie can calculate the odds that a horse will win a race much more accurately than an inexperienced one, but nevertheless cannot predict what will happen in any particular race.

### Statistical independence



a / The probability that one wheel will give a cherry is  $1/10$ . The probability that all three wheels will give cherries is  $1/10 \times 1/10 \times 1/10$ .

As an illustration of a general technique for calculating odds, suppose you are playing a 25-cent slot machine. Each of the three wheels has one chance in ten of coming up with a cherry. If all three wheels come up cherries, you win \$100. Even though the results of any particular trial are random, you can make certain quantitative predictions. First, you can calculate that your odds of winning on any given trial are  $1/10 \times 1/10 \times 1/10 = 1/1000 = 0.001$ . Here, I am representing the probabilities as numbers from 0 to 1, which is clearer than statements like “The odds are 999 to 1,” and makes the calculations easier. A probability of 0 represents something impossible, and a probability of 1 represents something that will definitely happen.

Also, you can say that any given trial is equally likely to result in a win, and it doesn’t matter whether you have won or lost in prior games. Mathematically, we say that each trial is statistically independent, or that separate games are uncorrelated. Most gamblers are mistakenly convinced that, to the contrary, games of chance are correlated. If they have been playing a slot machine all day, they are convinced that it is “getting ready to pay,” and they do not want anyone else playing the machine and “using up” the jackpot that they “have coming.” In other words, they are claiming that a series of trials at the slot machine is negatively correlated, that losing now makes you more likely to win later. Craps players claim that you should go to a table where the person rolling the dice is “hot,” because she is likely to keep on rolling good numbers. Craps players, then, believe that rolls of the dice are positively correlated, that winning now makes you more likely to win later.

My method of calculating the probability of winning on the slot machine was an example of the following important rule for calculations based on independent probabilities:

#### the law of independent probabilities

If the probability of one event happening is  $P_A$ , and the probability of a second statistically independent event happening is  $P_B$ , then the probability that they will both occur is the product of the probabilities,  $P_AP_B$ . If there are more than two events involved, you simply keep on multiplying.

Note that this only applies to independent probabilities. For instance, if you have a nickel and a dime in your pocket, and you randomly pull one out, there is a probability of 0.5 that it will be the nickel. If you then replace the coin and again pull one out randomly, there is again a probability of 0.5 of coming up with the nickel, because the probabilities are independent. Thus, there is a probability of 0.25 that you will get the nickel both times.

Suppose instead that you do not replace the first coin before pulling out the second one. Then you are bound to pull out the other coin the second time, and there is no way you could pull the nickel out twice. In this situation, the two trials are not independent, because the result of the first trial has an effect on the second trial. The law of independent probabilities does not apply, and the probability of getting the nickel twice is zero, not 0.25.

Experiments have shown that in the case of radioactive decay, the probability that any nucleus will decay during a given time interval is unaffected by what is happening to the other nuclei, and is also unrelated to how long it has gone without decaying. The first observation makes sense, because nuclei are isolated from each other at the centers of their respective atoms, and therefore have no physical way of influencing each other. The second fact is also reasonable, since all atoms are identical. Suppose we wanted to believe that certain atoms were “extra tough,” as demonstrated by their history of going an unusually long time without decaying. Those atoms would have to be different in some physical way, but nobody has ever succeeded in detecting differences among atoms. There is no way for an atom to be changed by the experiences it has in its lifetime.

### **Addition of probabilities**

The law of independent probabilities tells us to use multiplication to calculate the probability that both A and B will happen, assuming the probabilities are independent. What about the probability of an “or” rather than an “and?” If two events A and B are mutually exclusive, then the probability of one or the other occurring is the sum  $P_A + P_B$ . For instance, a bowler might have a 30% chance of getting a strike (knocking down all ten pins) and a 20% chance of knocking down nine of them. The bowler’s chance of knocking down either nine pins or ten pins is therefore 50%.

It does not make sense to add probabilities of things that are not mutually exclusive, i.e., that could both happen. Say I have a 90% chance of eating lunch on any given day, and a 90% chance of eating dinner. The probability that I will eat either lunch or dinner is not 180%.



b / Normalization: the probability of picking land plus the probability of picking water adds up to 1.

## Normalization

If I spin a globe and randomly pick a point on it, I have about a 70% chance of picking a point that's in an ocean and a 30% chance of picking a point on land. The probability of picking either water or land is  $70\% + 30\% = 100\%$ . Water and land are mutually exclusive, and there are no other possibilities, so the probabilities had to add up to 100%. It works the same if there are more than two possibilities — if you can classify all possible outcomes into a list of mutually exclusive results, then all the probabilities have to add up to 1, or 100%. This property of probabilities is known as normalization.

## Averages

Another way of dealing with randomness is to take averages. The casino knows that in the long run, the number of times you win will approximately equal the number of times you play multiplied by the probability of winning. In the game mentioned above, where the probability of winning is 0.001, if you spend a week playing, and pay \$2500 to play 10,000 times, you are likely to win about 10 times ( $10,000 \times 0.001 = 10$ ), and collect \$1000. On the average, the casino will make a profit of \$1500 from you. This is an example of the following rule.

### rule for calculating averages

If you conduct  $N$  identical, statistically independent trials, and the probability of success in each trial is  $P$ , then on the average, the total number of successful trials will be  $NP$ . If  $N$  is large enough, the relative error in this estimate will become small.

The statement that the rule for calculating averages gets more and more accurate for larger and larger  $N$  (known popularly as the “law of averages”) often provides a correspondence principle that connects classical and quantum physics. For instance, the amount of power produced by a nuclear power plant is not random at any detectable level, because the number of atoms in the reactor is so large. In general, random behavior at the atomic level tends to average out when we consider large numbers of atoms, which is why physics seemed deterministic before physicists learned techniques for studying atoms individually.

We can achieve great precision with averages in quantum physics because we can use identical atoms to reproduce exactly the same situation many times. If we were betting on horses or dice, we would be much more limited in our precision. After a thousand races, the horse would be ready to retire. After a million rolls, the dice would be worn out.

*self-check A*

Which of the following things *must* be independent, which *could* be independent, and which definitely are *not* independent?

- (1) the probability of successfully making two free-throws in a row in basketball
- (2) the probability that it will rain in London tomorrow and the probability that it will rain on the same day in a certain city in a distant galaxy
- (3) your probability of dying today and of dying tomorrow

▷ Answer, p. 980

### Discussion questions

**A** Newtonian physics is an essentially perfect approximation for describing the motion of a pair of dice. If Newtonian physics is deterministic, why do we consider the result of rolling dice to be random?

**B** Why isn't it valid to define randomness by saying that randomness is when all the outcomes are equally likely?

**C** The sequence of digits 1212121212121212 seems clearly nonrandom, and 41592653589793 seems random. The latter sequence, however, is the decimal form of  $\pi$ , starting with the third digit. There is a story about the Indian mathematician Ramanujan, a self-taught prodigy, that a friend came to visit him in a cab, and remarked that the number of the cab, 1729, seemed relatively uninteresting. Ramanujan replied that on the contrary, it was very interesting because it was the smallest number that could be represented in two different ways as the sum of two cubes. The Argentine author Jorge Luis Borges wrote a short story called "The Library of Babel," in which he imagined a library containing every book that could possibly be written using the letters of the alphabet. It would include a book containing only the repeated letter "a;" all the ancient Greek tragedies known today, all the lost Greek tragedies, and millions of Greek tragedies that were never actually written; your own life story, and various incorrect versions of your own life story; and countless anthologies containing a short story called "The Library of Babel." Of course, if you picked a book from the shelves of the library, it would almost certainly look like a nonsensical sequence of letters and punctuation, but it's always possible that the seemingly meaningless book would be a science-fiction screenplay written in the language of a Neanderthal tribe, or the lyrics to a set of incomparably beautiful love songs written in a language that never existed. In view of these examples, what does it really mean to say that something is random?



c / Why are dice random?



The area of a single square on the graph paper is then

(unitless area of a square)

$$= (\text{width of square with time units}) \times (\text{height of square})$$

If the units are to cancel out, then the height of the square must evidently be a quantity with units of inverse time. In other words, the  $y$  axis of the graph is to be interpreted as probability per unit time, not probability.

Figure f shows another example, a probability distribution for people's height. This kind of bell-shaped curve is quite common.

#### self-check B

Compare the number of people with heights in the range of 130-135 cm to the number in the range 135-140. ▷ Answer, p. 980

#### Looking for tall basketball players

#### example 1

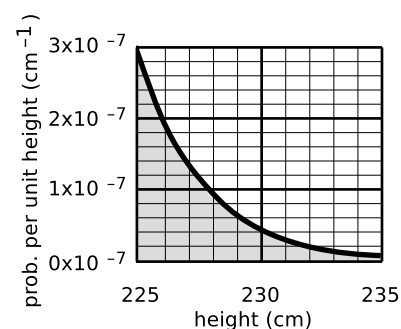
▷ A certain country with a large population wants to find very tall people to be on its Olympic basketball team and strike a blow against western imperialism. Out of a pool of  $10^8$  people who are the right age and gender, how many are they likely to find who are over 225 cm (7 feet 4 inches) in height? Figure g gives a close-up of the "tail" of the distribution shown previously in figure f.

▷ The shaded area under the curve represents the probability that a given person is tall enough. Each rectangle represents a probability of  $0.2 \times 10^{-7} \text{ cm}^{-1} \times 1 \text{ cm} = 2 \times 10^{-8}$ . There are about 35 rectangles covered by the shaded area, so the probability of having a height greater than 225 cm is  $7 \times 10^{-7}$ , or just under one in a million. Using the rule for calculating averages, the average, or expected number of people this tall is  $(10^8) \times (7 \times 10^{-7}) = 70$ .

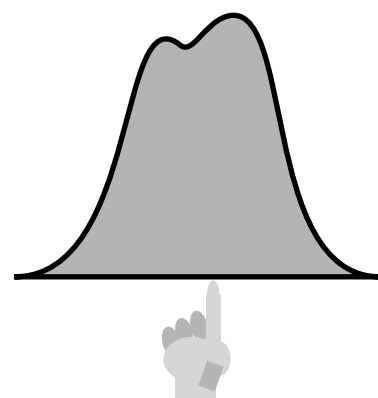
### Average and width of a probability distribution

If the next Martian you meet asks you, "How tall is an adult human?," you will probably reply with a statement about the average human height, such as "Oh, about 5 feet 6 inches." If you wanted to explain a little more, you could say, "But that's only an average. Most people are somewhere between 5 feet and 6 feet tall." Without bothering to draw the relevant bell curve for your new extraterrestrial acquaintance, you've summarized the relevant information by giving an average and a typical range of variation.

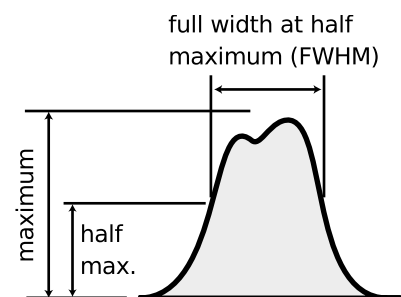
The average of a probability distribution can be defined geometrically as the horizontal position at which it could be balanced if it was constructed out of cardboard, h. A convenient numerical measure of the amount of variation about the average, or amount of uncertainty, is the full width at half maximum, or FWHM, defined in figure g. (The FWHM was introduced in chapter 2 of *Vibrations and Waves*.)



g / A close-up of the right-hand tail of the distribution shown in the figure f.



h / The average of a probability distribution.



i / The full width at half maximum (FWHM) of a probability distribution.



A great deal more could be said about this topic, and indeed an introductory statistics course could spend months on ways of defining the center and width of a distribution. Rather than force-feeding you on mathematical detail or techniques for calculating these things, it is perhaps more relevant to point out simply that there are various ways of defining them, and to inoculate you against the misuse of certain definitions.

The average is not the only possible way to say what is a typical value for a quantity that can vary randomly; another possible definition is the median, defined as the value that is exceeded with 50% probability. When discussing incomes of people living in a certain town, the average could be very misleading, since it can be affected massively if a single resident of the town is Bill Gates. Nor is the FWHM the only possible way of stating the amount of random variation; another possible way of measuring it is the standard deviation (defined as the square root of the average squared deviation from the average value).

### 33.4 Exponential decay and half-life

Most people know that radioactivity “lasts a certain amount of time,” but that simple statement leaves out a lot. As an example, consider the following medical procedure used to diagnose thyroid function. A very small quantity of the isotope  $^{131}\text{I}$ , produced in a nuclear reactor, is fed to or injected into the patient. The body’s biochemical systems treat this artificial, radioactive isotope exactly the same as  $^{127}\text{I}$ , which is the only naturally occurring type. (Nutritionally, iodine is a necessary trace element. Iodine taken into the body is partly excreted, but the rest becomes concentrated in the thyroid gland. Iodized salt has had iodine added to it to prevent the nutritional deficiency known as goiters, in which the iodine-starved thyroid becomes swollen.) As the  $^{131}\text{I}$  undergoes beta decay, it emits electrons, neutrinos, and gamma rays. The gamma rays can be measured by a detector passed over the patient’s body. As the radioactive iodine becomes concentrated in the thyroid, the amount of gamma radiation coming from the thyroid becomes greater, and that emitted by the rest of the body is reduced. The rate at which the iodine concentrates in the thyroid tells the doctor about the health of the thyroid.

If you ever undergo this procedure, someone will presumably explain a little about radioactivity to you, to allay your fears that you will turn into the Incredible Hulk, or that your next child will have an unusual number of limbs. Since iodine stays in your thyroid for a long time once it gets there, one thing you’ll want to know is whether your thyroid is going to become radioactive forever. They may just tell you that the radioactivity “only lasts a certain amount of time,” but we can now carry out a quantitative derivation of how

the radioactivity really will die out.

Let  $P_{\text{surv}}(t)$  be the probability that an iodine atom will survive without decaying for a period of at least  $t$ . It has been experimentally measured that half all  $^{131}\text{I}$  atoms decay in 8 hours, so we have

$$P_{\text{surv}}(8 \text{ hr}) = 0.5 \quad .$$

Now using the law of independent probabilities, the probability of surviving for 16 hours equals the probability of surviving for the first 8 hours multiplied by the probability of surviving for the second 8 hours,

$$\begin{aligned} P_{\text{surv}}(16 \text{ hr}) &= 0.50 \times 0.50 \\ &= 0.25 \quad . \end{aligned}$$

Similarly we have

$$\begin{aligned} P_{\text{surv}}(24 \text{ hr}) &= 0.50 \times 0.5 \times 0.5 \\ &= 0.125 \quad . \end{aligned}$$

Generalizing from this pattern, the probability of surviving for any time  $t$  that is a multiple of 8 hours is

$$P_{\text{surv}}(t) = 0.5^{t/8 \text{ hr}} \quad .$$

We now know how to find the probability of survival at intervals of 8 hours, but what about the points in time in between? What would be the probability of surviving for 4 hours? Well, using the law of independent probabilities again, we have

$$P_{\text{surv}}(8 \text{ hr}) = P_{\text{surv}}(4 \text{ hr}) \times P_{\text{surv}}(4 \text{ hr}) \quad ,$$

which can be rearranged to give

$$\begin{aligned} P_{\text{surv}}(4 \text{ hr}) &= \sqrt{P_{\text{surv}}(8 \text{ hr})} \\ &= \sqrt{0.5} \\ &= 0.707 \quad . \end{aligned}$$

This is exactly what we would have found simply by plugging in  $P_{\text{surv}}(t) = 0.5^{t/8 \text{ hr}}$  and ignoring the restriction to multiples of 8 hours. Since 8 hours is the amount of time required for half of the atoms to decay, it is known as the half-life, written  $t_{1/2}$ . The general rule is as follows:

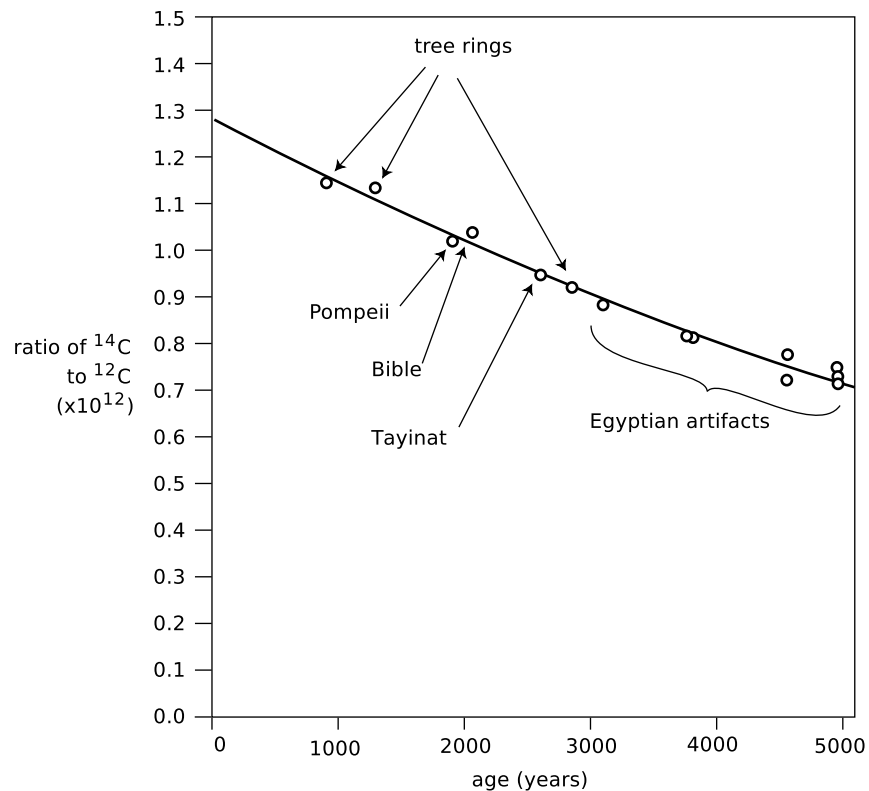
exponential decay equation

$$P_{\text{surv}}(t) = 0.5^{t/t_{1/2}}$$

Using the rule for calculating averages, we can also find the number of atoms,  $N(t)$ , remaining in a sample at time  $t$ :

$$N(t) = N(0) \times 0.5^{t/t_{1/2}}$$

Both of these equations have graphs that look like dying-out exponentials, as in the example below.



j / Calibration of the  $^{14}\text{C}$  dating method using tree rings and artifacts whose ages were known from other methods. Redrawn from Emilio Segrè, *Nuclei and Particles*, 1965.

### $^{14}\text{C}$ Dating

### example 2

Almost all the carbon on Earth is  $^{12}\text{C}$ , but not quite. The isotope  $^{14}\text{C}$ , with a half-life of 5600 years, is produced by cosmic rays in the atmosphere. It decays naturally, but is replenished at such a rate that the fraction of  $^{14}\text{C}$  in the atmosphere remains constant, at  $1.3 \times 10^{-12}$ . Living plants and animals take in both  $^{12}\text{C}$  and  $^{14}\text{C}$  from the atmosphere and incorporate both into their bodies. Once the living organism dies, it no longer takes in C atoms from the atmosphere, and the proportion of  $^{14}\text{C}$  gradually falls off as it undergoes radioactive decay. This effect can be used to find the age of dead organisms, or human artifacts made from plants or animals. Figure j shows the exponential decay curve of  $^{14}\text{C}$  in various objects. Similar methods, using longer-lived isotopes, prove the earth was billions of years old, not a few thousand as some had claimed on religious grounds.

*Radioactive contamination at Chernobyl* *example 3*

▷ One of the most dangerous radioactive isotopes released by the Chernobyl disaster in 1986 was  $^{90}\text{Sr}$ , whose half-life is 28 years. (a) How long will it be before the contamination is reduced to one tenth of its original level? (b) If a total of  $10^{27}$  atoms was released, about how long would it be before not a single atom was left?

▷ (a) We want to know the amount of time that a  $^{90}\text{Sr}$  nucleus has a probability of 0.1 of surviving. Starting with the exponential decay formula,

$$P_{\text{surv}} = 0.5^{t/t_{1/2}},$$

we want to solve for  $t$ . Taking natural logarithms of both sides,

$$\ln P = \frac{t}{t_{1/2}} \ln 0.5,$$

so

$$t = \frac{t_{1/2}}{\ln 0.5} \ln P$$

Plugging in  $P = 0.1$  and  $t_{1/2} = 28$  years, we get  $t = 93$  years.

(b) This is just like the first part, but  $P = 10^{-27}$ . The result is about 2500 years.

### Rate of decay

If you want to find how many radioactive decays occur within a time interval lasting from time  $t$  to time  $t + \Delta t$ , the most straightforward approach is to calculate it like this:

$$\begin{aligned} & \text{(number of decays between } t \text{ and } t + \Delta t) \\ &= N(t) - N(t + \Delta t) \\ &= N(0) [P_{\text{surv}}(t) - P_{\text{surv}}(t + \Delta t)] \\ &= N(0) \left[ 0.5^{t/t_{1/2}} - 0.5^{(t+\Delta t)/t_{1/2}} \right] \\ &= N(0) \left[ 1 - 0.5^{\Delta t/t_{1/2}} \right] 0.5^{t/t_{1/2}} \end{aligned}$$

A problem arises when  $\Delta t$  is small compared to  $t_{1/2}$ . For instance, suppose you have a hunk of  $10^{22}$  atoms of  $^{235}\text{U}$ , with a half-life of 700 million years, which is  $2.2 \times 10^{16}$  s. You want to know how many decays will occur in  $\Delta t = 1$  s. Since we're specifying the current number of atoms,  $t = 0$ . As you plug in to the formula above on your calculator, the quantity  $0.5^{\Delta t/t_{1/2}}$  comes out on your calculator to equal one, so the final result is zero. That's incorrect, though. In reality,  $0.5^{\Delta t/t_{1/2}}$  should equal 0.999999999999999968, but your calculator only gives eight digits of precision, so it rounded it off to one. In other words, the probability that a  $^{235}\text{U}$  atom will survive for 1 s is very close to one, but not equal to one. The number of decays in one second is therefore  $3.2 \times 10^5$ , not zero.

Well, my calculator only does eight digits of precision, just like yours, so how did I know the right answer? The way to do it is to use the following approximation:

$$a^b \approx 1 + b \ln a, \quad \text{if } b \ll 1$$

(The symbol  $\ll$  means “is much less than.”) Using it, we can find the following approximation:

$$\begin{aligned} & \text{(number of decays between } t \text{ and } t + \Delta t) \\ &= N(0) \left[ 1 - 0.5^{\Delta t/t_{1/2}} \right] 0.5^{t/t_{1/2}} \\ &\approx N(0) \left[ 1 - \left( 1 + \frac{\Delta t}{t_{1/2}} \ln 0.5 \right) \right] 0.5^{t/t_{1/2}} \\ &\approx (\ln 2) N(0) 0.5^{t/t_{1/2}} \frac{\Delta t}{t_{1/2}} \end{aligned}$$

This also gives us a way to calculate the rate of decay, i.e., the number of decays per unit time. Dividing by  $\Delta t$  on both sides, we have

$$\begin{aligned} & \text{(decays per unit time)} \approx \\ & \frac{(\ln 2) N(0)}{t_{1/2}} 0.5^{t/t_{1/2}}, \quad \text{if } \Delta t \ll t_{1/2} \end{aligned}$$

#### *The hot potato*

#### *example 4*

▷ A nuclear physicist with a demented sense of humor tosses you a cigar box, yelling “hot potato.” The label on the box says “contains  $10^{20}$  atoms of  $^{17}\text{F}$ , half-life of 66 s, produced today in our reactor at 1 p.m.” It takes you two seconds to read the label, after which you toss it behind some lead bricks and run away. The time is 1:40 p.m. Will you die?

▷ The time elapsed since the radioactive fluorine was produced in the reactor was 40 minutes, or 2400 s. The number of elapsed half-lives is therefore  $t/t_{1/2} = 36$ . The initial number of atoms was  $N(0) = 10^{20}$ . The number of decays per second is now about  $10^7 \text{ s}^{-1}$ , so it produced about  $2 \times 10^7$  high-energy electrons while you held it in your hands. Although twenty million electrons sounds like a lot, it is not really enough to be dangerous.

By the way, none of the equations we’ve derived so far was the actual probability distribution for the time at which a particular radioactive atom will decay. That probability distribution would be found by substituting  $N(0) = 1$  into the equation for the rate of decay.

If the sheer number of equations is starting to seem formidable, let’s pause and think for a second. The simple equation for  $P_{\text{surv}}$  is

something you can derive easily from the law of independent probabilities any time you need it. From that, you can quickly find the exact equation for the rate of decay. The derivation of the approximate equations for  $\Delta t \ll t$  is a little hairier, but note that except for the factors of  $\ln 2$ , everything in these equations can be found simply from considerations of logic and units. For instance, a longer half-life will obviously lead to a slower rate of decays, so it makes sense that we divide by it. As for the  $\ln 2$  factors, they are exactly the kind of thing that one looks up in a book when one needs to know them.

### Discussion questions

**A** In the medical procedure involving  $^{131}\text{I}$ , why is it the gamma rays that are detected, not the electrons or neutrinos that are also emitted?

**B** For 1 s, Fred holds in his hands 1 kg of radioactive stuff with a half-life of 1000 years. Ginger holds 1 kg of a different substance, with a half-life of 1 min, for the same amount of time. Did they place themselves in equal danger, or not?

**C** How would you interpret it if you calculated  $N(t)$ , and found it was less than one?

**D** Does the half-life depend on how much of the substance you have? Does the expected time until the sample decays completely depend on how much of the substance you have?

## 33.5 $\int$ Applications of calculus

The area under the probability distribution is of course an integral. If we call the random number  $x$  and the probability distribution  $D(x)$ , then the probability that  $x$  lies in a certain range is given by

$$(\text{probability of } a \leq x \leq b) = \int_a^b D(x)dx \quad .$$

What about averages? If  $x$  had a finite number of equally probable values, we would simply add them up and divide by how many we had. If they weren't equally likely, we'd make the weighted average  $x_1P_1 + x_2P_2 + \dots$ . But we need to generalize this to a variable  $x$  that can take on any of a continuum of values. The continuous version of a sum is an integral, so the average is

$$(\text{average value of } x) = \int xD(x)dx \quad ,$$

where the integral is over all possible values of  $x$ .

---

*Probability distribution for radioactive decay* *example 5*

Here is a rigorous justification for the statement in section 33.4 that the probability distribution for radioactive decay is found by substituting  $N(0) = 1$  into the equation for the rate of decay. We know that the probability distribution must be of the form

$$D(t) = k 0.5^{t/t_{1/2}} \quad ,$$

where  $k$  is a constant that we need to determine. The atom is guaranteed to decay eventually, so normalization gives us

$$\begin{aligned} (\text{probability of } 0 \leq t < \infty) &= 1 \\ &= \int_0^{\infty} D(t) dt \quad . \end{aligned}$$

The integral is most easily evaluated by converting the function into an exponential with  $e$  as the base

$$\begin{aligned} D(t) &= k \exp \left[ \ln \left( 0.5^{t/t_{1/2}} \right) \right] \\ &= k \exp \left[ \frac{t}{t_{1/2}} \ln 0.5 \right] \\ &= k \exp \left( -\frac{\ln 2}{t_{1/2}} t \right) \quad , \end{aligned}$$

which gives an integral of the familiar form  $\int e^{cx} dx = (1/c)e^{cx}$ . We thus have

$$1 = -\frac{k t_{1/2}}{\ln 2} \exp \left( -\frac{\ln 2}{t_{1/2}} t \right) \quad ,$$

which gives the desired result:

$$k = \frac{\ln 2}{t_{1/2}} \quad .$$

---

*Average lifetime* *example 6*

You might think that the half-life would also be the average lifetime of an atom, since half the atoms' lives are shorter and half longer. But the half whose lives are longer include some that survive for many half-lives, and these rare long-lived atoms skew the average. We can calculate the average lifetime as follows:

$$(\text{average lifetime}) = \int_0^{\infty} t D(t) dt$$

Using the convenient base- $e$  form again, we have

$$(\text{average lifetime}) = \frac{\ln 2}{t_{1/2}} \int_0^{\infty} t \exp \left( -\frac{\ln 2}{t_{1/2}} t \right) dt \quad .$$

This integral is of a form that can either be attacked with integration by parts or by looking it up in a table. The result is  $\int x e^{cx} dx = \frac{x}{c} e^{cx} - \frac{1}{c^2} e^{cx}$ , and the first term can be ignored for our purposes because it equals zero at both limits of integration. We end up with

$$\begin{aligned} (\text{average lifetime}) &= \frac{\ln 2}{t_{1/2}} \left( \frac{t_{1/2}}{\ln 2} \right)^2 \\ &= \frac{t_{1/2}}{\ln 2} \\ &= 1.443 t_{1/2} \quad , \end{aligned}$$

which is, as expected, longer than one half-life.



# Summary

## Selected vocabulary

probability . . . .	the likelihood that something will happen, expressed as a number between zero and one
normalization . .	the property of probabilities that the sum of the probabilities of all possible outcomes must equal one
independence . .	the lack of any relationship between two random events
probability distribution . . . . .	a curve that specifies the probabilities of various random values of a variable; areas under the curve correspond to probabilities
FWHM . . . . .	the full width at half-maximum of a probability distribution; a measure of the width of the distribution
half-life . . . . .	the amount of time that a radioactive atom will survive with probability 1/2 without decaying

## Notation

$P$ . . . . .	probability
$t_{1/2}$ . . . . .	half-life
$D$ . . . . .	a probability distribution (used only in optional section 33.5; not a standardized notation)

## Summary

Quantum physics differs from classical physics in many ways, the most dramatic of which is that certain processes at the atomic level, such as radioactive decay, are random rather than deterministic. There is a method to the madness, however: quantum physics still rules out any process that violates conservation laws, and it also offers methods for calculating probabilities numerically.

In this chapter we focused on certain generic methods of working with probabilities, without concerning ourselves with any physical details. Without knowing any of the details of radioactive decay, for example, we were still able to give a fairly complete treatment of the relevant probabilities. The most important of these generic methods is the law of independent probabilities, which states that if two random events are not related in any way, then the probability that they will both occur equals the product of the two probabilities,

$$\begin{aligned} &\text{probability of A and B} \\ &= P_A P_B \quad \text{[if A and B are independent]} \end{aligned}$$

The most important application is to radioactive decay. The time that a radioactive atom has a 50% chance of surviving is called the half-life,  $t_{1/2}$ . The probability of surviving for two half-lives is

$(1/2)(1/2) = 1/4$ , and so on. In general, the probability of surviving a time  $t$  is given by

$$P_{surv}(t) = 0.5^{t/t_{1/2}} \quad .$$

Related quantities such as the rate of decay and probability distribution for the time of decay are given by the same type of exponential function, but multiplied by certain constant factors.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** If a radioactive substance has a half-life of one year, does this mean that it will be completely decayed after two years? Explain.

**2** Many individuals carry the recessive gene for albinism, but they are not albino unless they receive the gene from both their parents. In the U.S., an individual's probability of receiving the gene from a given parent is about 0.014. What is the probability that a given child will be born albino? ✓

**3** Problem 3 has been deleted.

**4** Use a calculator to check the approximation that

$$a^b \approx 1 + b \ln a \quad ,$$

if  $b \ll 1$ , using some arbitrary numbers. Then see how good the approximation is for values of  $b$  that are not quite as small compared to one.

**5** Make up an example of a numerical problem involving a rate of decay where  $\Delta t \ll t_{1/2}$ , but the exact expression for the rate of decay on page 897 can still be evaluated on a calculator without getting something that rounds off to zero. Check that you get approximately the same result using both methods on pp. 897-898 to calculate the number of decays between  $t$  and  $t + \Delta t$ . Keep plenty of significant figures in your results, in order to show the difference between them.

**6** Devise a method for testing experimentally the hypothesis that a gambler's chance of winning at craps is independent of her previous record of wins and losses.

**7** Refer to the probability distribution for people's heights in figure f on page 892.

(a) Show that the graph is properly normalized.

(b) Estimate the fraction of the population having heights between 140 and 150 cm. ✓

8 (a) A nuclear physicist is studying a nuclear reaction caused in an accelerator experiment, with a beam of ions from the accelerator striking a thin metal foil and causing nuclear reactions when a nucleus from one of the beam ions happens to hit one of the nuclei in the target. After the experiment has been running for a few hours, a few billion radioactive atoms have been produced, embedded in the target. She does not know what nuclei are being produced, but she suspects they are an isotope of some heavy element such as Pb, Bi, Fr or U. Following one such experiment, she takes the target foil out of the accelerator, sticks it in front of a detector, measures the activity every 5 min, and makes a graph (figure). The isotopes she thinks may have been produced are:

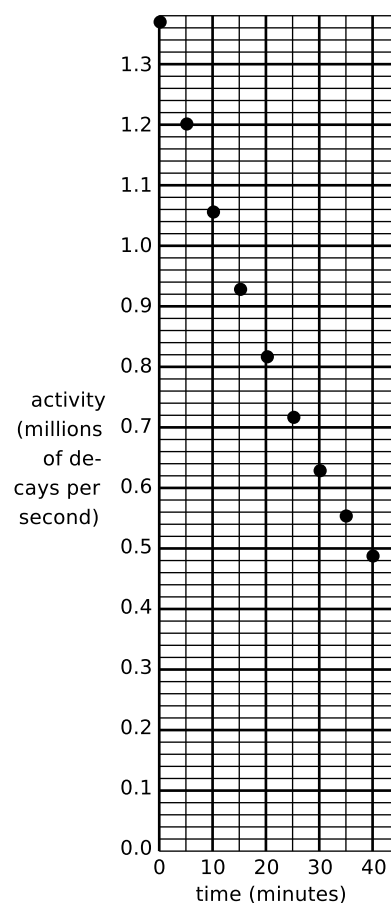
isotope	half-life (minutes)
$^{211}\text{Pb}$	36.1
$^{214}\text{Pb}$	26.8
$^{214}\text{Bi}$	19.7
$^{223}\text{Fr}$	21.8
$^{239}\text{U}$	23.5

Which one is it?

(b) Having decided that the original experimental conditions produced one specific isotope, she now tries using beams of ions traveling at several different speeds, which may cause different reactions. The following table gives the activity of the target 10, 20 and 30 minutes after the end of the experiment, for three different ion speeds.

	activity (millions of decays/s) after...		
	10 min	20 min	30 min
first ion speed	1.933	0.832	0.382
second ion speed	1.200	0.545	0.248
third ion speed	6.544	1.296	0.248

Since such a large number of decays is being counted, assume that the data are only inaccurate due to rounding off when writing down the table. Which are consistent with the production of a single isotope, and which imply that more than one isotope was being created?



Problem 8.

**9** All helium on earth is from the decay of naturally occurring heavy radioactive elements such as uranium. Each alpha particle that is emitted ends up claiming two electrons, which makes it a helium atom. If the original  $^{238}\text{U}$  atom is in solid rock (as opposed to the earth's molten regions), the He atoms are unable to diffuse out of the rock. This problem involves dating a rock using the known decay properties of uranium 238. Suppose a geologist finds a sample of hardened lava, melts it in a furnace, and finds that it contains 1230 mg of uranium and 2.3 mg of helium.  $^{238}\text{U}$  decays by alpha emission, with a half-life of  $4.5 \times 10^9$  years. The subsequent chain of alpha and electron (beta) decays involves much shorter half-lives, and terminates in the stable nucleus  $^{206}\text{Pb}$ . (You may want to review alpha and beta decay.) Almost all natural uranium is  $^{238}\text{U}$ , and the chemical composition of this rock indicates that there were no decay chains involved other than that of  $^{238}\text{U}$ .

(a) How many alphas are emitted in decay chain of a single  $^{238}\text{U}$  atom?

[Hint: Use conservation of mass.]

(b) How many electrons are emitted per decay chain?

[Hint: Use conservation of charge.]

(c) How long has it been since the lava originally hardened? ✓

**10** Physicists thought for a long time that bismuth-209 was the heaviest stable isotope. (Very heavy elements decay by alpha emission because of the strong electrical repulsion of all their protons.) However, a 2003 paper by Marcillac et al. describes an experiment in which bismuth-209 lost its claim to fame — it actually undergoes alpha decay with a half-life of  $1.9 \times 10^{19}$  years.

(a) After the alpha particle is emitted, what is the isotope left over?

(b) Compare the half-life to the age of the universe, which is about 14 billion years.

(c) A tablespoon of Pepto-Bismol contains about  $4 \times 10^{20}$  bismuth-209 atoms. Once you've swallowed it, how much time will it take, on the average, before the first atomic decay? ✓

**11** A blindfolded person fires a gun at a circular target of radius  $b$ , and is allowed to continue firing until a shot actually hits it. Any part of the target is equally likely to get hit. We measure the random distance  $r$  from the center of the circle to where the bullet went in.

(a) Show that the probability distribution of  $r$  must be of the form  $D(r) = kr$ , where  $k$  is some constant. (Of course we have  $D(r) = 0$  for  $r > b$ .)

(b) Determine  $k$  by requiring  $D$  to be properly normalized. ✓

(c) Find the average value of  $r$ . ✓

(d) Interpreting your result from part c, how does it compare with  $b/2$ ? Does this make sense? Explain.  $\int$

**12** We are given some atoms of a certain radioactive isotope, with half-life  $t_{1/2}$ . We pick one atom at random, and observe it for one half-life, starting at time zero. If it decays during that one-half-life period, we record the time  $t$  at which the decay occurred. If it doesn't, we reset our clock to zero and keep trying until we get an atom that cooperates. The final result is a time  $0 \leq t \leq t_{1/2}$ , with a distribution that looks like the usual exponential decay curve, but with its tail chopped off.

(a) Find the distribution  $D(t)$ , with the proper normalization.  $\checkmark$

(b) Find the average value of  $t$ .  $\checkmark$

(c) Interpreting your result from part b, how does it compare with  $t_{1/2}/2$ ? Does this make sense? Explain.  $\int$

**13** The speed,  $v$ , of an atom in an ideal gas has a probability distribution of the form  $D(v) = bve^{-cv^2}$ , where  $0 \leq v < \infty$ ,  $c$  relates to the temperature, and  $b$  is determined by normalization.

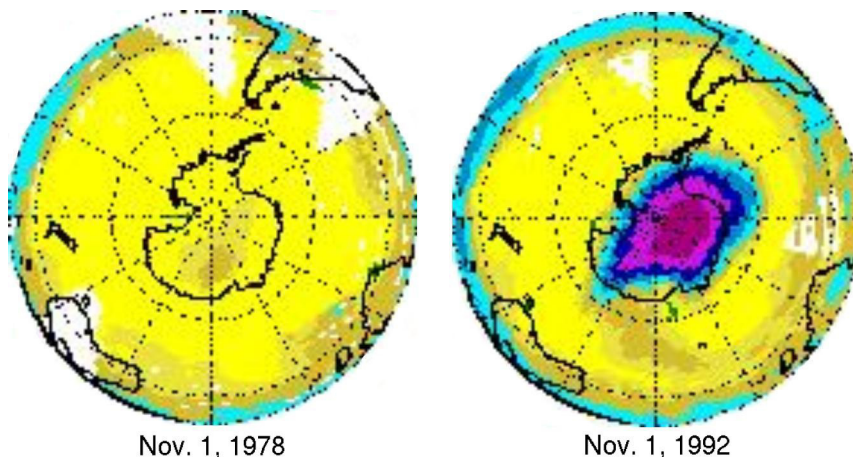
(a) Sketch the distribution.

(b) Find  $b$  in terms of  $c$ .  $\checkmark$

(c) Find the average speed in terms of  $c$ , eliminating  $b$ . (Don't try to do the indefinite integral, because it can't be done in closed form. The relevant definite integral can be found in tables or done with computer software.)  $\checkmark \int$

**14** Neutrinos interact so weakly with normal matter that, of the neutrinos from the sun that enter the earth from the day side, only about  $10^{-10}$  of them fail to reemerge on the night side. From this fact, estimate the thickness of matter, in units of light-years, that would be required in order to block half of them. This "half-distance" is analogous to a half-life for radioactive decay.





In recent decades, a huge hole in the ozone layer has spread out from Antarctica.

## Chapter 34

# Light as a particle

The only thing that interferes with my learning is my education.

*Albert Einstein*

Radioactivity is random, but do the laws of physics exhibit randomness in other contexts besides radioactivity? Yes. Radioactive decay was just a good playpen to get us started with concepts of randomness, because all atoms of a given isotope are identical. By stocking the playpen with an unlimited supply of identical atom-toys, nature helped us to realize that their future behavior could be different regardless of their original identity. We are now ready to leave the playpen, and see how randomness fits into the structure of physics at the most fundamental level.

The laws of physics describe light and matter, and the quantum revolution rewrote both descriptions. Radioactivity was a good example of matter's behaving in a way that was inconsistent with classical physics, but if we want to get under the hood and understand how nonclassical things happen, it will be easier to focus on light rather than matter. A radioactive atom such as uranium-235 is after all an extremely complex system, consisting of 92 protons, 143 neutrons, and 92 electrons. Light, however, can be a simple sine wave.

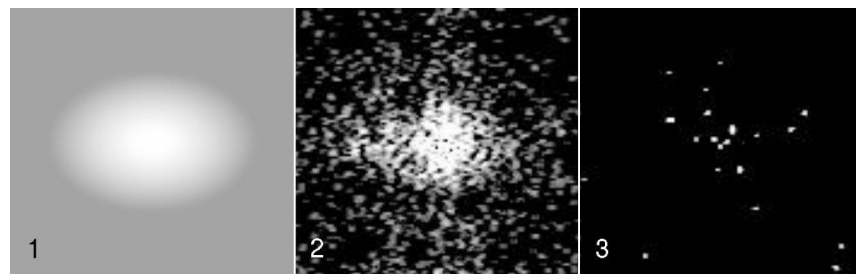
However successful the classical wave theory of light had been — allowing the creation of radio and radar, for example — it still failed



to describe many important phenomena. An example that is currently of great interest is the way the ozone layer protects us from the dangerous short-wavelength ultraviolet part of the sun's spectrum. In the classical description, light is a wave. When a wave passes into and back out of a medium, its frequency is unchanged, and although its wavelength is altered while it is in the medium, it returns to its original value when the wave reemerges. Luckily for us, this is not at all what ultraviolet light does when it passes through the ozone layer, or the layer would offer no protection at all!

## 34.1 Evidence for light as a particle

a / Images made by a digital camera. In each successive image, the dim spot of light has been made even dimmer.



For a long time, physicists tried to explain away the problems with the classical theory of light as arising from an imperfect understanding of atoms and the interaction of light with individual atoms and molecules. The ozone paradox, for example, could have been attributed to the incorrect assumption that the ozone layer was a smooth, continuous substance, when in reality it was made of individual ozone molecules. It wasn't until 1905 that Albert Einstein threw down the gauntlet, proposing that the problem had nothing to do with the details of light's interaction with atoms and everything to do with the fundamental nature of light itself.

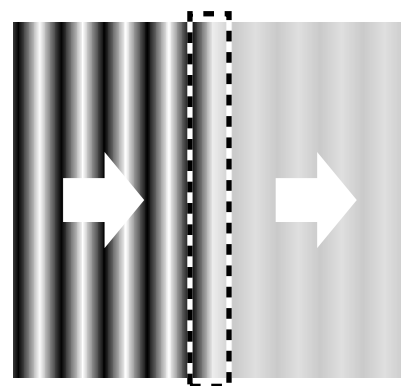
In those days the data were sketchy, the ideas vague, and the experiments difficult to interpret; it took a genius like Einstein to cut through the thicket of confusion and find a simple solution. Today, however, we can get right to the heart of the matter with a piece of ordinary consumer electronics, the digital camera. Instead of film, a digital camera has a computer chip with its surface divided up into a grid of light-sensitive squares, called "pixels." Compared to a grain of the silver compound used to make regular photographic film, a digital camera pixel is activated by an amount of light energy orders of magnitude smaller. We can learn something new about light by using a digital camera to detect smaller and smaller amounts of light, as shown in figures a/1 through a/3. Figure 1 is fake, but 2 and 3 are real digital-camera images made by Prof. Lyman Page of Princeton University as a classroom demonstration. Figure 1 is

what we would see if we used the digital camera to take a picture of a fairly dim source of light. In figures 2 and 3, the intensity of the light was drastically reduced by inserting semitransparent absorbers like the tinted plastic used in sunglasses. Going from 1 to 2 to 3, more and more light energy is being thrown away by the absorbers.

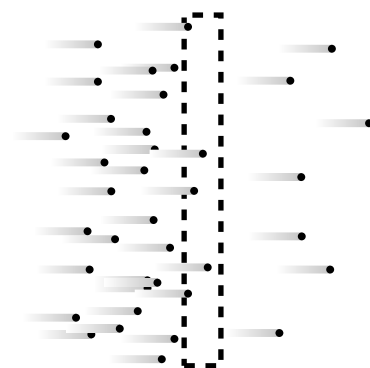
The results are dramatically different from what we would expect based on the wave theory of light. If light was a wave and nothing but a wave, b, then the absorbers would simply cut down the wave's amplitude across the whole wavefront. The digital camera's entire chip would be illuminated uniformly, and weakening the wave with an absorber would just mean that every pixel would take a long time to soak up enough energy to register a signal.

But figures a/2 and a/3 show that some pixels take strong hits while others pick up no energy at all. Instead of the wave picture, the image that is naturally evoked by the data is something more like a hail of bullets from a machine gun, c. Each "bullet" of light apparently carries only a tiny amount of energy, which is why detecting them individually requires a sensitive digital camera rather than an eye or a piece of film.

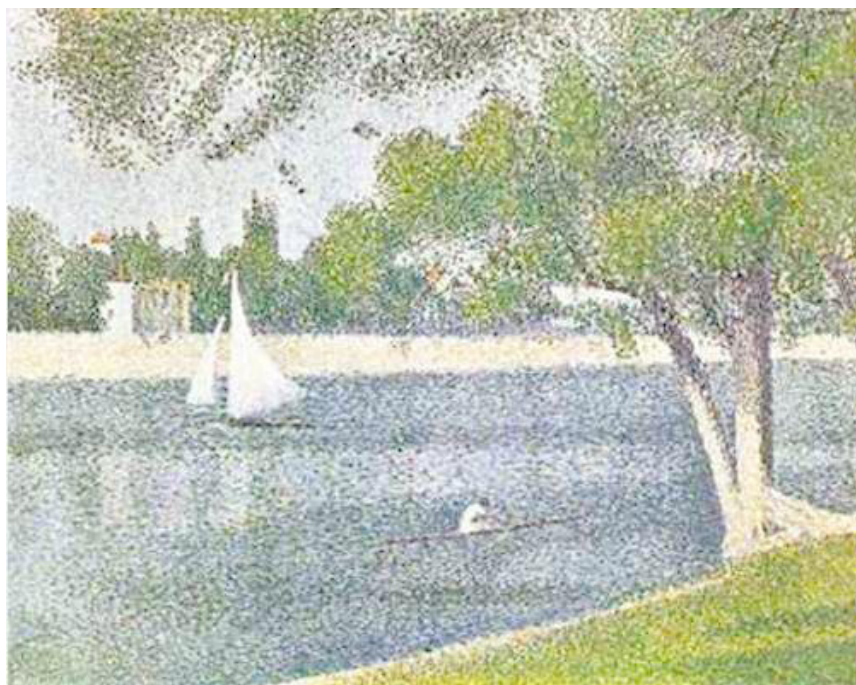
Although Einstein was interpreting different observations, this is the conclusion he reached in his 1905 paper: that the pure wave theory of light is an oversimplification, and that the energy of a beam of light comes in finite chunks rather than being spread smoothly throughout a region of space.



b / A water wave is partially absorbed.



c / A stream of bullets is partially absorbed.



d / Einstein and Seurat: twins separated at birth? Detail from *Seine Grande Jatte* by Georges Seurat, 1886.

We now think of these chunks as particles of light, and call them “photons,” although Einstein avoided the word “particle,” and the word “photon” was invented later. Regardless of words, the trouble was that waves and particles seemed like inconsistent categories. The reaction to Einstein’s paper could be kindly described as vigorously skeptical. Even twenty years later, Einstein wrote, “There are therefore now two theories of light, both indispensable, and — as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists — without any logical connection.” In the remainder of this chapter we will learn how the seeming paradox was eventually resolved.

### Discussion questions

**A** Suppose someone rebuts the digital camera data in figure a, claiming that the random pattern of dots occurs not because of anything fundamental about the nature of light but simply because the camera’s pixels are not all exactly the same — some are just more sensitive than others. How could we test this interpretation?

**B** Discuss how the correspondence principle applies to the observations and concepts discussed in this section.

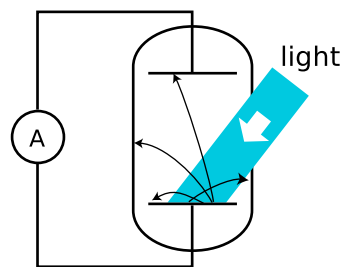
## 34.2 How much light is one photon?

### The photoelectric effect

We have seen evidence that light energy comes in little chunks, so the next question to be asked is naturally how much energy is in one chunk. The most straightforward experimental avenue for addressing this question is a phenomenon known as the photoelectric effect. The photoelectric effect occurs when a photon strikes the surface of a solid object and knocks out an electron. It occurs continually all around you. It is happening right now at the surface of your skin and on the paper or computer screen from which you are reading these words. It does not ordinarily lead to any observable electrical effect, however, because on the average, free electrons are wandering back in just as frequently as they are being ejected. (If an object did somehow lose a significant number of electrons, its growing net positive charge would begin attracting the electrons back more and more strongly.)

Figure e shows a practical method for detecting the photoelectric effect. Two very clean parallel metal plates (the electrodes of a capacitor) are sealed inside a vacuum tube, and only one plate is exposed to light. Because there is a good vacuum between the plates, any ejected electron that happens to be headed in the right direction will almost certainly reach the other capacitor plate without colliding with any air molecules.

The illuminated (bottom) plate is left with a net positive charge, and the unilluminated (top) plate acquires a negative charge from



**e** / Apparatus for observing the photoelectric effect. A beam of light strikes a capacitor plate inside a vacuum tube, and electrons are ejected (black arrows).

the electrons deposited on it. There is thus an electric field between the plates, and it is because of this field that the electrons' paths are curved, as shown in the diagram. However, since vacuum is a good insulator, any electrons that reach the top plate are prevented from responding to the electrical attraction by jumping back across the gap. Instead they are forced to make their way around the circuit, passing through an ammeter. The ammeter measures the strength of the photoelectric effect.

### An unexpected dependence on frequency

The photoelectric effect was discovered serendipitously by Heinrich Hertz in 1887, as he was experimenting with radio waves. He was not particularly interested in the phenomenon, but he did notice that the effect was produced strongly by ultraviolet light and more weakly by lower frequencies. Light whose frequency was lower than a certain critical value did not eject any electrons at all.<sup>1</sup> This dependence on frequency didn't make any sense in terms of the classical wave theory of light. A light wave consists of electric and magnetic fields. The stronger the fields, i.e., the greater the wave's amplitude, the greater the forces that would be exerted on electrons that found themselves bathed in the light. It should have been amplitude (brightness) that was relevant, not frequency. The dependence on frequency not only proves that the wave model of light needs modifying, but with the proper interpretation it allows us to determine how much energy is in one photon, and it also leads to a connection between the wave and particle models that we need in order to reconcile them.

To make any progress, we need to consider the physical process by which a photon would eject an electron from the metal electrode. A metal contains electrons that are free to move around. Ordinarily, in the interior of the metal, such an electron feels attractive forces from atoms in every direction around it. The forces cancel out. But if the electron happens to find itself at the surface of the metal, the attraction from the interior side is not balanced out by any attraction from outside. In popping out through the surface the electron therefore loses some amount of energy  $E_s$ , which depends on the type of metal used.

Suppose a photon strikes an electron, annihilating itself and giving up all its energy to the electron.<sup>2</sup> The electron will (1) lose kinetic energy through collisions with other electrons as it plows through the metal on its way to the surface; (2) lose an amount of kinetic energy equal to  $E_s$  as it emerges through the surface; and (3) lose more energy on its way across the gap between the plates, due to

<sup>1</sup>In fact this was all prior to Thomson's discovery of the electron, so Hertz would not have described the effect in terms of electrons — we are discussing everything with the benefit of hindsight.

<sup>2</sup>We now know that this is what always happens in the photoelectric effect, although it had not yet been established in 1905 whether or not the photon was completely annihilated.



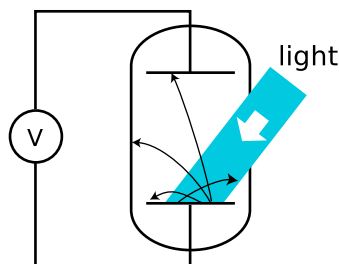
f / The hamster in her hamster ball is like an electron emerging from the metal (tiled kitchen floor) into the surrounding vacuum (wood floor). The wood floor is higher than the tiled floor, so as she rolls up the step, the hamster will lose a certain amount of kinetic energy, analogous to  $E_s$ . If her kinetic energy is too small, she won't even make it up the step.

the electric field between the plates. Even if the electron happens to be right at the surface of the metal when it absorbs the photon, and even if the electric field between the plates has not yet built up very much,  $E_s$  is the bare minimum amount of energy that the electron must receive from the photon if it is to contribute to a measurable current. The reason for using very clean electrodes is to minimize  $E_s$  and make it have a definite value characteristic of the metal surface, not a mixture of values due to the various types of dirt and crud that are present in tiny amounts on all surfaces in everyday life.

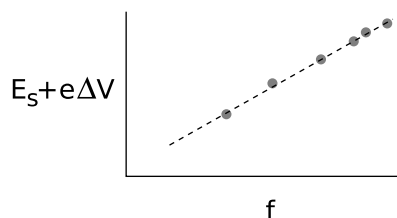
We can now interpret the frequency dependence of the photoelectric effect in a simple way: apparently the amount of energy possessed by a photon is related to its frequency. A low-frequency red or infrared photon has an energy less than  $E_s$ , so a beam of them will not produce any current. A high-frequency blue or violet photon, on the other hand, packs enough of a punch to allow an electron to get out of the electrode. At frequencies higher than the minimum, the photoelectric current continues to increase with the frequency of the light because of effects (1) and (3).

### Numerical relationship between energy and frequency

Prompted by Einstein's photon paper, Robert Millikan (whom we encountered in ch. 26) figured out how to use the photoelectric effect to probe precisely the link between frequency and photon energy. Rather than going into the historical details of Millikan's actual experiments (a lengthy experimental program that occupied a large part of his professional career) we will describe a simple version, shown in figure g, that is used sometimes in college laboratory courses. The idea is simply to illuminate one plate of the vacuum tube with light of a single wavelength and monitor the voltage difference between the two plates as they charge up. Since the resistance of a voltmeter is very high (much higher than the resistance of an ammeter), we can assume to a good approximation that electrons reaching the top plate are stuck there permanently, so the voltage will keep on increasing for as long as electrons are making it across the vacuum tube.



g / A different way of studying the photoelectric effect.



h / The quantity  $E_s + e\Delta V$  indicates the energy of one photon. It is found to be proportional to the frequency of the light.

At a moment when the voltage difference has reached a value  $\Delta V$ , the minimum energy required by an electron to make it out of the bottom plate and across the gap to the other plate is  $E_s + e\Delta V$ . As  $\Delta V$  increases, we eventually reach a point at which  $E_s + e\Delta V$  equals the energy of one photon. No more electrons can cross the gap, and the reading on the voltmeter stops rising. The quantity  $E_s + e\Delta V$  now tells us the energy of one photon. If we determine this energy for a variety of frequencies,  $h$ , we find the following simple relationship between the energy of a photon and the frequency of the light:

$$E = hf$$

where  $h$  is a constant with a numerical value of  $6.63 \times 10^{-34}$  J·s.

Note how the equation brings the wave and particle models of light under the same roof: the left side is the energy of one *particle* of light, while the right side is the frequency of the same light, interpreted as a *wave*. The constant  $h$  is known as Planck's constant (see historical note on page 915).

**self-check A**

How would you extract  $h$  from the graph in figure h? What if you didn't even know  $E_s$  in advance, and could only graph  $e\Delta V$  versus  $f$ ? ▷

Answer, p. 980

Since the energy of a photon is  $hf$ , a beam of light can only have energies of  $hf$ ,  $2hf$ ,  $3hf$ , etc. Its energy is quantized — there is no such thing as a fraction of a photon. Quantum physics gets its name from the fact that it quantizes things like energy, momentum, and angular momentum that had previously been thought to be smooth, continuous and infinitely divisible.

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**Historical Note**

What I'm presenting in this chapter is a simplified explanation of how the photon could have been discovered. The actual history is more complex. Max Planck (1858-1947) began the photon saga with a theoretical investigation of the spectrum of light emitted by a hot, glowing object. He introduced quantization of the energy of light waves, in multiples of  $hf$ , purely as a mathematical trick that happened to produce the right results. Planck did not believe that his procedure could have any physical significance. In his 1905 paper Einstein took Planck's quantization as a description of reality, and applied it to various theoretical and experimental puzzles, including the photoelectric effect. Millikan then subjected Einstein's ideas to a series of rigorous experimental tests. Although his results matched Einstein's predictions perfectly, Millikan was skeptical about photons, and his papers conspicuously omit any reference to them. Only in his autobiography did Millikan rewrite history and claim that he had given experimental proof for photons.

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**Number of photons emitted by a lightbulb per second example 1**

▷ Roughly how many photons are emitted by a 100-W lightbulb in 1 second?

▷ People tend to remember wavelengths rather than frequencies for visible light. The bulb emits photons with a range of frequencies and wavelengths, but let's take 600 nm as a typical wavelength for purposes of estimation. The energy of a single photon

is

$$\begin{aligned}E_{\text{photon}} &= hf \\ &= \frac{hc}{\lambda}\end{aligned}$$

A power of 100 W means 100 joules per second, so the number of photons is

$$\frac{100 \text{ J}}{E_{\text{photon}}} = \frac{100 \text{ J}}{hc/\lambda} \approx 3 \times 10^{20} \quad .$$

#### *Momentum of a photon*

*example 2*

▷ According to the theory of relativity, the momentum of a beam of light is given by  $p = E/c$  (see homework problem 12 on page 759). Apply this to find the momentum of a single photon in terms of its frequency, and in terms of its wavelength.

▷ Combining the equations  $p = E/c$  and  $E = hf$ , we find

$$\begin{aligned}p &= \frac{E}{c} \\ &= \frac{hf}{c} \quad .\end{aligned}$$

To reexpress this in terms of wavelength, we use  $c = f\lambda$ :

$$\begin{aligned}p &= \frac{hf}{f\lambda} \\ &= \frac{h}{\lambda}\end{aligned}$$

The second form turns out to be simpler.

#### **Discussion questions**

**A** The photoelectric effect only ever ejects a very tiny percentage of the electrons available near the surface of an object. How well does this agree with the wave model of light, and how well with the particle model? Consider the two different distance scales involved: the wavelength of the light, and the size of an atom, which is on the order of  $10^{-10}$  or  $10^{-9}$  m.

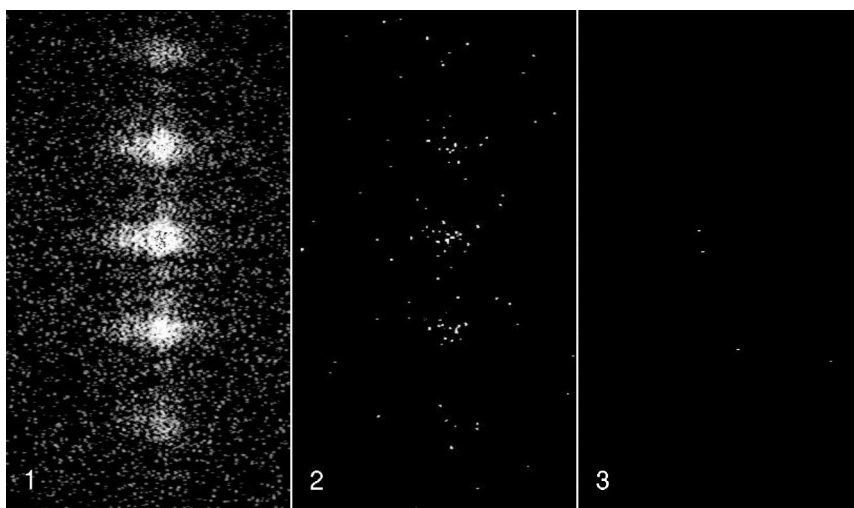
**B** What is the significance of the fact that Planck's constant is numerically very small? How would our everyday experience of light be different if it was not so small?

**C** How would the experiments described above be affected if a single electron was likely to get hit by more than one photon?

**D** Draw some representative trajectories of electrons for  $\Delta V = 0$ ,  $\Delta V$  less than the maximum value, and  $\Delta V$  greater than the maximum value.

**E** Does  $E = hf$  imply that a photon changes its energy when it passes from one transparent material into another substance with a different index of refraction?





i / Wave interference patterns photographed by Prof. Lyman Page with a digital camera. Laser light with a single well-defined wavelength passed through a series of absorbers to cut down its intensity, then through a set of slits to produce interference, and finally into a digital camera chip. (A triple slit was actually used, but for conceptual simplicity we discuss the results in the main text as if it was a double slit.) In panel 2 the intensity has been reduced relative to 1, and even more so for panel 3.

### 34.3 Wave-particle duality

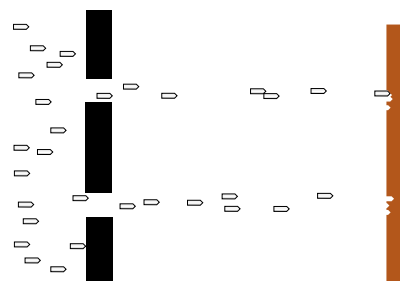
How can light be both a particle and a wave? We are now ready to resolve this seeming contradiction. Often in science when something seems paradoxical, it's because we either don't define our terms carefully, or don't test our ideas against any specific real-world situation. Let's define particles and waves as follows:

- Waves exhibit superposition, and specifically interference phenomena.
- Particles can only exist in whole numbers, not fractions.

As a real-world check on our philosophizing, there is one particular experiment that works perfectly. We set up a double-slit interference experiment that we know will produce a diffraction pattern if light is an honest-to-goodness wave, but we detect the light with a detector that is capable of sensing individual photons, e.g., a digital camera. To make it possible to pick out individual dots from individual photons, we must use filters to cut down the intensity of the light to a very low level, just as in the photos by Prof. Page in section 34.1. The whole thing is sealed inside a light-tight box. The results are shown in figure i. (In fact, the similar figures in section 34.1 are simply cutouts from these figures.)

Neither the pure wave theory nor the pure particle theory can explain the results. If light was only a particle and not a wave, there would be no interference effect. The result of the experiment would be like firing a hail of bullets through a double slit, j. Only two spots directly behind the slits would be hit.

If, on the other hand, light was only a wave and not a particle, we would get the same kind of diffraction pattern that would happen



j / Bullets pass through a double slit.



k / A water wave passes through a double slit.



with a water wave,  $k$ . There would be no discrete dots in the photo, only a diffraction pattern that shaded smoothly between light and dark.

Applying the definitions to this experiment, light must be both a particle and a wave. It is a wave because it exhibits interference effects. At the same time, the fact that the photographs contain discrete dots is a direct demonstration that light refuses to be split into units of less than a single photon. There can only be whole numbers of photons: four photons in figure i/3, for example.

### A wrong interpretation: photons interfering with each other

One possible interpretation of wave-particle duality that occurred to physicists early in the game was that perhaps the interference effects came from photons interacting with each other. By analogy, a water wave consists of moving water molecules, and interference of water waves results ultimately from all the mutual pushes and pulls of the molecules. This interpretation was conclusively disproved by G.I. Taylor, a student at Cambridge. The demonstration by Prof. Page that we've just been discussing is essentially a modernized version of Taylor's work. Taylor reasoned that if interference effects came from photons interacting with each other, a bare minimum of two photons would have to be present at the same time to produce interference. By making the light source extremely dim, we can be virtually certain that there are never two photons in the box at the same time. In figure i, the intensity of the light has been cut down so much by the absorbers that if it was in the open, the average separation between photons would be on the order of a kilometer! At any given moment, the number of photons in the box is most likely to be zero. It is virtually certain that there were never two photons in the box at once.



I / A single photon can go through both slits.

### The concept of a photon's path is undefined.

If a single photon can demonstrate double-slit interference, then which slit did it pass through? The unavoidable answer must be that it passes through both! This might not seem so strange if we think of the photon as a wave, but it is highly counterintuitive if we try to visualize it as a particle. The moral is that we should not think in terms of the *path* of a photon. Like the fully human and fully divine Jesus of Christian theology, a photon is supposed to be 100% wave and 100% particle. If a photon had a well defined path, then it would not demonstrate wave superposition and interference effects, contradicting its wave nature. (In the next chapter we will discuss the Heisenberg uncertainty principle, which gives a numerical way of approaching this issue.)

### Another wrong interpretation: the pilot wave hypothesis

A second possible explanation of wave-particle duality was taken seriously in the early history of quantum mechanics. What if the photon *particle* is like a surfer riding on top of its accompanying *wave*? As the wave travels along, the particle is pushed, or “piloted” by it. Imagining the particle and the wave as two separate entities allows us to avoid the seemingly paradoxical idea that a photon is both at once. The wave happily does its wave tricks, like superposition and interference, and the particle acts like a respectable particle, resolutely refusing to be in two different places at once. If the wave, for instance, undergoes destructive interference, becoming nearly zero in a particular region of space, then the particle simply is not guided into that region.

The problem with the pilot wave interpretation is that the only way it can be experimentally tested or verified is if someone manages to detach the particle from the wave, and show that there really are two entities involved, not just one. Part of the scientific method is that hypotheses are supposed to be experimentally testable. Since nobody has ever managed to separate the wavelike part of a photon from the particle part, the interpretation is not useful or meaningful in a scientific sense.

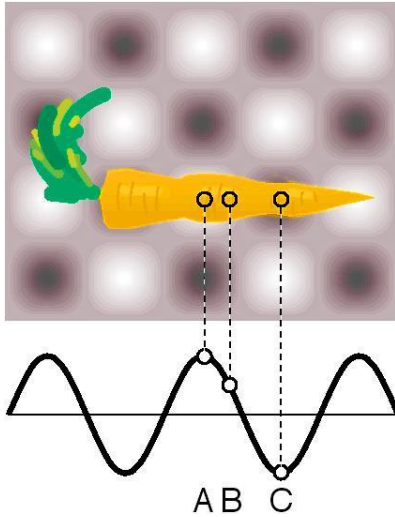
### The probability interpretation

The correct interpretation of wave-particle duality is suggested by the random nature of the experiment we’ve been discussing: even though every photon wave/particle is prepared and released in the same way, the location at which it is eventually detected by the digital camera is different every time. The idea of the probability interpretation of wave-particle duality is that the location of the photon-particle is random, but the probability that it is in a certain location is higher where the photon-wave’s amplitude is greater.

More specifically, the probability distribution of the particle must be proportional to the *square* of the wave’s amplitude,

$$(\text{probability distribution}) \propto (\text{amplitude})^2 \quad .$$

This follows from the correspondence principle and from the fact that a wave’s energy density is proportional to the square of its amplitude. If we run the double-slit experiment for a long enough time, the pattern of dots fills in and becomes very smooth as would have been expected in classical physics. To preserve the correspondence between classical and quantum physics, the amount of energy deposited in a given region of the picture over the long run must be proportional to the square of the wave’s amplitude. The amount of energy deposited in a certain area depends on the number of pho-



m / Example 3.

tons picked up, which is proportional to the probability of finding any given photon there.

#### A microwave oven

#### example 3

▷ The figure shows two-dimensional (top) and one-dimensional (bottom) representations of the standing wave inside a microwave oven. Gray represents zero field, and white and black signify the strongest fields, with white being a field that is in the opposite direction compared to black. Compare the probabilities of detecting a microwave photon at points A, B, and C.

▷ A and C are both extremes of the wave, so the probabilities of detecting a photon at A and C are equal. It doesn't matter that we have represented C as negative and A as positive, because it is the square of the amplitude that is relevant. The amplitude at B is about 1/2 as much as the others, so the probability of detecting a photon there is about 1/4 as much.

The probability interpretation was disturbing to physicists who had spent their previous careers working in the deterministic world of classical physics, and ironically the most strenuous objections against it were raised by Einstein, who had invented the photon concept in the first place. The probability interpretation has nevertheless passed every experimental test, and is now as well established as any part of physics.

An aspect of the probability interpretation that has made many people uneasy is that the process of detecting and recording the photon's position seems to have a magical ability to get rid of the wavelike side of the photon's personality and force it to decide for once and for all where it really wants to be. But detection or measurement is after all only a physical process like any other, governed by the same laws of physics. We will postpone a detailed discussion of this issue until the following chapter, since a measuring device like a digital camera is made of matter, but we have so far only discussed how quantum mechanics relates to light.

#### What is the proportionality constant?

#### example 4

▷ What is the proportionality constant that would make an actual equation out of (probability distribution)  $\propto$  (amplitude)<sup>2</sup>?

▷ The probability that the photon is in a certain small region of volume  $v$  should equal the fraction of the wave's energy that is within that volume:

$$\begin{aligned}
 P &= \frac{\text{energy in volume } v}{\text{energy of photon}} \\
 &= \frac{\text{energy in volume } v}{hf}
 \end{aligned}$$

We assume  $v$  is small enough so that the electric and magnetic

fields are nearly constant throughout it. We then have

$$P = \frac{\left( \frac{1}{8\pi k} |\mathbf{E}|^2 + \frac{c^2}{8\pi k} |\mathbf{B}|^2 \right) v}{hf} .$$

We can simplify this formidable looking expression by recognizing that in an electromagnetic wave,  $|\mathbf{E}|$  and  $|\mathbf{B}|$  are related by  $|\mathbf{E}| = c|\mathbf{B}|$ . With some algebra, it turns out that the electric and magnetic fields each contribute half the total energy, so we can simplify this to

$$\begin{aligned} P &= 2 \frac{\left( \frac{1}{8\pi k} |\mathbf{E}|^2 \right) v}{hf} \\ &= \frac{v}{4\pi k hf} |\mathbf{E}|^2 . \end{aligned}$$

As advertised, the probability is proportional to the square of the wave's amplitude.

### Discussion questions

**A** In example 3 on page 920, about the carrot in the microwave oven, show that it would be nonsensical to have probability be proportional to the field itself, rather than the square of the field.

**B** Einstein did not try to reconcile the wave and particle theories of light, and did not say much about their apparent inconsistency. Einstein basically visualized a beam of light as a stream of bullets coming from a machine gun. In the photoelectric effect, a photon “bullet” would only hit one atom, just as a real bullet would only hit one person. Suppose someone reading his 1905 paper wanted to interpret it by saying that Einstein’s so-called particles of light are simply short wave-trains that only occupy a small region of space. Comparing the wavelength of visible light (a few hundred nm) to the size of an atom (on the order of 0.1 nm), explain why this poses a difficulty for reconciling the particle and wave theories.

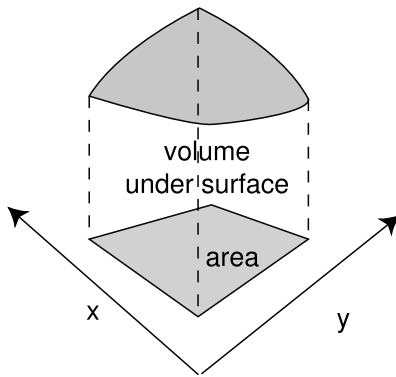
**C** Can a white photon exist?

**D** In double-slit diffraction of photons, would you get the same pattern of dots on the digital camera image if you covered one slit? Why should it matter whether you give the photon two choices or only one?

## 34.4 Photons in three dimensions

Up until now I've been sneaky and avoided a full discussion of the three-dimensional aspects of the probability interpretation. The example of the carrot in the microwave oven, for example, reduced to a one-dimensional situation because we were considering three points along the same line and because we were only comparing ratios of probabilities. The purpose of bringing it up now is to head off any feeling that you've been cheated conceptually rather than to prepare you for mathematical problem solving in three dimensions, which would not be appropriate for the level of this course.

A typical example of a probability distribution in section 33.3 was the distribution of heights of human beings. The thing that varied randomly, height,  $h$ , had units of meters, and the probability distribution was a graph of a function  $D(h)$ . The units of the probability distribution had to be  $\text{m}^{-1}$  (inverse meters) so that areas under the curve, interpreted as probabilities, would be unitless:  $(\text{area}) = (\text{height})(\text{width}) = \text{m}^{-1} \cdot \text{m}$ .



n / The volume under a surface.

Now suppose we have a two-dimensional problem, e.g., the probability distribution for the place on the surface of a digital camera chip where a photon will be detected. The point where it is detected would be described with two variables,  $x$  and  $y$ , each having units of meters. The probability distribution will be a function of both variables,  $D(x, y)$ . A probability is now visualized as the volume under the surface described by the function  $D(x, y)$ , as shown in figure n. The units of  $D$  must be  $\text{m}^{-2}$  so that probabilities will be unitless:  $(\text{probability}) = (\text{depth})(\text{length})(\text{width}) = \text{m}^{-2} \cdot \text{m} \cdot \text{m}$ .

Generalizing finally to three dimensions, we find by analogy that the probability distribution will be a function of all three coordinates,  $D(x, y, z)$ , and will have units of  $\text{m}^{-3}$ . It is, unfortunately, impossible to visualize the graph unless you are a mutant with a natural feel for life in four dimensions. If the probability distribution is nearly constant within a certain volume of space  $v$ , the probability that the photon is in that volume is simply  $vD$ . If you know enough calculus, it should be clear that this can be generalized to  $P = \int D dx dy dz$  if  $D$  is not constant.

## Summary

### Selected vocabulary

photon . . . . .	a particle of light
photoelectric effect . . . .	the ejection, by a photon, of an electron from the surface of an object
wave-particle duality . . .	the idea that light is both a wave and a particle

### Summary

Around the turn of the twentieth century, experiments began to show problems with the classical wave theory of light. In any experiment sensitive enough to detect very small amounts of light energy, it becomes clear that light energy cannot be divided into chunks smaller than a certain amount. Measurements involving the photoelectric effect demonstrate that this smallest unit of light energy equals  $hf$ , where  $f$  is the frequency of the light and  $h$  is a number known as Planck's constant. We say that light energy is quantized in units of  $hf$ , and we interpret this quantization as evidence that light has particle properties as well as wave properties. Particles of light are called photons.

The only method of reconciling the wave and particle natures of light that has stood the test of experiment is the probability interpretation: the probability that the particle is at a given location is proportional to the square of the amplitude of the wave at that location.

One important consequence of wave-particle duality is that we must abandon the concept of the path the particle takes through space. To hold on to this concept, we would have to contradict the well established wave nature of light, since a wave can spread out in every direction simultaneously.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

*For some of these homework problems, you may find it convenient to refer to the diagram of the electromagnetic spectrum shown on p. 667.*

**1** Give a numerical comparison of the number of photons per second emitted by a hundred-watt FM radio transmitter and a hundred-watt lightbulb. ✓

**2** Two different flashes of light each have the same energy. One consists of photons with a wavelength of 600 nm, the other 400 nm. If the number of photons in the 600-nm flash is  $3.0 \times 10^{18}$ , how many photons are in the 400-nm flash? ✓

**3** When light is reflected from a mirror, perhaps only 80% of the energy comes back. The rest is converted to heat. One could try to explain this in two different ways: (1) 80% of the photons are reflected, or (2) all the photons are reflected, but each loses 20% of its energy. Based on your everyday knowledge about mirrors, how can you tell which interpretation is correct? [Based on a problem from PSSC Physics.]

**4** Suppose we want to build an electronic light sensor using an apparatus like the one described in the section on the photoelectric effect. How would its ability to detect different parts of the spectrum depend on the type of metal used in the capacitor plates?

**5** The photoelectric effect can occur not just for metal cathodes but for any substance, including living tissue. Ionization of DNA molecules can cause cancer or birth defects. If the energy required to ionize DNA is on the same order of magnitude as the energy required to produce the photoelectric effect in a metal, which of these types of electromagnetic waves might pose such a hazard? Explain.

60 Hz waves from power lines

100 MHz FM radio

1900 MHz radio waves from a cellular phone

2450 MHz microwaves from a microwave oven

visible light

ultraviolet light

x-rays

**6** The beam of a 100-W overhead projector covers an area of  $1 \text{ m} \times 1 \text{ m}$  when it hits the screen 3 m away. Estimate the number

of photons that are in flight at any given time. (Since this is only an estimate, we can ignore the fact that the beam is not parallel.)

**7** The two diffraction patterns were made by sending a flash of light through the same double slit. Give a numerical comparison of the amounts of energy in the two flashes. ✓

**8** Three of the four graphs are properly normalized to represent single photons. Which one isn't? Explain.

**9** Photon Fred has a greater energy than photon Ginger. For each of the following quantities, explain whether Fred's value of that quantity is greater than Ginger's, less than Ginger's, or equal to Ginger's. If there is no way to tell, explain why.

frequency

speed

wavelength

period

electric field strength

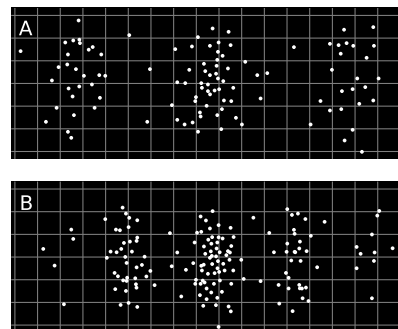
magnetic field strength

**10** Give experimental evidence to disprove the following interpretation of wave-particle duality: *A photon is really a particle, but it travels along a wavy path, like a zigzag with rounded corners.* Cite a specific, real experiment.

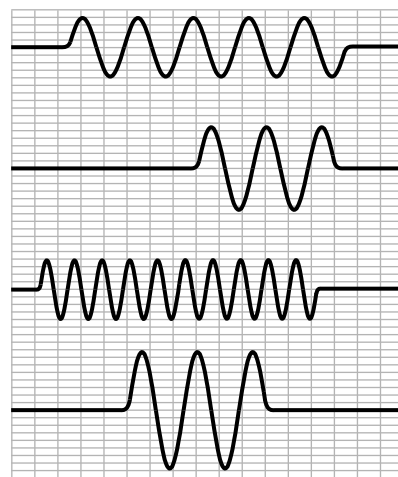
**11** In the photoelectric effect, electrons are observed with virtually no time delay ( $\sim 10$  ns), even when the light source is very weak. (A weak light source does however only produce a small number of ejected electrons.) The purpose of this problem is to show that the lack of a significant time delay contradicted the classical wave theory of light, so throughout this problem you should put yourself in the shoes of a classical physicist and pretend you don't know about photons at all. At that time, it was thought that the electron might have a radius on the order of  $10^{-15}$  m. (Recent experiments have shown that if the electron has any finite size at all, it is far smaller.)

(a) Estimate the power that would be soaked up by a single electron in a beam of light with an intensity of  $1 \text{ mW/m}^2$ . ✓

(b) The energy,  $E_s$ , required for the electron to escape through the surface of the cathode is on the order of  $10^{-19}$  J. Find how long it would take the electron to absorb this amount of energy, and explain why your result constitutes strong evidence that there is something wrong with the classical theory. ✓

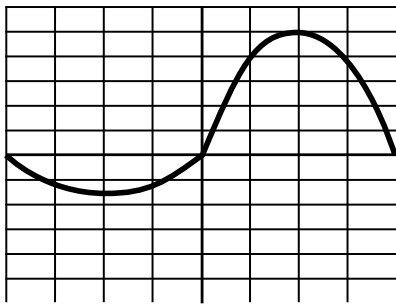


Problem 7.



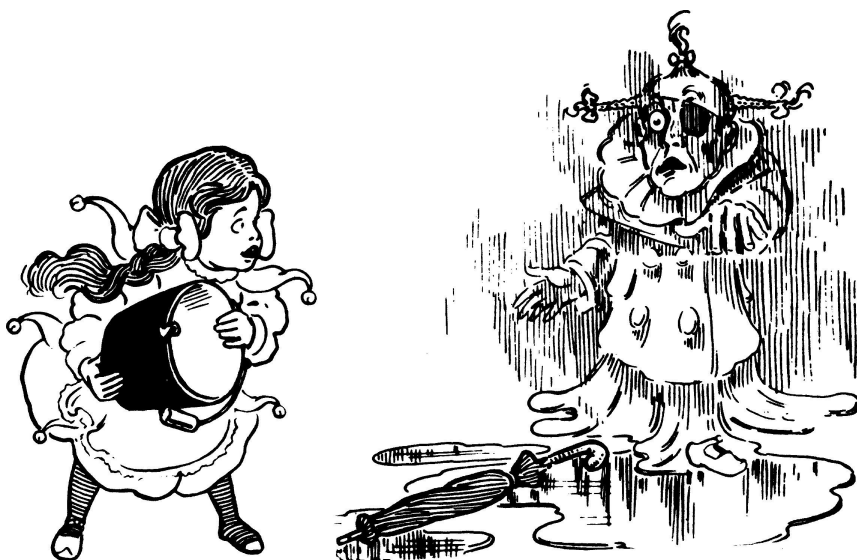
Problem 8.





**12** Many radio antennas are designed so that they preferentially emit or receive electromagnetic waves in a certain direction. However, no antenna is perfectly directional. The wave shown in the figure represents a single photon being emitted by an antenna at the center. The antenna is directional, so there is a stronger wave on the right than on the left. What is the probability that the photon will be observed on the right?

Problem 12.



Dorothy melts the Wicked Witch of the West.

## Chapter 35

# Matter as a wave

[In] a few minutes I shall be all melted... I have been wicked in my day, but I never thought a little girl like you would ever be able to melt me and end my wicked deeds. Look out — here I go!

### *The Wicked Witch of the West*

As the Wicked Witch learned the hard way, losing molecular cohesion can be unpleasant. That's why we should be very grateful that the concepts of quantum physics apply to matter as well as light. If matter obeyed the laws of classical physics, molecules wouldn't exist.

Consider, for example, the simplest atom, hydrogen. Why does one hydrogen atom form a chemical bond with another hydrogen atom? Roughly speaking, we'd expect a neighboring pair of hydrogen atoms, A and B, to exert no force on each other at all, attractive or repulsive: there are two repulsive interactions (proton A with proton B and electron A with electron B) and two attractive interactions (proton A with electron B and electron A with proton B). Thinking a little more precisely, we should even expect that once the two atoms got close enough, the interaction would be repulsive. For instance, if you squeezed them so close together that the two protons were almost on top of each other, there would be a tremendously strong repulsion between them due to the  $1/r^2$  nature of the

electrical force. The repulsion between the electrons would not be as strong, because each electron ranges over a large area, and is not likely to be found right on top of the other electron. This was only a rough argument based on averages, but the conclusion is validated by a more complete classical analysis: hydrogen molecules should not exist according to classical physics.

Quantum physics to the rescue! As we'll see shortly, the whole problem is solved by applying the same quantum concepts to electrons that we have already used for photons.

## 35.1 Electrons as waves

We started our journey into quantum physics by studying the random behavior of *matter* in radioactive decay, and then asked how randomness could be linked to the basic laws of nature governing *light*. The probability interpretation of wave-particle duality was strange and hard to accept, but it provided such a link. It is now natural to ask whether the same explanation could be applied to matter. If the fundamental building block of light, the photon, is a particle as well as a wave, is it possible that the basic units of matter, such as electrons, are waves as well as particles?

A young French aristocrat studying physics, Louis de Broglie (pronounced “broylee”), made exactly this suggestion in his 1923 Ph.D. thesis. His idea had seemed so farfetched that there was serious doubt about whether to grant him the degree. Einstein was asked for his opinion, and with his strong support, de Broglie got his degree.

Only two years later, American physicists C.J. Davisson and L. Germer confirmed de Broglie's idea by accident. They had been studying the scattering of electrons from the surface of a sample of nickel, made of many small crystals. (One can often see such a crystalline pattern on a brass doorknob that has been polished by repeated handling.) An accidental explosion occurred, and when they put their apparatus back together they observed something entirely different: the scattered electrons were now creating an interference pattern! This dramatic proof of the wave nature of matter came about because the nickel sample had been melted by the explosion and then resolidified as a single crystal. The nickel atoms, now nicely arranged in the regular rows and columns of a crystalline lattice, were acting as the lines of a diffraction grating. The new crystal was analogous to the type of ordinary diffraction grating in which the lines are etched on the surface of a mirror (a reflection grating) rather than the kind in which the light passes through the transparent gaps between the lines (a transmission grating).

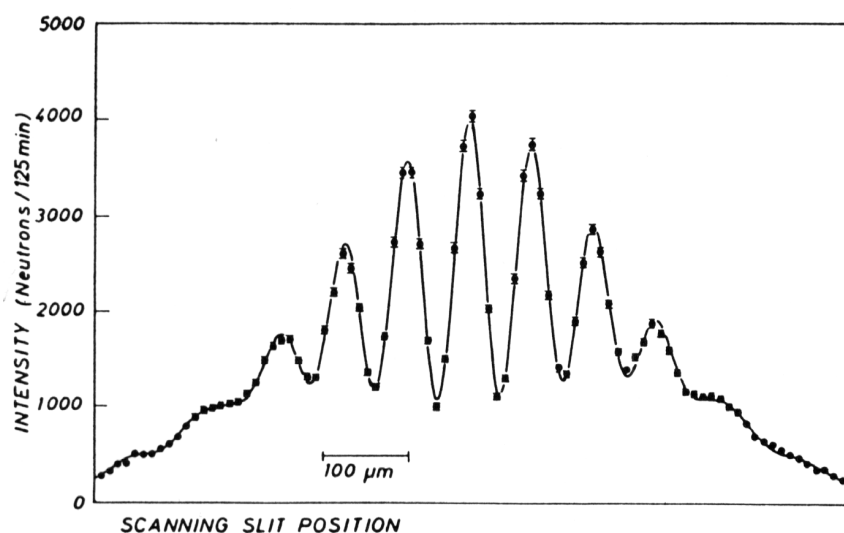
Although we will concentrate on the wave-particle duality of electrons because it is important in chemistry and the physics of atoms,

all the other “particles” of matter you’ve learned about show wave properties as well. Figure a, for instance, shows a wave interference pattern of neutrons.

It might seem as though all our work was already done for us, and there would be nothing new to understand about electrons: they have the same kind of funny wave-particle duality as photons. That’s almost true, but not quite. There are some important ways in which electrons differ significantly from photons:

1. Electrons have mass, and photons don’t.
2. Photons always move at the speed of light, but electrons can move at any speed less than  $c$ .
3. Photons don’t have electric charge, but electrons do, so electric forces can act on them. The most important example is the atom, in which the electrons are held by the electric force of the nucleus.
4. Electrons cannot be absorbed or emitted as photons are. Destroying an electron, or creating one out of nothing, would violate conservation of charge.

(In chapter 36 we will learn of one more fundamental way in which electrons differ from photons, for a total of five.)



a / A double-slit interference pattern made with neutrons. (A. Zeilinger, R. Gähler, C.G. Shull, W. Treimer, and W. Mampe, *Reviews of Modern Physics*, Vol. 60, 1988.)

Because electrons are different from photons, it is not immediately obvious which of the photon equations from chapter 34 can be applied to electrons as well. A particle property, the energy of one photon, is related to its wave properties via  $E = hf$  or, equivalently,  $E = hc/\lambda$ . The momentum of a photon was given by  $p = hf/c$  or

$p = h/\lambda$  (example 2 on page 916). Ultimately it was a matter of experiment to determine which of these equations, if any, would work for electrons, but we can make a quick and dirty guess simply by noting that some of the equations involve  $c$ , the speed of light, and some do not. Since  $c$  is irrelevant in the case of an electron, we might guess that the equations of general validity are those that do not have  $c$  in them:

$$E = hf$$

$$p = \frac{h}{\lambda}$$

This is essentially the reasoning that de Broglie went through, and experiments have confirmed these two equations for all the fundamental building blocks of light and matter, not just for photons and electrons.

The second equation, which I soft-pedaled in chapter 34, takes on a greater importance for electrons. This is first of all because the momentum of matter is more likely to be significant than the momentum of light under ordinary conditions, and also because force is the transfer of momentum, and electrons are affected by electrical forces.

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*The wavelength of an elephant* example 1

▷ What is the wavelength of a trotting elephant?

▷ One may doubt whether the equation should be applied to an elephant, which is not just a single particle but a rather large collection of them. Throwing caution to the wind, however, we estimate the elephant's mass at  $10^3$  kg and its trotting speed at 10 m/s. Its wavelength is therefore roughly

$$\begin{aligned}\lambda &= \frac{h}{p} \\ &= \frac{h}{mv} \\ &= \frac{6.63 \times 10^{-34} \text{ J}\cdot\text{s}}{(10^3 \text{ kg})(10 \text{ m/s})} \\ &\sim 10^{-37} \frac{(\text{kg}\cdot\text{m}^2/\text{s}^2)\cdot\text{s}}{\text{kg}\cdot\text{m/s}} \\ &= 10^{-37} \text{ m} \quad .\end{aligned}$$

The wavelength found in this example is so fantastically small that we can be sure we will never observe any measurable wave phenomena with elephants. The result is numerically small because Planck's constant is so small, and as in some examples encountered previously, this smallness is in accord with the correspondence principle.

Although a smaller mass in the equation  $\lambda = h/mv$  does result in a longer wavelength, the wavelength is still quite short even for individual electrons under typical conditions, as shown in the following example.

*The typical wavelength of an electron* *example 2*

▷ Electrons in circuits and in atoms are typically moving through voltage differences on the order of 1 V, so that a typical energy is  $(e)(1 \text{ V})$ , which is on the order of  $10^{-19} \text{ J}$ . What is the wavelength of an electron with this amount of kinetic energy?

▷ This energy is nonrelativistic, since it is much less than  $mc^2$ . Momentum and energy are therefore related by the nonrelativistic equation  $KE = p^2/2m$ . Solving for  $p$  and substituting in to the equation for the wavelength, we find

$$\begin{aligned}\lambda &= \frac{h}{\sqrt{2m \cdot KE}} \\ &= 1.6 \times 10^{-9} \text{ m} \quad .\end{aligned}$$

This is on the same order of magnitude as the size of an atom, which is no accident: as we will discuss in the next chapter in more detail, an electron in an atom can be interpreted as a standing wave. The smallness of the wavelength of a typical electron also helps to explain why the wave nature of electrons wasn't discovered until a hundred years after the wave nature of light. To scale the usual wave-optics devices such as diffraction gratings down to the size needed to work with electrons at ordinary energies, we need to make them so small that their parts are comparable in size to individual atoms. This is essentially what Davisson and Germer did with their nickel crystal.

*self-check A*

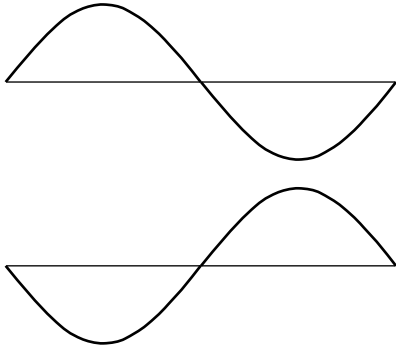
These remarks about the inconvenient smallness of electron wavelengths apply only under the assumption that the electrons have typical energies. What kind of energy would an electron have to have in order to have a longer wavelength that might be more convenient to work with?

▷ Answer, p. 980

### What kind of wave is it?

If a sound wave is a vibration of matter, and a photon is a vibration of electric and magnetic fields, what kind of a wave is an electron made of? The disconcerting answer is that there is no experimental “observable,” i.e., directly measurable quantity, to correspond to the electron wave itself. In other words, there are devices like microphones that detect the oscillations of air pressure in a sound wave, and devices such as radio receivers that measure the oscillation of the electric and magnetic fields in a light wave, but nobody has ever found any way to measure an electron wave directly.

We can of course detect the energy (or momentum) possessed by an electron just as we could detect the energy of a photon using a digital



b / These two electron waves are not distinguishable by any measuring device.

camera. (In fact I'd imagine that an unmodified digital camera chip placed in a vacuum chamber would detect electrons just as handily as photons.) But this only allows us to determine where the wave carries high probability and where it carries low probability. Probability is proportional to the square of the wave's amplitude, but measuring its square is not the same as measuring the wave itself. In particular, we get the same result by squaring either a positive number or its negative, so there is no way to determine the positive or negative sign of an electron wave.

Most physicists tend toward the school of philosophy known as operationalism, which says that a concept is only meaningful if we can define some set of operations for observing, measuring, or testing it. According to a strict operationalist, then, the electron wave itself is a meaningless concept. Nevertheless, it turns out to be one of those concepts like love or humor that is impossible to measure and yet very useful to have around. We therefore give it a symbol,  $\Psi$  (the capital Greek letter psi), and a special name, the electron *wavefunction* (because it is a function of the coordinates  $x, y$ , and  $z$  that specify where you are in space). It would be impossible, for example, to calculate the shape of the electron wave in a hydrogen atom without having some symbol for the wave. But when the calculation produces a result that can be compared directly to experiment, the final algebraic result will turn out to involve only  $\Psi^2$ , which is what is observable, not  $\Psi$  itself.

Since  $\Psi$ , unlike  $\mathbf{E}$  and  $\mathbf{B}$ , is not directly measurable, we are free to make the probability equations have a simple form: instead of having the probability distribution equal to some funny constant multiplied by  $\Psi^2$ , we simply define  $\Psi$  so that the constant of proportionality is one:

$$(\text{probability distribution}) = \Psi^2 \quad .$$

Since the probability distribution has units of  $\text{m}^{-3}$ , the units of  $\Psi$  must be  $\text{m}^{-3/2}$ .

### Discussion question

**A** Frequency is oscillations per second, whereas wavelength is meters per oscillation. How could the equations  $E = hf$  and  $p = h/\lambda$  be made to look more alike by using quantities that were more closely analogous? (This more symmetric treatment makes it easier to incorporate relativity into quantum mechanics, since relativity says that space and time are not entirely separate.)

## 35.2 $\int \star$ Dispersive waves

A colleague of mine who teaches chemistry loves to tell the story about an exceptionally bright student who, when told of the equation  $p = h/\lambda$ , protested, “But when I derived it, it had a factor of 2!” The issue that’s involved is a real one, albeit one that could be glossed over (and is, in most textbooks) without raising any alarms in the mind of the average student. The present optional section addresses this point; it is intended for the student who wishes to delve a little deeper.

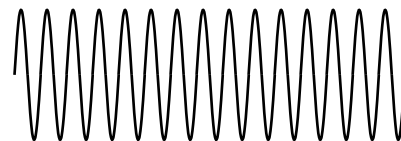
Here’s how the now-legendary student was presumably reasoning. We start with the equation  $v = f\lambda$ , which is valid for any sine wave, whether it’s quantum or classical. Let’s assume we already know  $E = hf$ , and are trying to derive the relationship between wavelength and momentum:

$$\begin{aligned}\lambda &= \frac{v}{f} \\ &= \frac{vh}{E} \\ &= \frac{vh}{\frac{1}{2}mv^2} \\ &= \frac{2h}{mv} \\ &= \frac{2h}{p}\end{aligned}$$

The reasoning seems valid, but the result does contradict the accepted one, which is after all solidly based on experiment.

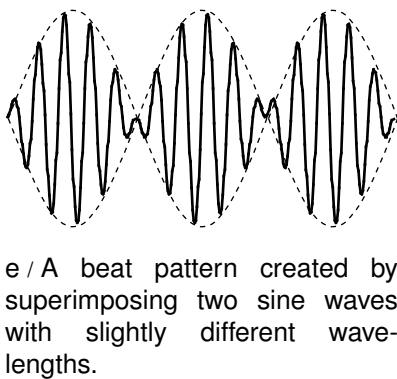
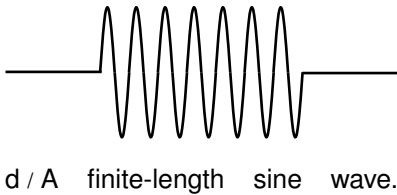
The mistaken assumption is that we can figure everything out in terms of pure sine waves. Mathematically, the only wave that has a perfectly well defined wavelength and frequency is a sine wave, and not just any sine wave but an infinitely long one, *c.* The unphysical thing about such a wave is that it has no leading or trailing edge, so it can never be said to enter or leave any particular region of space. Our derivation made use of the velocity,  $v$ , and if velocity is to be a meaningful concept, it must tell us how quickly stuff (mass, energy, momentum,...) is transported from one region of space to another. Since an infinitely long sine wave doesn’t remove any stuff from one region and take it to another, the “velocity of its stuff” is not a well defined concept.

Of course the individual wave peaks do travel through space, and one might think that it would make sense to associate their speed with the “speed of stuff,” but as we will see, the two velocities are in general unequal when a wave’s velocity depends on wavelength. Such a wave is called a *dispersive* wave, because a wave pulse consisting of a superposition of waves of different wavelengths will separate (disperse) into its separate wavelengths as the waves move through



c / Part of an infinite sine wave.





space at different speeds. Nearly all the waves we have encountered have been nondispersive. For instance, sound waves and light waves (in a vacuum) have speeds independent of wavelength. A water wave is one good example of a dispersive wave. Long-wavelength water waves travel faster, so a ship at sea that encounters a storm typically sees the long-wavelength parts of the wave first. When dealing with dispersive waves, we need symbols and words to distinguish the two speeds. The speed at which wave peaks move is called the phase velocity,  $v_p$ , and the speed at which “stuff” moves is called the group velocity,  $v_g$ .

An infinite sine wave can only tell us about the phase velocity, not the group velocity, which is really what we would be talking about when we referred to the speed of an electron. If an infinite sine wave is the simplest possible wave, what’s the next best thing? We might think the runner up in simplicity would be a wave train consisting of a chopped-off segment of a sine wave,  $d$ . However, this kind of wave has kinks in it at the end. A simple wave should be one that we can build by superposing a small number of infinite sine waves, but a kink can never be produced by superposing any number of infinitely long sine waves.

Actually the simplest wave that transports stuff from place to place is the pattern shown in figure  $e$ . Called a beat pattern, it is formed by superposing two sine waves whose wavelengths are similar but not quite the same. If you have ever heard the pulsating howling sound of musicians in the process of tuning their instruments to each other, you have heard a beat pattern. The beat pattern gets stronger and weaker as the two sine waves go in and out of phase with each other. The beat pattern has more “stuff” (energy, for example) in the areas where constructive interference occurs, and less in the regions of cancellation. As the whole pattern moves through space, stuff is transported from some regions and into other ones.

If the frequency of the two sine waves differs by 10%, for instance, then ten periods will occur between times when they are in phase. Another way of saying it is that the sinusoidal “envelope” (the dashed lines in figure  $e$ ) has a frequency equal to the difference in frequency between the two waves. For instance, if the waves had frequencies of 100 Hz and 110 Hz, the frequency of the envelope would be 10 Hz.

To apply similar reasoning to the wavelength, we must define a quantity  $z = 1/\lambda$  that relates to wavelength in the same way that frequency relates to period. In terms of this new variable, the  $z$  of the envelope equals the difference between the  $z$ ’s of the two sine waves.

The group velocity is the speed at which the envelope moves through space. Let  $\Delta f$  and  $\Delta z$  be the differences between the frequencies and  $z$ ’s of the two sine waves, which means that they equal the frequency

and  $z$  of the envelope. The group velocity is  $v_g = f_{\text{envelope}} \lambda_{\text{envelope}} = \Delta f / \Delta z$ . If  $\Delta f$  and  $\Delta z$  are sufficiently small, we can approximate this expression as a derivative,

$$v_g = \frac{df}{dz} \quad .$$

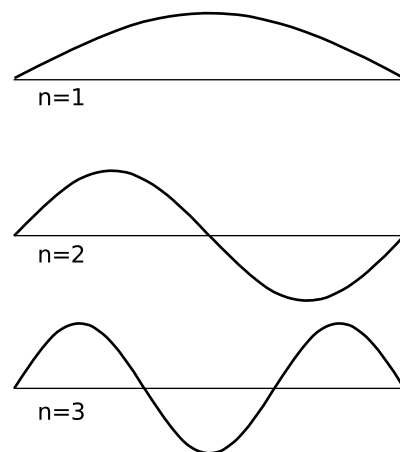
This expression is usually taken as the definition of the group velocity for wave patterns that consist of a superposition of sine waves having a narrow range of frequencies and wavelengths. In quantum mechanics, with  $f = E/h$  and  $z = p/h$ , we have  $v_g = dE/dp$ . In the case of a nonrelativistic electron the relationship between energy and momentum is  $E = p^2/2m$ , so the group velocity is  $dE/dp = p/m = v$ , exactly what it should be. It is only the phase velocity that differs by a factor of two from what we would have expected, but the phase velocity is not the physically important thing.

### 35.3 Bound states

Electrons are at their most interesting when they're in atoms, that is, when they are bound within a small region of space. We can understand a great deal about atoms and molecules based on simple arguments about such bound states, without going into any of the realistic details of atom. The simplest model of a bound state is known as the particle in a box: like a ball on a pool table, the electron feels zero force while in the interior, but when it reaches an edge it encounters a wall that pushes back inward on it with a large force. In particle language, we would describe the electron as bouncing off of the wall, but this incorrectly assumes that the electron has a certain path through space. It is more correct to describe the electron as a wave that undergoes 100% reflection at the boundaries of the box.

Like a generation of physics students before me, I rolled my eyes when initially introduced to the unrealistic idea of putting a particle in a box. It seemed completely impractical, an artificial textbook invention. Today, however, it has become routine to study electrons in rectangular boxes in actual laboratory experiments. The “box” is actually just an empty cavity within a solid piece of silicon, amounting in volume to a few hundred atoms. The methods for creating these electron-in-a-box setups (known as “quantum dots”) were a by-product of the development of technologies for fabricating computer chips.

For simplicity let's imagine a one-dimensional electron in a box, i.e., we assume that the electron is only free to move along a line. The resulting standing wave patterns, of which the first three are shown in figure f, are just like some of the patterns we encountered with sound waves in musical instruments. The wave patterns must be zero at the ends of the box, because we are assuming the walls



f / Three possible standing-wave patterns for a particle in a box.

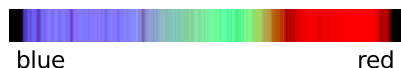
are impenetrable, and there should therefore be zero probability of finding the electron outside the box. Each wave pattern is labeled according to  $n$ , the number of peaks and valleys it has. In quantum physics, these wave patterns are referred to as “states” of the particle-in-the-box system.

The following seemingly innocuous observations about the particle in the box lead us directly to the solutions to some of the most vexing failures of classical physics:

*The particle’s energy is quantized (can only have certain values).* Each wavelength corresponds to a certain momentum, and a given momentum implies a definite kinetic energy,  $E = p^2/2m$ . (This is the second type of energy quantization we have encountered. The type we studied previously had to do with restricting the number of particles to a whole number, while assuming some specific wavelength and energy for each particle. This type of quantization refers to the energies that a single particle can have. Both photons and matter particles demonstrate both types of quantization under the appropriate circumstances.)

*The particle has a minimum kinetic energy.* Long wavelengths correspond to low momenta and low energies. There can be no state with an energy lower than that of the  $n = 1$  state, called the ground state.

*The smaller the space in which the particle is confined, the higher its kinetic energy must be.* Again, this is because long wavelengths give lower energies.



g / The spectrum of the light from the star Sirius. Photograph by the author.

### Spectra of thin gases

### example 3

A fact that was inexplicable by classical physics was that thin gases absorb and emit light only at certain wavelengths. This was observed both in earthbound laboratories and in the spectra of stars. Figure g shows the example of the spectrum of the star Sirius, in which there are “gap teeth” at certain wavelengths. Taking this spectrum as an example, we can give a straightforward explanation using quantum physics.

Energy is released in the dense interior of the star, but the outer layers of the star are thin, so the atoms are far apart and electrons are confined within individual atoms. Although their standing-wave patterns are not as simple as those of the particle in the box, their energies are quantized.

When a photon is on its way out through the outer layers, it can be absorbed by an electron in an atom, but only if the amount of energy it carries happens to be the right amount to kick the electron from one of the allowed energy levels to one of the higher levels. The photon energies that are missing from the spectrum are the ones that equal the difference in energy between two electron energy levels. (The most prominent of the absorption lines in

Sirius's spectrum are absorption lines of the hydrogen atom.)

#### *The stability of atoms*

*example 4*

In many Star Trek episodes the Enterprise, in orbit around a planet, suddenly lost engine power and began spiraling down toward the planet's surface. This was utter nonsense, of course, due to conservation of energy: the ship had no way of getting rid of energy, so it did not need the engines to replenish it.

Consider, however, the electron in an atom as it orbits the nucleus. The electron *does* have a way to release energy: it has an acceleration due to its continuously changing direction of motion, and according to classical physics, any accelerating charged particle emits electromagnetic waves. According to classical physics, atoms should collapse!

The solution lies in the observation that a bound state has a minimum energy. An electron in one of the higher-energy atomic states can and does emit photons and hop down step by step in energy. But once it is in the ground state, it cannot emit a photon because there is no lower-energy state for it to go to.

#### *Chemical bonds in hydrogen molecules*

*example 5*

I began this chapter with a classical argument that chemical bonds, as in an  $H_2$  molecule, should not exist. Quantum physics explains why this type of bonding does in fact occur. When the atoms are next to each other, the electrons are shared between them. The “box” is about twice as wide, and a larger box allows a smaller energy. Energy is required in order to separate the atoms. (A qualitatively different type of bonding is discussed in on page 965.)

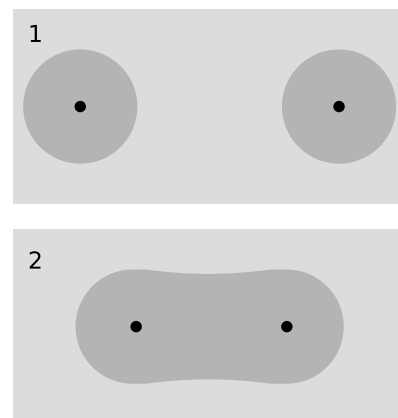
### Discussion questions

**A** Neutrons attract each other via the strong nuclear force, so according to classical physics it should be possible to form nuclei out of clusters of two or more neutrons, with no protons at all. Experimental searches, however, have failed to turn up evidence of a stable two-neutron system (dineutron) or larger stable clusters. These systems are apparently not just unstable in the sense of being able to beta decay but unstable in the sense that they don't hold together at all. Explain based on quantum physics why a dineutron might spontaneously fly apart.

**B** The following table shows the energy gap between the ground state and the first excited state for four nuclei, in units of picojoules. (The nuclei were chosen to be ones that have similar structures, e.g., they are all spherical in shape.)

nucleus	energy gap (picojoules)
$^4\text{He}$	3.234
$^{16}\text{O}$	0.968
$^{40}\text{Ca}$	0.536
$^{208}\text{Pb}$	0.418

Explain the trend in the data.



h / Example 5: Two hydrogen atoms bond to form an  $H_2$  molecule. In the molecule, the two electrons' wave patterns overlap, and are about twice as wide.

## 35.4 The uncertainty principle

### The uncertainty principle

#### *Eliminating randomness through measurement?*

A common reaction to quantum physics, among both early-twentieth-century physicists and modern students, is that we should be able to get rid of randomness through accurate measurement. If I say, for example, that it is meaningless to discuss the path of a photon or an electron, you might suggest that we simply measure the particle's position and velocity many times in a row. This series of snapshots would amount to a description of its path.

A practical objection to this plan is that the process of measurement will have an effect on the thing we are trying to measure. This may not be of much concern, for example, when a traffic cop measures your car's motion with a radar gun, because the energy and momentum of the radar pulses aren't enough to change the car's motion significantly. But on the subatomic scale it is a very real problem. Making a videotape of an electron orbiting a nucleus is not just difficult, it is theoretically impossible, even with the video camera hooked up to the best imaginable microscope. The video camera makes pictures of things using light that has bounced off them and come into the camera. If even a single photon of the right wavelength was to bounce off of the electron we were trying to study, the electron's recoil would be enough to change its behavior significantly (see homework problem 4).

#### *The heisenberg uncertainty principle*

This insight, that measurement changes the thing being measured, is the kind of idea that clove-cigarette-smoking intellectuals outside of the physical sciences like to claim they knew all along. If only, they say, the physicists had made more of a habit of reading literary journals, they could have saved a lot of work. The anthropologist Margaret Mead has recently been accused of inadvertently encouraging her teenaged Samoan informants to exaggerate the freedom of youthful sexual experimentation in their society. If this is considered a damning critique of her work, it is because she could have done better: other anthropologists claim to have been able to eliminate the observer-as-participant problem and collect untainted data.

The German physicist Werner Heisenberg, however, showed that in quantum physics, *any* measuring technique runs into a brick wall when we try to improve its accuracy beyond a certain point. Heisenberg showed that the limitation is a question of *what there is to be known*, even in principle, about the system itself, not of the inability of a particular measuring device to ferret out information that is knowable.

Suppose, for example, that we have constructed an electron in a box

(quantum dot) setup in our laboratory, and we are able to adjust the length  $L$  of the box as desired. All the standing wave patterns pretty much fill the box, so our knowledge of the electron's position is of limited accuracy. If we write  $\Delta x$  for the range of uncertainty in our knowledge of its position, then  $\Delta x$  is roughly the same as the length of the box:

$$\Delta x \approx L$$

If we wish to know its position more accurately, we can certainly squeeze it into a smaller space by reducing  $L$ , but this has an unintended side-effect. A standing wave is really a superposition of two traveling waves going in opposite directions. The equation  $p = h/\lambda$  only gives the magnitude of the momentum vector, not its direction, so we should really interpret the wave as a 50/50 mixture of a right-going wave with momentum  $p = h/\lambda$  and a left-going one with momentum  $p = -h/\lambda$ . The uncertainty in our knowledge of the electron's momentum is  $\Delta p = 2h/\lambda$ , covering the range between these two values. Even if we make sure the electron is in the ground state, whose wavelength  $\lambda = 2L$  is the longest possible, we have an uncertainty in momentum of  $\Delta p = h/L$ . In general, we find

$$\Delta p \gtrsim h/L \quad ,$$

with equality for the ground state and inequality for the higher-energy states. Thus if we reduce  $L$  to improve our knowledge of the electron's position, we do so at the cost of knowing less about its momentum. This trade-off is neatly summarized by multiplying the two equations to give

$$\Delta p \Delta x \gtrsim h \quad .$$

Although we have derived this in the special case of a particle in a box, it is an example of a principle of more general validity:

### **the Heisenberg uncertainty principle**

It is not possible, even in principle, to know the momentum and the position of a particle simultaneously and with perfect accuracy. The uncertainties in these two quantities are always such that

$$\Delta p \Delta x \gtrsim h \quad .$$

(This approximation can be made into a strict inequality,  $\Delta p \Delta x > h/4\pi$ , but only with more careful definitions, which we will not bother with.<sup>1</sup>)

Note that although I encouraged you to think of this derivation in terms of a specific real-world system, the quantum dot, I never

<sup>1</sup>See homework problems 6 and 7.

made any reference to specific measuring equipment. The argument is simply that we cannot *know* the particle's position very accurately unless it *has* a very well defined position, it cannot have a very well defined position unless its wave-pattern covers only a very small amount of space, and its wave-pattern cannot be thus compressed without giving it a short wavelength and a correspondingly uncertain momentum. The uncertainty principle is therefore a restriction on how much there is to know about a particle, not just on what we can know about it with a certain technique.

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*An estimate for electrons in atoms* *example 6*

▷ A typical energy for an electron in an atom is on the order of  $(1 \text{ volt}) \cdot e$ , which corresponds to a speed of about 1% of the speed of light. If a typical atom has a size on the order of 0.1 nm, how close are the electrons to the limit imposed by the uncertainty principle?

▷ If we assume the electron moves in all directions with equal probability, the uncertainty in its momentum is roughly twice its typical momentum. This only an order-of-magnitude estimate, so we take  $\Delta p$  to be the same as a typical momentum:

$$\begin{aligned}\Delta p \Delta x &= p_{\text{typical}} \Delta x \\ &= (m_{\text{electron}})(0.01c)(0.1 \times 10^{-9} \text{ m}) \\ &= 3 \times 10^{-34} \text{ J} \cdot \text{s}\end{aligned}$$

This is on the same order of magnitude as Planck's constant, so evidently the electron is "right up against the wall." (The fact that it is somewhat less than  $h$  is of no concern since this was only an estimate, and we have not stated the uncertainty principle in its most exact form.)

*self-check B*

If we were to apply the uncertainty principle to human-scale objects, what would be the significance of the small numerical value of Planck's constant?

▷ Answer, p. 981

*self-check C*

Suppose rain is falling on your roof, and there is a tiny hole that lets raindrops into your living room now and then. All these drops hit the same spot on the floor, so they have the same value of  $x$ . Not only that, but if the rain is falling straight down, they all have zero horizontal momentum. Thus it seems that the raindrops have  $\Delta p = 0$ ,  $\Delta x = 0$ , and  $\Delta p \Delta x = 0$ , violating the uncertainty principle. To look for the hole in this argument, consider how it would be acted out on the microscopic scale: an electron wave comes along and hits a narrow slit. What really happens?

▷ Answer, p. 981

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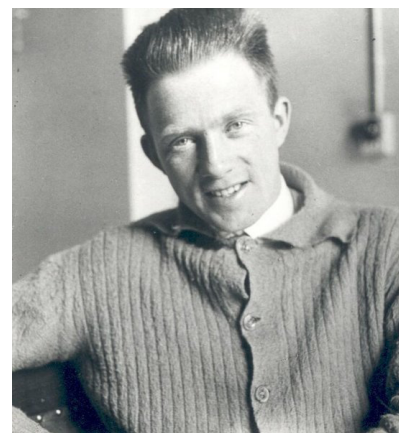
**Historical Note**

The true nature of Heisenberg's role in the Nazi atomic bomb effort is a fascinating question, and dramatic enough to have inspired a well-received 1998 theatrical play, "Copenhagen." The real story, however,

may never be completely unraveled. Heisenberg was the scientific leader of the German bomb program up until its cancellation in 1942, when the German military decided that it was too ambitious a project to undertake in wartime, and too unlikely to produce results.

Some historians believe that Heisenberg intentionally delayed and obstructed the project because he secretly did not want the Nazis to get the bomb. Heisenberg's apologists point out that he never joined the Nazi party, and was not anti-Semitic. He actively resisted the government's *Deutsche-Physik* policy of eliminating supposed Jewish influences from physics, and as a result was denounced by the S.S. as a traitor, escaping punishment only because Himmler personally declared him innocent. One strong piece of evidence is a secret message carried to the U.S. in 1941, by one of the last Jews to escape from Berlin, and eventually delivered to the chairman of the Uranium Committee, which was then studying the feasibility of a bomb. The message stated "...that a large number of German physicists are working intensively on the problem of the uranium bomb under the direction of Heisenberg, [and] that Heisenberg himself tries to delay the work as much as possible, fearing the catastrophic results of success. But he cannot help fulfilling the orders given to him, and if the problem can be solved, it will be solved probably in the near future. So he gave the advice to us to hurry up if U.S.A. will not come too late." The message supports the view that Heisenberg intentionally misled his government about the bomb's technical feasibility; German Minister of Armaments Albert Speer wrote that he was convinced to drop the project after a 1942 meeting with Heisenberg because "the physicists themselves didn't want to put too much into it." Heisenberg also may have warned Danish physicist Niels Bohr personally in September 1941 about the existence of the Nazi bomb effort.

On the other side of the debate, critics of Heisenberg say that he clearly wanted Germany to win the war, that he visited German-occupied territories in a semi-official role, and that he simply may not have been very good at his job directing the bomb project. On a visit to the occupied Netherlands in 1943, he told a colleague, "Democracy cannot develop sufficient energy to rule Europe. There are, therefore, only two alternatives: Germany and Russia. And then a Europe under German leadership would be the lesser evil." Some historians<sup>2</sup> argue that the real point of Heisenberg's meeting with Bohr was to try to convince the U.S. not to try to build a bomb, so that Germany, possessing a nuclear monopoly, would defeat the Soviets — this was after the June 1941 entry of the U.S.S.R. into the war, but before the December 1941 Pearl Harbor attack brought the U.S. in. Bohr apparently considered Heisenberg's account of the meeting, published after the war was over, to be inaccurate.<sup>3</sup> The secret 1941 message also has a curious moral passivity to it, as if Heisenberg was saying "I hope you stop me before I do something bad," but we should also consider the great risk Heisenberg would have been running if he actually originated the message.



i / Werner Heisenberg.

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<sup>2</sup>A Historical Perspective on Copenhagen, David C. Cassidy, *Physics Today*, July 2000, p. 28, <http://www.aip.org/pt/vol-53/iss-7/p28.html>

<sup>3</sup>Bohr drafted several replies, but never published them for fear of hurting Heisenberg and his family. Bohr's papers were to be sealed for 50 years after his death, but his family released them early, in February 2002.



## Measurement and Schrödinger's cat

In chapter 34 I briefly mentioned an issue concerning measurement that we are now ready to address carefully. If you hang around a laboratory where quantum-physics experiments are being done and secretly record the physicists' conversations, you'll hear them say many things that assume the probability interpretation of quantum mechanics. Usually they will speak as though the randomness of quantum mechanics enters the picture when something is measured. In the digital camera experiments of chapter 34, for example, they would casually describe the detection of a photon at one of the pixels as if the moment of detection was when the photon was forced to "make up its mind." Although this mental cartoon usually works fairly well as a description of things one experiences in the lab, it cannot ultimately be correct, because it attributes a special role to measurement, which is really just a physical process like all other physical processes.

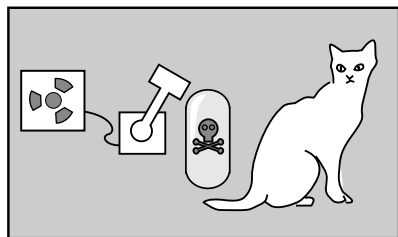
If we are to find an interpretation that avoids giving any special role to measurement processes, then we must think of the entire laboratory, including the measuring devices and the physicists themselves, as one big quantum-mechanical system made out of protons, neutrons, electrons, and photons. In other words, we should take quantum physics seriously as a description not just of microscopic objects like atoms but of human-scale ("macroscopic") things like the apparatus, the furniture, and the people.

The most celebrated example is called the Schrödinger's cat experiment. Luckily for the cat, there probably was no actual experiment — it was simply a "thought experiment" that the German theorist Schrödinger discussed with his colleagues. Schrödinger wrote:

One can even construct quite burlesque cases. A cat is shut up in a steel container, together with the following diabolical apparatus (which one must keep out of the direct clutches of the cat): In a [radiation detector] there is a tiny mass of radioactive substance, so little that in the course of an hour perhaps one atom of it disintegrates, but also with equal probability not even one; if it does happen, the [detector] responds and ... activates a hammer that shatters a little flask of prussic acid [filling the chamber with poison gas]. If one has left this entire system to itself for an hour, then one will say to himself that the cat is still living, if in that time no atom has disintegrated. The first atomic disintegration would have poisoned it.

Now comes the strange part. Quantum mechanics says that the particles the cat is made of have wave properties, including the property of superposition. Schrödinger describes the wavefunction of the box's contents at the end of the hour:

The wavefunction of the entire system would express this situation by having the living and the dead cat mixed ... in equal parts [50/50 proportions]. The uncertainty originally restricted to the atomic domain has been transformed into a macroscopic uncertainty...



j / Schrödinger's cat.

At first Schrödinger's description seems like nonsense. When you opened the box, would you see two ghostlike cats, as in a doubly exposed photograph, one dead and one alive? Obviously not. You would have a single, fully material cat, which would either be dead or very, very upset. But Schrödinger has an equally strange and logical answer for that objection. In the same way that the quantum randomness of the radioactive atom spread to the cat and made its wavefunction a random mixture of life and death, the randomness spreads wider once you open the box, and your own wavefunction becomes a mixture of a person who has just killed a cat and a person who hasn't.

### Discussion questions

**A** Compare  $\Delta p$  and  $\Delta x$  for the two lowest energy levels of the one-dimensional particle in a box, and discuss how this relates to the uncertainty principle.

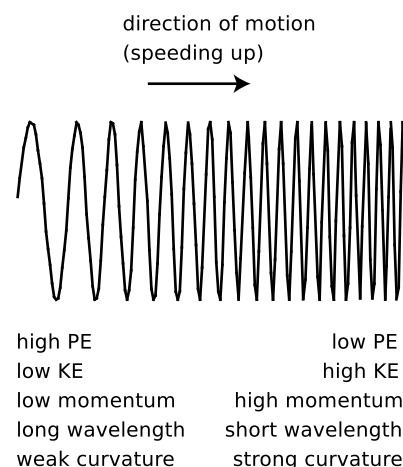
**B** On a graph of  $\Delta p$  versus  $\Delta x$ , sketch the regions that are allowed and forbidden by the Heisenberg uncertainty principle. Interpret the graph: Where does an atom lie on it? An elephant? Can either  $p$  or  $x$  be measured with perfect accuracy if we don't care about the other?

## 35.5 Electrons in electric fields

So far the only electron wave patterns we've considered have been simple sine waves, but whenever an electron finds itself in an electric field, it must have a more complicated wave pattern. Let's consider the example of an electron being accelerated by the electron gun at the back of a TV tube. The electron is moving from a region of low voltage into a region of higher voltage. Since its charge is negative, it loses PE by moving to a higher voltage, so its KE increases. As its potential energy goes down, its kinetic energy goes up by an equal amount, keeping the total energy constant. Increasing kinetic energy implies a growing momentum, and therefore a shortening wavelength,  $k$ .

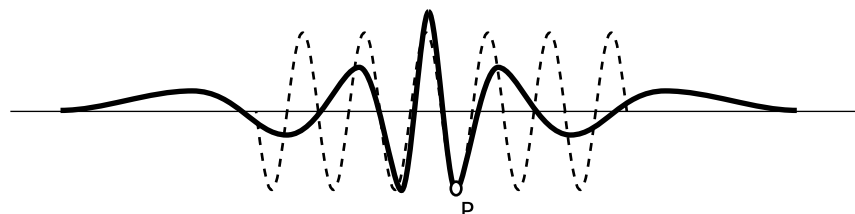
The wavefunction as a whole does not have a single well-defined wavelength, but the wave changes so gradually that if you only look at a small part of it you can still pick out a wavelength and relate it to the momentum and energy. (The picture actually exaggerates by many orders of magnitude the rate at which the wavelength changes.)

But what if the electric field was stronger? The electric field in a TV is only  $\sim 10^5$  N/C, but the electric field within an atom is more like  $10^{12}$  N/C. In figure 1, the wavelength changes so rapidly that there is nothing that looks like a sine wave at all. We could get a general idea of the wavelength in a given region by measuring the distance between two peaks, but that would only be a rough approximation. Suppose we want to know the wavelength at point P. The trick is

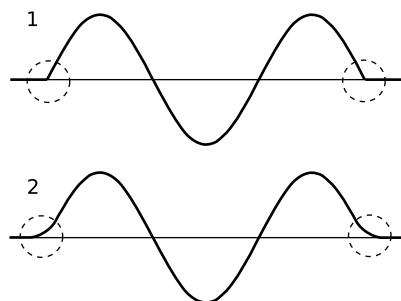


**k /** An electron in a gentle electric field gradually shortens its wavelength as it gains energy.

to construct a sine wave, like the one shown with the dashed line, which matches the curvature of the actual wavefunction as closely as possible near P. The sine wave that matches as well as possible is called the “osculating” curve, from a Latin word meaning “to kiss.” The wavelength of the osculating curve is the wavelength that will relate correctly to conservation of energy.



! / A typical wavefunction of an electron in an atom (heavy curve) and the osculating sine wave (dashed curve) that matches its curvature at point P.



m / 1. Kinks like this don't happen. 2. The wave actually penetrates into the classically forbidden region.

## Tunneling

We implicitly assumed that the particle-in-a-box wavefunction would cut off abruptly at the sides of the box, m/1, but that would be unphysical. A kink has infinite curvature, and curvature is related to energy, so it can't be infinite. A physically realistic wavefunction must always “tail off” gradually, m/2. In classical physics, a particle can never enter a region in which its potential energy would be greater than the amount of energy it has available. But in quantum physics the wavefunction will always have a tail that reaches into the classically forbidden region. If it was not for this effect, called tunneling, the fusion reactions that power the sun would not occur due to the high potential energy that nuclei need in order to get close together! Tunneling is discussed in more detail in the next section.

## 35.6 ∫ ★ The Schrödinger equation

In section 35.5 we were able to apply conservation of energy to an electron's wavefunction, but only by using the clumsy graphical technique of osculating sine waves as a measure of the wave's curvature. You have learned a more convenient measure of curvature in calculus: the second derivative. To relate the two approaches, we take the second derivative of a sine wave:

$$\begin{aligned}\frac{d^2}{dx^2} \sin\left(\frac{2\pi x}{\lambda}\right) &= \frac{d}{dx} \left( \frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda} \right) \\ &= - \left( \frac{2\pi}{\lambda} \right)^2 \sin \frac{2\pi x}{\lambda}\end{aligned}$$

Taking the second derivative gives us back the same function, but with a minus sign and a constant out in front that is related to the wavelength. We can thus relate the second derivative to the osculating wavelength:

$$[1] \quad \frac{d^2\Psi}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \Psi$$

This could be solved for  $\lambda$  in terms of  $\Psi$ , but it will turn out to be more convenient to leave it in this form.

Using conservation of energy, we have

$$[2] \quad \begin{aligned} E &= KE + PE \\ &= \frac{p^2}{2m} + PE \\ &= \frac{(h/\lambda)^2}{2m} + PE \end{aligned}$$

Note that both equation [1] and equation [2] have  $\lambda^2$  in the denominator. We can simplify our algebra by multiplying both sides of equation [2] by  $\Psi$  to make it look more like equation [1]:

$$\begin{aligned} E \cdot \Psi &= \frac{(h/\lambda)^2}{2m} \Psi + PE \cdot \Psi \\ &= \frac{1}{2m} \left(\frac{h}{2\pi}\right)^2 \left(\frac{2\pi}{\lambda}\right)^2 \Psi + PE \cdot \Psi \\ &= -\frac{1}{2m} \left(\frac{h}{2\pi}\right)^2 \frac{d^2\Psi}{dx^2} + PE \cdot \Psi \end{aligned}$$

Further simplification is achieved by using the symbol  $\hbar$  ( $h$  with a slash through it, read “h-bar”) as an abbreviation for  $h/2\pi$ . We then have the important result known as the **Schrödinger equation**:

$$E \cdot \Psi = -\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + PE \cdot \Psi$$

(Actually this is a simplified version of the Schrödinger equation, applying only to standing waves in one dimension.) Physically it is a statement of conservation of energy. The total energy  $E$  must be constant, so the equation tells us that a change in potential energy must be accompanied by a change in the curvature of the wavefunction. This change in curvature relates to a change in wavelength, which corresponds to a change in momentum and kinetic energy.

*self-check D*

Considering the assumptions that were made in deriving the Schrödinger equation, would it be correct to apply it to a photon? To an electron moving at relativistic speeds?

▷ Answer, p.

981

Usually we know right off the bat how the potential energy depends on  $x$ , so the basic mathematical problem of quantum physics is to find a function  $\Psi(x)$  that satisfies the Schrödinger equation for a given function  $PE(x)$ . An equation, such as the Schrödinger equation, that specifies a relationship between a function and its derivatives is known as a differential equation.

The study of differential equations in general is beyond the mathematical level of this book, but we can gain some important insights by considering the easiest version of the Schrödinger equation, in which the potential energy is constant. We can then rearrange the Schrödinger equation as follows:

$$\frac{d^2\Psi}{dx^2} = \frac{2m(PE - E)}{\hbar^2}\Psi \quad ,$$

which boils down to

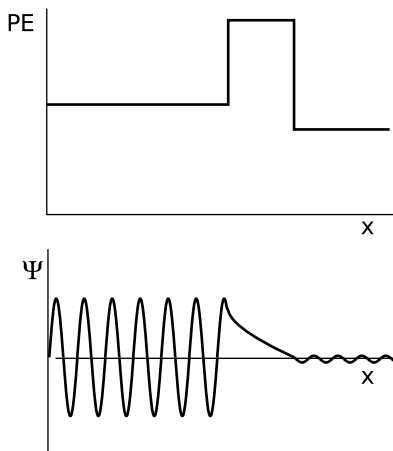
$$\frac{d^2\Psi}{dx^2} = a\Psi \quad ,$$

where, according to our assumptions,  $a$  is independent of  $x$ . We need to find a function whose second derivative is the same as the original function except for a multiplicative constant. The only functions with this property are sine waves and exponentials:

$$\begin{aligned} \frac{d^2}{dx^2} [q \sin(rx + s)] &= -qr^2 \sin(rx + s) \\ \frac{d^2}{dx^2} [qe^{rx+s}] &= qr^2 e^{rx+s} \end{aligned}$$

The sine wave gives negative values of  $a$ ,  $a = -r^2$ , and the exponential gives positive ones,  $a = r^2$ . The former applies to the classically allowed region with  $PE < E$ .

This leads us to a quantitative calculation of the tunneling effect discussed briefly in the preceding subsection. The wavefunction evidently tails off exponentially in the classically forbidden region. Suppose, as shown in figure n, a wave-particle traveling to the right encounters a barrier that it is classically forbidden to enter. Although the form of the Schrödinger equation we're using technically does not apply to traveling waves (because it makes no reference to time), it turns out that we can still use it to make a reasonable calculation of the probability that the particle will make it through the barrier. If we let the barrier's width be  $w$ , then the ratio of the



n / Tunneling through a barrier.

wavefunction on the left side of the barrier to the wavefunction on the right is

$$\frac{qe^{rx+s}}{qe^{r(x+w)+s}} = e^{-rw} \quad .$$

Probabilities are proportional to the squares of wavefunctions, so the probability of making it through the barrier is

$$\begin{aligned} P &= e^{-2rw} \\ &= \exp\left(-\frac{2w}{\hbar}\sqrt{2m(PE - E)}\right) \end{aligned}$$

#### *self-check E*

If we were to apply this equation to find the probability that a person can walk through a wall, what would the small value of Planck's constant imply? ▷ Answer, p. 981

### **Use of complex numbers**

In a classically forbidden region, a particle's total energy,  $PE + KE$ , is less than its  $PE$ , so its  $KE$  must be negative. If we want to keep believing in the equation  $KE = p^2/2m$ , then apparently the momentum of the particle is the square root of a negative number. This is a symptom of the fact that the Schrödinger equation fails to describe all of nature unless the wavefunction and various other quantities are allowed to be complex numbers. In particular it is not possible to describe traveling waves correctly without using complex wavefunctions.

This may seem like nonsense, since real numbers are the only ones that are, well, real! Quantum mechanics can always be related to the real world, however, because its structure is such that the results of measurements always come out to be real numbers. For example, we may describe an electron as having non-real momentum in classically forbidden regions, but its average momentum will always come out to be real (the imaginary parts average out to zero), and it can never transfer a non-real quantity of momentum to another particle.

A complete investigation of these issues is beyond the scope of this book, and this is why we have normally limited ourselves to standing waves, which can be described with real-valued wavefunctions.

# Summary

## Selected vocabulary

wavefunction . . . the numerical measure of an electron wave, or in general of the wave corresponding to any quantum mechanical particle

## Notation

$\hbar$  . . . . . Planck's constant divided by  $2\pi$  (used only in optional section 35.6)  
 $\Psi$  . . . . . the wavefunction of an electron

## Summary

Light is both a particle and a wave. Matter is both a particle and a wave. The equations that connect the particle and wave properties are the same in all cases:

$$E = hf$$
$$p = h/\lambda$$

Unlike the electric and magnetic fields that make up a photon-wave, the electron wavefunction is not directly measurable. Only the square of the wavefunction, which relates to probability, has direct physical significance.

A particle that is bound within a certain region of space is a standing wave in terms of quantum physics. The two equations above can then be applied to the standing wave to yield some important general observations about bound particles:

1. The particle's energy is quantized (can only have certain values).
2. The particle has a minimum energy.
3. The smaller the space in which the particle is confined, the higher its kinetic energy must be.

These immediately resolve the difficulties that classical physics had encountered in explaining observations such as the discrete spectra of atoms, the fact that atoms don't collapse by radiating away their energy, and the formation of chemical bonds.

A standing wave confined to a small space must have a short wavelength, which corresponds to a large momentum in quantum physics. Since a standing wave consists of a superposition of two traveling waves moving in opposite directions, this large momentum should actually be interpreted as an equal mixture of two possible momenta: a large momentum to the left, or a large momentum to the right. Thus it is not possible for a quantum wave-particle to be confined to a small space without making its momentum very uncertain. In general, the Heisenberg uncertainty principle states that

it is not possible to know the position and momentum of a particle simultaneously with perfect accuracy. The uncertainties in these two quantities must satisfy the approximate inequality

$$\Delta p \Delta x \gtrsim h \quad .$$

When an electron is subjected to electric forces, its wavelength cannot be constant. The “wavelength” to be used in the equation  $p = h/\lambda$  should be thought of as the wavelength of the sine wave that most closely approximates the curvature of the wavefunction at a specific point.

Infinite curvature is not physically possible, so realistic wavefunctions cannot have kinks in them, and cannot just cut off abruptly at the edge of a region where the particle’s energy would be insufficient to penetrate according to classical physics. Instead, the wavefunction “tails off” in the classically forbidden region, and as a consequence it is possible for particles to “tunnel” through regions where according to classical physics they should not be able to penetrate. If this quantum tunneling effect did not exist, there would be no fusion reactions to power our sun, because the energies of the nuclei would be insufficient to overcome the electrical repulsion between them.

### Exploring further

**The New World of Mr. Tompkins: George Gamow’s Classic Mr. Tompkins in Paperback**, George Gamow. Mr. Tompkins finds himself in a world where the speed of light is only 30 miles per hour, making relativistic effects obvious. Later parts of the book play similar games with Planck’s constant.

**The First Three Minutes: A Modern View of the Origin of the Universe**, Steven Weinberg. Surprisingly simple ideas allow us to understand the infancy of the universe surprisingly well.

**Three Roads to Quantum Gravity**, Lee Smolin. The greatest embarrassment of physics today is that we are unable to fully reconcile general relativity (the theory of gravity) with quantum mechanics. This book does a good job of introducing the lay reader to a difficult, speculative subject, and showing that even though we don’t have a full theory of quantum gravity, we do have a clear outline of what such a theory must look like.



## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** In a television, suppose the electrons are accelerated from rest through a voltage difference of  $10^4$  V. What is their final wavelength?

✓

**2** Use the Heisenberg uncertainty principle to estimate the minimum velocity of a proton or neutron in a  $^{208}\text{Pb}$  nucleus, which has a diameter of about 13 fm ( $1 \text{ fm} = 10^{-15} \text{ m}$ ). Assume that the speed is nonrelativistic, and then check at the end whether this assumption was warranted.

✓

**3** A free electron that contributes to the current in an ohmic material typically has a speed of  $10^5$  m/s (much greater than the drift velocity).

(a) Estimate its de Broglie wavelength, in nm.

✓

(b) If a computer memory chip contains  $10^8$  electric circuits in a  $1 \text{ cm}^2$  area, estimate the linear size, in nm, of one such circuit.

✓

(c) Based on your answers from parts a and b, does an electrical engineer designing such a chip need to worry about wave effects such as diffraction?

(d) Estimate the maximum number of electric circuits that can fit on a  $1 \text{ cm}^2$  computer chip before quantum-mechanical effects become important.

**4** On page 938, I discussed the idea of hooking up a video camera to a visible-light microscope and recording the trajectory of an electron orbiting a nucleus. An electron in an atom typically has a speed of about 1% of the speed of light.

(a) Calculate the momentum of the electron.

✓

(b) When we make images with photons, we can't resolve details that are smaller than the photons' wavelength. Suppose we wanted to map out the trajectory of the electron with an accuracy of 0.01 nm. What part of the electromagnetic spectrum would we have to use?

(c) As found in homework problem 12 on page 759, the momentum of a photon is given by  $p = E/c$ . Estimate the momentum of a photon having the necessary wavelength.

✓

(d) Comparing your answers from parts a and c, what would be the effect on the electron if the photon bounced off of it? What does this tell you about the possibility of mapping out an electron's orbit around a nucleus?

**5** Find the energy of a particle in a one-dimensional box of length  $L$ , expressing your result in terms of  $L$ , the particle's mass  $m$ , the

number of peaks and valleys  $n$  in the wavefunction, and fundamental constants. ✓

**6** The Heisenberg uncertainty principle,  $\Delta p \Delta x \gtrsim h$ , can only be made into a strict inequality if we agree on a rigorous mathematical definition of  $\Delta x$  and  $\Delta p$ . Suppose we define the deltas in terms of the full width at half maximum (FWHM), which we first encountered in *Vibrations and Waves*, and revisited on page 893 of this book. Now consider the lowest-energy state of the one-dimensional particle in a box. As argued on page 939, the momentum has equal probability of being  $h/L$  or  $-h/L$ , so the FWHM definition gives  $\Delta p = 2h/L$ .  
 (a) Find  $\Delta x$  using the FWHM definition. Keep in mind that the probability distribution depends on the square of the wavefunction.  
 (b) Find  $\Delta x \Delta p$ . ✓

**7** If  $x$  has an average value of zero, then the standard deviation of the probability distribution  $D(x)$  is defined by

$$\sigma^2 = \sqrt{\int D(x) x^2 dx} \quad ,$$

where the integral ranges over all possible values of  $x$ .

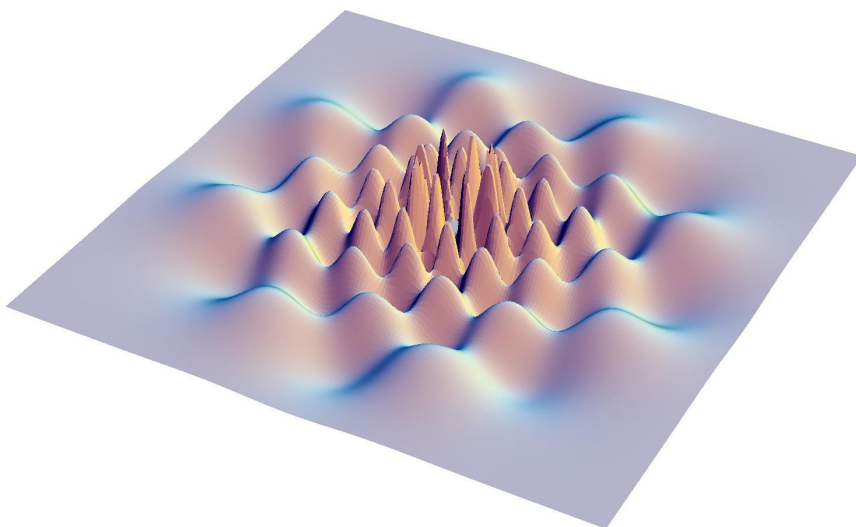
Interpretation: if  $x$  only has a high probability of having values close to the average (i.e., small positive and negative values), the thing being integrated will always be small, because  $x^2$  is always a small number; the standard deviation will therefore be small. Squaring  $x$  makes sure that either a number below the average ( $x < 0$ ) or a number above the average ( $x > 0$ ) will contribute a positive amount to the standard deviation. We take the square root of the whole thing so that it will have the same units as  $x$ , rather than having units of  $x^2$ .

Redo problem 6 using the standard deviation rather than the FWHM.

Hints: (1) You need to determine the amplitude of the wave based on normalization. (2) You'll need the following definite integral:  $\int_{-\pi/2}^{\pi/2} u^2 \cos^2 u du = (\pi^3 - 6\pi)/24$ . ✓  $\int$

**8** In section 35.6 we derived an expression for the probability that a particle would tunnel through a rectangular potential barrier. Generalize this to a barrier of any shape. [Hints: First try generalizing to two rectangular barriers in a row, and then use a series of rectangular barriers to approximate the actual curve of an arbitrary potential. Note that the width and height of the barrier in the original equation occur in such a way that all that matters is the area under the  $PE$ -versus- $x$  curve. Show that this is still true for a series of rectangular barriers, and generalize using an integral.] If you had done this calculation in the 1930's you could have become a famous physicist.  $\int$  ★





A wavefunction of an electron in a hydrogen atom.

## Chapter 36

# The atom

You can learn a lot by taking a car engine apart, but you will have learned a lot more if you can put it all back together again and make it run. Half the job of reductionism is to break nature down into its smallest parts and understand the rules those parts obey. The second half is to show how those parts go together, and that is our goal in this chapter. We have seen how certain features of all atoms can be explained on a generic basis in terms of the properties of bound states, but this kind of argument clearly cannot tell us any details of the behavior of an atom or explain why one atom acts differently from another.

The biggest embarrassment for reductionists is that the job of putting things back together is usually much harder than the taking them apart. Seventy years after the fundamentals of atomic physics were solved, it is only beginning to be possible to calculate accurately the properties of atoms that have many electrons. Systems consisting of many atoms are even harder. Supercomputer manufacturers point to the folding of large protein molecules as a process whose calculation is just barely feasible with their fastest machines. The goal of this chapter is to give a gentle and visually oriented guide to some of the simpler results about atoms.

## 36.1 Classifying states

We'll focus our attention first on the simplest atom, hydrogen, with one proton and one electron. We know in advance a little of what we should expect for the structure of this atom. Since the electron is bound to the proton by electrical forces, it should display a set of discrete energy states, each corresponding to a certain standing wave pattern. We need to understand what states there are and what their properties are.

What properties should we use to classify the states? The most sensible approach is to use conserved quantities. Energy is one conserved quantity, and we already know to expect each state to have a specific energy. It turns out, however, that energy alone is not sufficient. Different standing wave patterns of the atom can have the same energy.

Momentum is also a conserved quantity, but it is not particularly appropriate for classifying the states of the electron in a hydrogen atom. The reason is that the force between the electron and the proton results in the continual exchange of momentum between them. (Why wasn't this a problem for energy as well? Kinetic energy and momentum are related by  $KE = p^2/2m$ , so the much more massive proton never has very much kinetic energy. We are making an approximation by assuming all the kinetic energy is in the electron, but it is quite a good approximation.)

Angular momentum does help with classification. There is no transfer of angular momentum between the proton and the electron, since the force between them is a center-to-center force, producing no torque.

Like energy, angular momentum is quantized in quantum physics. As an example, consider a quantum wave-particle confined to a circle, like a wave in a circular moat surrounding a castle. A sine wave in such a "quantum moat" cannot have any old wavelength, because an integer number of wavelengths must fit around the circumference,  $C$ , of the moat. The larger this integer is, the shorter the wavelength, and a shorter wavelength relates to greater momentum and angular momentum. Since this integer is related to angular momentum, we use the symbol  $\ell$  for it:

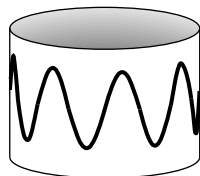
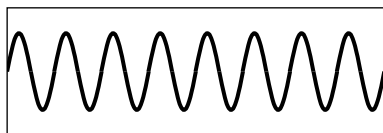
$$\lambda = \frac{C}{\ell}$$

The angular momentum is

$$L = rp \quad .$$

Here,  $r = C/2\pi$ , and  $p = h/\lambda = h\ell/C$ , so

$$\begin{aligned} L &= \frac{C}{2\pi} \cdot \frac{h\ell}{C} \\ &= \frac{h}{2\pi} \ell \end{aligned}$$



a / Eight wavelengths fit around this circle:  $\ell = 8$ .

In the example of the quantum moat, angular momentum is quantized in units of  $\hbar/2\pi$ , and this turns out to be a completely general fact about quantum physics. That makes  $\hbar/2\pi$  a pretty important number, so we define the abbreviation  $\hbar = h/2\pi$ . This symbol is read “h-bar.”

### quantization of angular momentum

The angular momentum of a particle due to its motion through space is quantized in units of  $\hbar$ .

#### self-check A

What is the angular momentum of the wavefunction shown on page 701?

▷ Answer, p. 981

## 36.2 Angular momentum in three dimensions

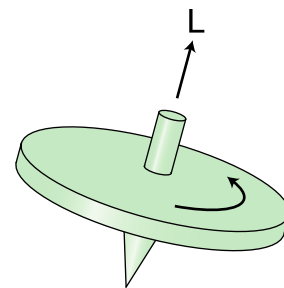
Up until now we’ve only worked with angular momentum in the context of rotation in a plane, for which we could simply use positive and negative signs to indicate clockwise and counterclockwise directions of rotation. A hydrogen atom, however, is unavoidably three-dimensional. Let’s first consider the generalization of angular momentum to three dimensions in the classical case, and then consider how it carries over into quantum physics.

### Three-dimensional angular momentum in classical physics

If we are to completely specify the angular momentum of a classical object like a top,  $\mathbf{b}$ , in three dimensions, it’s not enough to say whether the rotation is clockwise or counterclockwise. We must also give the orientation of the plane of rotation or, equivalently, the direction of the top’s axis. The convention is to specify the direction of the axis. There are two possible directions along the axis, and as a matter of convention we use the direction such that if we sight along it, the rotation appears clockwise.

Angular momentum can, in fact, be defined as a vector pointing along this direction. This might seem like a strange definition, since nothing actually moves in that direction, but it wouldn’t make sense to define the angular momentum vector as being in the direction of motion, because every part of the top has a different direction of motion. Ultimately it’s not just a matter of picking a definition that is convenient and unambiguous: the definition we’re using is the only one that makes the total angular momentum of a system a conserved quantity if we let “total” mean the vector sum.

As with rotation in one dimension, we cannot define what we mean by angular momentum in a particular situation unless we pick a point as an axis. This is really a different use of the word “axis” than the one in the previous paragraphs. Here we simply mean a



$\mathbf{b}$  / The angular momentum vector of a spinning top.

point from which we measure the distance  $r$ . In the hydrogen atom, the nearly immobile proton provides a natural choice of axis.

### Three-dimensional angular momentum in quantum physics

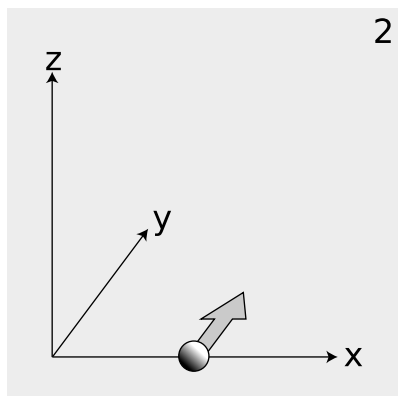
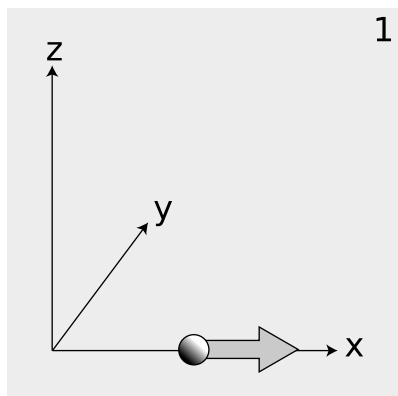
Once we start to think more carefully about the role of angular momentum in quantum physics, it may seem that there is a basic problem: the angular momentum of the electron in a hydrogen atom depends on both its distance from the proton and its momentum, so in order to know its angular momentum precisely it would seem we would need to know both its position and its momentum simultaneously with good accuracy. This, however, might seem to be forbidden by the Heisenberg uncertainty principle.

Actually the uncertainty principle does place limits on what can be known about a particle's angular momentum vector, but it does not prevent us from knowing its magnitude as an exact integer multiple of  $\hbar$ . The reason is that in three dimensions, there are really three separate uncertainty principles:

$$\Delta p_x \Delta x \gtrsim \hbar$$

$$\Delta p_y \Delta y \gtrsim \hbar$$

$$\Delta p_z \Delta z \gtrsim \hbar$$



- c/1. This particle is moving directly away from the axis, and has no angular momentum.  
2. This particle has angular momentum.

Now consider a particle, c/1, that is moving along the  $x$  axis at position  $x$  and with momentum  $p_x$ . We may not be able to know both  $x$  and  $p_x$  with unlimited accuracy, but we can still know the particle's angular momentum about the origin exactly. Classically, it is zero, because the particle is moving directly away from the origin: if it was to be nonzero, we would need both a nonzero  $x$  and a nonzero  $p_y$ . In quantum terms, the uncertainty principle does not place any constraint on  $\Delta x \Delta p_y$ .

Suppose, on the other hand, a particle finds itself, as in figure c/2, at a position  $x$  along the  $x$  axis, and it is moving parallel to the  $y$  axis with momentum  $p_y$ . It has angular momentum  $x p_y$  about the  $z$  axis, and again we can know its angular momentum with unlimited accuracy, because the uncertainty principle only relates  $x$  to  $p_x$  and  $y$  to  $p_y$ . It does not relate  $x$  to  $p_y$ .

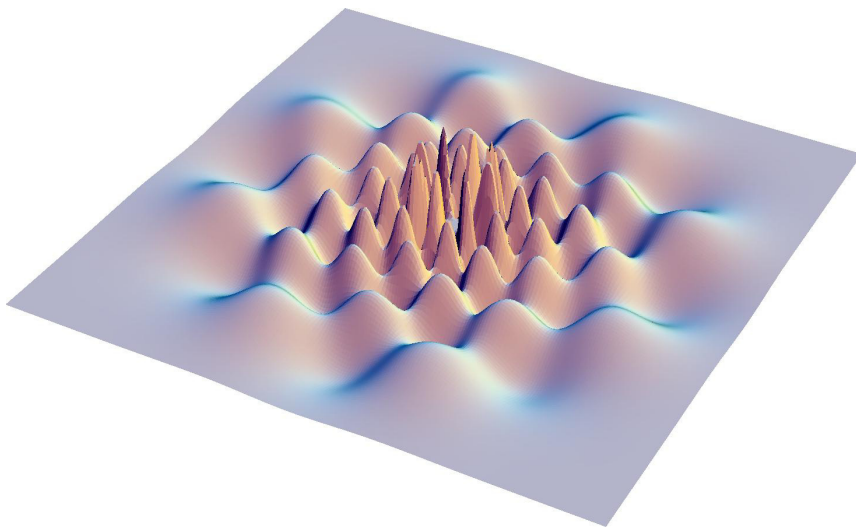
As shown by these examples, the uncertainty principle does not restrict the accuracy of our knowledge of angular momenta as severely as might be imagined. However, it does prevent us from knowing all three components of an angular momentum vector simultaneously. The most general statement about this is the following theorem, which we present without proof:

### the angular momentum vector in quantum physics

The most that can be known about an angular momentum vector is its magnitude and one of its three vector components. Both are quantized in units of  $\hbar$ .

## 36.3 The hydrogen atom

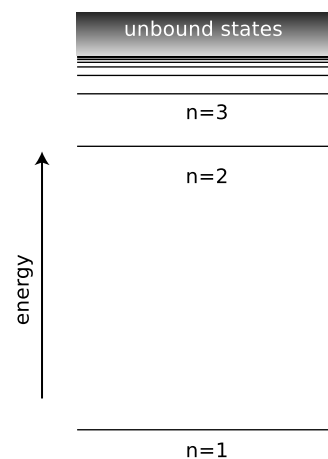
Deriving the wavefunctions of the states of the hydrogen atom from first principles would be mathematically too complex for this book, but it's not hard to understand the logic behind such a wavefunction in visual terms. Consider the wavefunction from the beginning of the chapter, which is reproduced below. Although the graph looks three-dimensional, it is really only a representation of the part of the wavefunction lying within a two-dimensional plane. The third (up-down) dimension of the plot represents the value of the wavefunction at a given point, not the third dimension of space. The plane chosen for the graph is the one perpendicular to the angular momentum vector.



Each ring of peaks and valleys has eight wavelengths going around in a circle, so this state has  $L = 8\hbar$ , i.e., we label it  $\ell = 8$ . The wavelength is shorter near the center, and this makes sense because when the electron is close to the nucleus it has a lower PE, a higher KE, and a higher momentum.

Between each ring of peaks in this wavefunction is a nodal circle, i.e., a circle on which the wavefunction is zero. The full three-dimensional wavefunction has nodal spheres: a series of nested spherical surfaces on which it is zero. The number of radii at which nodes occur, including  $r = \infty$ , is called  $n$ , and  $n$  turns out to be closely

d / A wavefunction of a hydrogen atom.



e / The energy of a state in the hydrogen atom depends only on its  $n$  quantum number.



related to energy. The ground state has  $n = 1$  (a single node only at  $r = \infty$ ), and higher-energy states have higher  $n$  values. There is a simple equation relating  $n$  to energy, which we will discuss in section 36.4.

The numbers  $n$  and  $\ell$ , which identify the state, are called its quantum numbers. A state of a given  $n$  and  $\ell$  can be oriented in a variety of directions in space. We might try to indicate the orientation using the three quantum numbers  $\ell_x = L_x/\hbar$ ,  $\ell_y = L_y/\hbar$ , and  $\ell_z = L_z/\hbar$ . But we have already seen that it is impossible to know all three of these simultaneously. To give the most complete possible description of a state, we choose an arbitrary axis, say the  $z$  axis, and label the state according to  $n$ ,  $\ell$ , and  $\ell_z$ .

Angular momentum requires motion, and motion implies kinetic energy. Thus it is not possible to have a given amount of angular momentum without having a certain amount of kinetic energy as well. Since energy relates to the  $n$  quantum number, this means that for a given  $n$  value there will be a maximum possible  $\ell$ . It turns out that this maximum value of  $\ell$  equals  $n - 1$ .

In general, we can list the possible combinations of quantum numbers as follows:

$n$ can equal 1, 2, 3, ...
$\ell$ can range from 0 to $n - 1$ , in steps of 1
$\ell_z$ can range from $-\ell$ to $\ell$ , in steps of 1

Applying these rules, we have the following list of states:

$n = 1, \quad \ell = 0, \quad \ell_z = 0$	one state
$n = 2, \quad \ell = 0, \quad \ell_z = 0$	one state
$n = 2, \quad \ell = 1, \quad \ell_z = -1, 0, \text{ or } 1$	three states
...	...

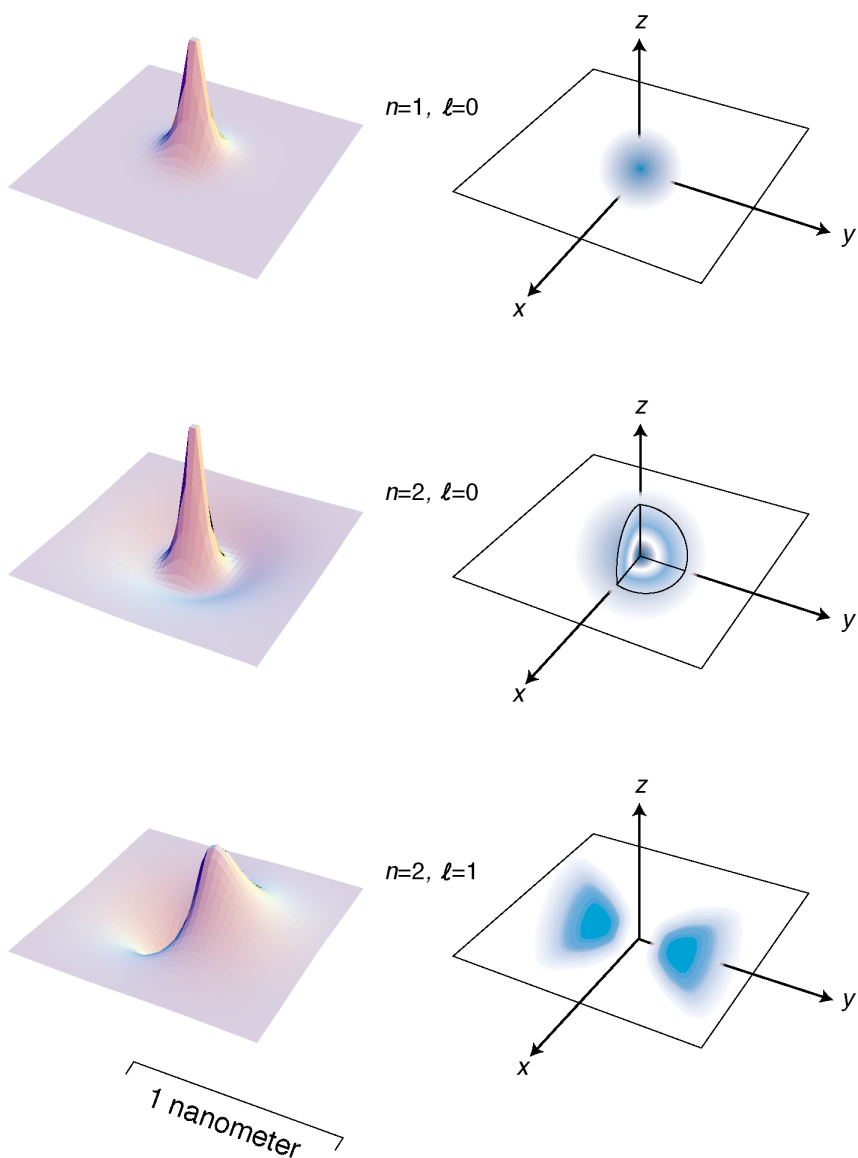
*self-check B*  
Continue the list for  $n = 3$ .

▷ Answer, p. 981

Figure f shows the lowest-energy states of the hydrogen atom. The left-hand column of graphs displays the wavefunctions in the  $x - y$  plane, and the right-hand column shows the probability distribution in a three-dimensional representation.

### Discussion questions

- A**
The quantum number  $n$  is defined as the number of radii at which the wavefunction is zero, including  $r = \infty$ . Relate this to the features of the figures on the facing page.
- B**
Based on the definition of  $n$ , why can't there be any such thing as an  $n = 0$  state?
- C**
Relate the features of the wavefunction plots in figure f to the corresponding features of the probability distribution pictures.



f / The three lowest-energy states of hydrogen.

**D** How can you tell from the wavefunction plots in figure f which ones have which angular momenta?

**E** Criticize the following incorrect statement: “The  $\ell = 8$  wavefunction in figure d has a shorter wavelength in the center because in the center the electron is in a higher energy level.”

**F** Discuss the implications of the fact that the probability cloud in of the  $n = 2, \ell = 1$  state is split into two parts.

## 36.4 ★ Energies of states in hydrogen

### History

The experimental technique for measuring the energy levels of an atom accurately is spectroscopy: the study of the spectrum of light emitted (or absorbed) by the atom. Only photons with certain energies can be emitted or absorbed by a hydrogen atom, for example, since the amount of energy gained or lost by the atom must equal the difference in energy between the atom's initial and final states. Spectroscopy had actually become a highly developed art several decades before Einstein even proposed the photon, and the Swiss spectroscopist Johann Balmer determined in 1885 that there was a simple equation that gave all the wavelengths emitted by hydrogen. In modern terms, we think of the photon wavelengths merely as indirect evidence about the underlying energy levels of the atom, and we rework Balmer's result into an equation for these atomic energy levels:

$$E_n = -\frac{2.2 \times 10^{-18} \text{ J}}{n^2} \quad ,$$

This energy includes both the kinetic energy of the electron and the electrical energy. The zero-level of the electrical energy scale is chosen to be the energy of an electron and a proton that are infinitely far apart. With this choice, negative energies correspond to bound states and positive energies to unbound ones.

Where does the mysterious numerical factor of  $2.2 \times 10^{-18} \text{ J}$  come from? In 1913 the Danish theorist Niels Bohr realized that it was exactly numerically equal to a certain combination of fundamental physical constants:

$$E_n = -\frac{mk^2e^4}{2\hbar^2} \cdot \frac{1}{n^2} \quad ,$$

where  $m$  is the mass of the electron, and  $k$  is the Coulomb force constant for electric forces.

Bohr was able to cook up a derivation of this equation based on the incomplete version of quantum physics that had been developed by that time, but his derivation is today mainly of historical interest. It assumes that the electron follows a circular path, whereas the whole concept of a path for a particle is considered meaningless in our more complete modern version of quantum physics. Although Bohr was able to produce the right equation for the energy levels, his model also gave various wrong results, such as predicting that the atom would be flat, and that the ground state would have  $\ell = 1$  rather than the correct  $\ell = 0$ .

### Approximate treatment

A full and correct treatment is impossible at the mathematical level of this book, but we can provide a straightforward explanation for the form of the equation using approximate arguments.

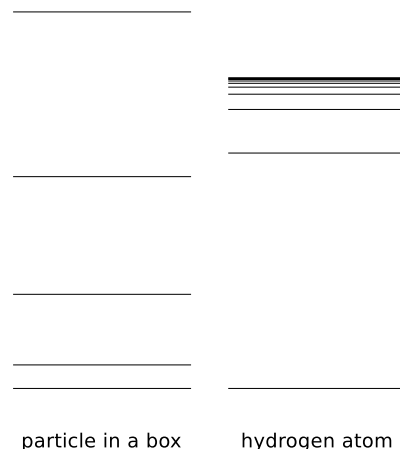
A typical standing-wave pattern for the electron consists of a central oscillating area surrounded by a region in which the wavefunction tails off. As discussed in section 35.6, the oscillating type of pattern is typically encountered in the classically allowed region, while the tailing off occurs in the classically forbidden region where the electron has insufficient kinetic energy to penetrate according to classical physics. We use the symbol  $r$  for the radius of the spherical boundary between the classically allowed and classically forbidden regions. Classically,  $r$  would be the distance from the proton at which the electron would have to stop, turn around, and head back in.

If  $r$  had the same value for every standing-wave pattern, then we'd essentially be solving the particle-in-a-box problem in three dimensions, with the box being a spherical cavity. Consider the energy levels of the particle in a box compared to those of the hydrogen atom, g. They're qualitatively different. The energy levels of the particle in a box get farther and farther apart as we go higher in energy, and this feature doesn't even depend on the details of whether the box is two-dimensional or three-dimensional, or its exact shape. The reason for the spreading is that the box is taken to be completely impenetrable, so its size,  $r$ , is fixed. A wave pattern with  $n$  humps has a wavelength proportional to  $r/n$ , and therefore a momentum proportional to  $n$ , and an energy proportional to  $n^2$ . In the hydrogen atom, however, the force keeping the electron bound isn't an infinite force encountered when it bounces off of a wall, it's the attractive electrical force from the nucleus. If we put more energy into the electron, it's like throwing a ball upward with a higher energy — it will get farther out before coming back down. This means that in the hydrogen atom, we expect  $r$  to increase as we go to states of higher energy. This tends to keep the wavelengths of the high energy states from getting too short, reducing their kinetic energy. The closer and closer crowding of the energy levels in hydrogen also makes sense because we know that there is a certain energy that would be enough to make the electron escape completely, and therefore the sequence of bound states cannot extend above that energy.

When the electron is at the maximum classically allowed distance  $r$  from the proton, it has zero kinetic energy. Thus when the electron is at distance  $r$ , its energy is purely electrical:

$$[1] \quad E = -\frac{ke^2}{r}$$

Now comes the approximation. In reality, the electron's wavelength cannot be constant in the classically allowed region, but we pretend that it is. Since  $n$  is the number of nodes in the wavefunction, we can interpret it approximately as the number of wavelengths that fit across the diameter  $2r$ . We are not even attempting a derivation



g / The energy levels of a particle in a box, contrasted with those of the hydrogen atom.

that would produce all the correct numerical factors like 2 and  $\pi$  and so on, so we simply make the approximation

$$[2] \quad \lambda \sim \frac{r}{n} \quad .$$

Finally we assume that the typical kinetic energy of the electron is on the same order of magnitude as the absolute value of its total energy. (This is true to within a factor of two for a typical classical system like a planet in a circular orbit around the sun.) We then have

$$[3] \quad \begin{aligned} &\text{absolute value of total energy} \\ &= \frac{ke^2}{r} \\ &\sim K \\ &= p^2/2m \\ &= (h/\lambda)^2/2m \\ &= h^2 n^2 / 2mr^2 \end{aligned}$$

We now solve the equation  $ke^2/r \sim h^2 n^2 / 2mr^2$  for  $r$  and throw away numerical factors we can't hope to have gotten right, yielding

$$[4] \quad r \sim \frac{h^2 n^2}{mke^2} \quad .$$

Plugging  $n = 1$  into this equation gives  $r = 2$  nm, which is indeed on the right order of magnitude. Finally we combine equations [4] and [1] to find

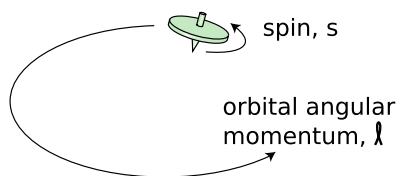
$$E \sim -\frac{mk^2 e^4}{h^2 n^2} \quad ,$$

which is correct except for the numerical factors we never aimed to find.

### Discussion questions

**A** States of hydrogen with  $n$  greater than about 10 are never observed in the sun. Why might this be?

**B** Sketch graphs of  $r$  and  $E$  versus  $n$  for the hydrogen, and compare with analogous graphs for the one-dimensional particle in a box.



$\hbar$  / The top has angular momentum both because of the motion of its center of mass through space and due to its internal rotation. Electron spin is roughly analogous to the intrinsic spin of the top.

## 36.5 Electron spin

It's disconcerting to the novice ping-pong player to encounter for the first time a more skilled player who can put spin on the ball. Even though you can't see that the ball is spinning, you can tell something is going on by the way it interacts with other objects in its environment. In the same way, we can tell from the way electrons interact with other things that they have an intrinsic spin of their own. Experiments show that even when an electron is not moving through space, it still has angular momentum amounting to  $\hbar/2$ .

This may seem paradoxical because the quantum moat, for instance, gave only angular momenta that were integer multiples of  $\hbar$ , not half-units, and I claimed that angular momentum was always quantized in units of  $\hbar$ , not just in the case of the quantum moat. That whole discussion, however, assumed that the angular momentum would come from the motion of a particle through space. The  $\hbar/2$  angular momentum of the electron is simply a property of the particle, like its charge or its mass. It has nothing to do with whether the electron is moving or not, and it does not come from any internal motion within the electron. Nobody has ever succeeded in finding any internal structure inside the electron, and even if there was internal structure, it would be mathematically impossible for it to result in a half-unit of angular momentum.

We simply have to accept this  $\hbar/2$  angular momentum, called the “spin” of the electron — Mother Nature rubs our noses in it as an observed fact. Protons and neutrons have the same  $\hbar/2$  spin, while photons have an intrinsic spin of  $\hbar$ . In general, half-integer spins are typical of material particles. Integral values are found for the particles that carry forces: photons, which embody the electric and magnetic fields of force, as well as the more exotic messengers of the nuclear and gravitational forces.

As was the case with ordinary angular momentum, we can describe spin angular momentum in terms of its magnitude, and its component along a given axis. We notate these quantities, in units of  $\hbar$ , as  $s$  and  $s_z$ , so an electron has  $s = 1/2$  and  $s_z = +1/2$  or  $-1/2$ .

Taking electron spin into account, we need a total of four quantum numbers to label a state of an electron in the hydrogen atom:  $n$ ,  $\ell$ ,  $\ell_z$ , and  $s_z$ . (We omit  $s$  because it always has the same value.) The symbols  $\ell$  and  $\ell_z$  include only the angular momentum the electron has because it is moving through space, not its spin angular momentum. The availability of two possible spin states of the electron leads to a doubling of the numbers of states:

$n = 1,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 1,$	$\ell_z = -1, 0, \text{ or } 1,$	$s_z = +1/2$ or $-1/2$	six states
...				...

## 36.6 Atoms with more than one electron

What about other atoms besides hydrogen? It would seem that things would get much more complex with the addition of a second electron. A hydrogen atom only has one particle that moves around much, since the nucleus is so heavy and nearly immobile. Helium, with two, would be a mess. Instead of a wavefunction whose square tells us the probability of finding a single electron at any given location in space, a helium atom would need to have a wavefunction

whose square would tell us the probability of finding two electrons at any given combination of points. Ouch! In addition, we would have the extra complication of the electrical interaction between the two electrons, rather than being able to imagine everything in terms of an electron moving in a static field of force created by the nucleus alone.

Despite all this, it turns out that we can get a surprisingly good description of many-electron atoms simply by assuming the electrons can occupy the same standing-wave patterns that exist in a hydrogen atom. The ground state of helium, for example, would have both electrons in states that are very similar to the  $n = 1$  states of hydrogen. The second-lowest-energy state of helium would have one electron in an  $n = 1$  state, and the other in an  $n = 2$  state. The relatively complex spectra of elements heavier than hydrogen can be understood as arising from the great number of possible combinations of states for the electrons.

A surprising thing happens, however, with lithium, the three-electron atom. We would expect the ground state of this atom to be one in which all three electrons settle down into  $n = 1$  states. What really happens is that two electrons go into  $n = 1$  states, but the third stays up in an  $n = 2$  state. This is a consequence of a new principle of physics:

#### **the Pauli exclusion principle**

Only one electron can ever occupy a given state.

There are two  $n = 1$  states, one with  $s_z = +1/2$  and one with  $s_z = -1/2$ , but there is no third  $n = 1$  state for lithium's third electron to occupy, so it is forced to go into an  $n = 2$  state.

It can be proved mathematically that the Pauli exclusion principle applies to any type of particle that has half-integer spin. Thus two neutrons can never occupy the same state, and likewise for two protons. Photons, however, are immune to the exclusion principle because their spin is an integer. Material objects can't pass through each other, but beams of light can, and the basic reason is that the exclusion principle applies to one but not to the other.<sup>1</sup>

---

<sup>1</sup>There are also electrical forces between atoms, but as argued on page 928, the attractions and repulsions tend to cancel out.

## Deriving the periodic table

We can now account for the structure of the periodic table, which seemed so mysterious even to its inventor Mendeleev. The first row consists of atoms with electrons only in the  $n = 1$  states:

- H 1 electron in an  $n = 1$  state
- He 2 electrons in the two  $n = 1$  states

The next row is built by filling the  $n = 2$  energy levels:

- Li 2 electrons in  $n = 1$  states, 1 electron in an  $n = 2$  state
- Be 2 electrons in  $n = 1$  states, 2 electrons in  $n = 2$  states
- ...
- O 2 electrons in  $n = 1$  states, 6 electrons in  $n = 2$  states
- F 2 electrons in  $n = 1$  states, 7 electrons in  $n = 2$  states
- Ne 2 electrons in  $n = 1$  states, 8 electrons in  $n = 2$  states

In the third row we start in on the  $n = 3$  levels:

- Na 2 electrons in  $n = 1$  states, 8 electrons in  $n = 2$  states, 1 electron in an  $n = 3$  state
- ...

We can now see a logical link between the filling of the energy levels and the structure of the periodic table. Column 0, for example, consists of atoms with the right number of electrons to fill all the available states up to a certain value of  $n$ . Column I contains atoms like lithium that have just one electron more than that.

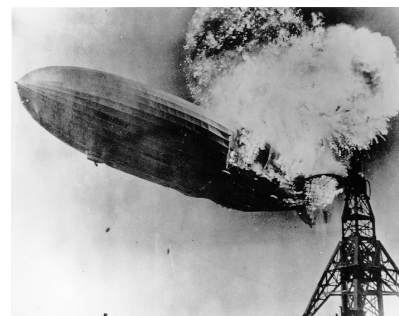
This shows that the columns relate to the filling of energy levels, but why does that have anything to do with chemistry? Why, for example, are the elements in columns I and VII dangerously reactive? Consider, for example, the element sodium (Na), which is so reactive that it may burst into flames when exposed to air. The electron in the  $n = 3$  state has an unusually high energy. If we let a sodium atom come in contact with an oxygen atom, energy can be released by transferring the  $n = 3$  electron from the sodium to one of the vacant lower-energy  $n = 2$  states in the oxygen. This energy is transformed into heat. Any atom in column I is highly reactive for the same reason: it can release energy by giving away the electron that has an unusually high energy.

Column VII is spectacularly reactive for the opposite reason: these atoms have a single vacancy in a low-energy state, so energy is released when these atoms steal an electron from another atom.

It might seem as though these arguments would only explain reactions of atoms that are in different rows of the periodic table, because only in these reactions can a transferred electron move from a higher- $n$  state to a lower- $n$  state. This is incorrect. An  $n = 2$  electron in fluorine (F), for example, would have a different energy than an  $n = 2$  electron in lithium (Li), due to the different number of protons and electrons with which it is interacting. Roughly speak-

I	II	III	IV	V	VI	VII	0
<sup>1</sup> H							<sup>2</sup> He
<sup>3</sup> Li	<sup>4</sup> Be	<sup>5</sup> B	<sup>6</sup> C	<sup>7</sup> N	<sup>8</sup> O	<sup>9</sup> F	<sup>10</sup> Ne
<sup>11</sup> Na	...						

i / The beginning of the periodic table.



j / Hydrogen is highly reactive.



ing, the  $n = 2$  electron in fluorine is more tightly bound (lower in energy) because of the larger number of protons attracting it. The effect of the increased number of attracting protons is only partly counteracted by the increase in the number of repelling electrons, because the forces exerted on an electron by the other electrons are in many different directions and cancel out partially.

#### *Neutron stars*

#### *example 1*

Here's an exotic example that doesn't even involve atoms. When a star runs out of fuel for its nuclear reactions, it begins to collapse under its own weight. Since Newton's law of gravity depends on the inverse square of the distance, the gravitational forces become stronger as the star collapses, which encourages it to collapse even further. The final result depends on the mass of the star, but let's consider a star that's only a little more massive than our own sun. Such a star will collapse to the point where the gravitational energy being released is sufficient to cause the reaction  $p + e^- \rightarrow n + \nu$  to occur. (As you found in homework problem 10 on page 759, this reaction can only occur when there is some source of energy, because the mass-energy of the products is greater than the mass-energy of the things being consumed.) The neutrinos fly off and are never heard from again, so we're left with a star consisting only of neutrons!

Now the exclusion principle comes into play. The collapse can't continue indefinitely. The situation is in fact closely analogous to that of an atom. A lead atom's cloud of 82 electrons can't shrink down to the size of a hydrogen atom, because only two electrons can have the lowest-energy wave pattern. The same happens with the neutron star. No physical repulsion keeps the neutrons apart. They're electrically neutral, so they don't repel or attract one another electrically. The gravitational force is attractive, and as the collapse proceeds to the point where the neutrons come within range of the strong nuclear force ( $\sim 10^{-15}$  m), we even start getting nuclear attraction. The only thing that stops the whole process is the exclusion principle. The star ends up being only a few kilometers across, and has the same billion-ton-per-teaspoon density as an atomic nucleus. Indeed, we can think of it as one big nucleus (with atomic number zero, because there are no protons).

As with a spinning figure skater pulling her arms in, conservation of angular momentum makes the star spin faster and faster. The whole object may end up with a rotational period of a fraction of a second! Such a star sends out radio pulses with each revolution, like a sort of lighthouse. The first time such a signal was detected, radio astronomers thought that it was a signal from aliens.

## Summary

### Selected vocabulary

quantum number	a numerical label used to classify a quantum state
spin . . . . .	the built-in angular momentum possessed by a particle even when at rest

### Notation

$n$ . . . . .	the number of radial nodes in the wavefunction, including the one at $r = \infty$
$\hbar$ . . . . .	$h/2\pi$
$\mathbf{L}$ . . . . .	the angular momentum vector of a particle, not including its spin
$\ell$ . . . . .	the magnitude of the $L$ vector, divided by $\hbar$
$\ell_z$ . . . . .	the $z$ component of the $L$ vector, divided by $\hbar$ ; this is the standard notation in nuclear physics, but not in atomic physics
$s$ . . . . .	the magnitude of the spin angular momentum vector, divided by $\hbar$
$s_z$ . . . . .	the $z$ component of the spin angular momentum vector, divided by $\hbar$ ; this is the standard notation in nuclear physics, but not in atomic physics

### Other terminology and notation

$m_\ell$ . . . . .	a less obvious notation for $\ell_z$ , standard in atomic physics
$m_s$ . . . . .	a less obvious notation for $s_z$ , standard in atomic physics

## Summary

Hydrogen, with one proton and one electron, is the simplest atom, and more complex atoms can often be analyzed to a reasonably good approximation by assuming their electrons occupy states that have the same structure as the hydrogen atom's. The electron in a hydrogen atom exchanges very little energy or angular momentum with the proton, so its energy and angular momentum are nearly constant, and can be used to classify its states. The energy of a hydrogen state depends only on its  $n$  quantum number.

In quantum physics, the angular momentum of a particle moving in a plane is quantized in units of  $\hbar$ . Atoms are three-dimensional, however, so the question naturally arises of how to deal with angular momentum in three dimensions. In three dimensions, angular momentum is a vector in the direction perpendicular to the plane of motion, such that the motion appears clockwise if viewed along the direction of the vector. Since angular momentum depends on both position and momentum, the Heisenberg uncertainty principle limits the accuracy with which one can know it. The most the can

be known about an angular momentum vector is its magnitude and one of its three vector components, both of which are quantized in units of  $\hbar$ .

In addition to the angular momentum that an electron carries by virtue of its motion through space, it possesses an intrinsic angular momentum with a magnitude of  $\hbar/2$ . Protons and neutrons also have spins of  $\hbar/2$ , while the photon has a spin equal to  $\hbar$ .

Particles with half-integer spin obey the Pauli exclusion principle: only one such particle can exist in a given state, i.e., with a given combination of quantum numbers.

We can enumerate the lowest-energy states of hydrogen as follows:

$n = 1,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2 \text{ or } -1/2$	two states
$n = 2,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2 \text{ or } -1/2$	two states
$n = 2,$	$\ell = 1,$	$\ell_z = -1, 0, \text{ or } 1,$	$s_z = +1/2 \text{ or } -1/2$	six states
...				...

The periodic table can be understood in terms of the filling of these states. The nonreactive noble gases are those atoms in which the electrons are exactly sufficient to fill all the states up to a given  $n$  value. The most reactive elements are those with one more electron than a noble gas element, which can release a great deal of energy by giving away their high-energy electron, and those with one electron fewer than a noble gas, which release energy by accepting an electron.

## Problems

### Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

**1** (a) A distance scale is shown below the wavefunctions and probability densities illustrated in figure f on page 959. Compare this with the order-of-magnitude estimate derived in section 36.4 for the radius  $r$  at which the wavefunction begins tailing off. Was the estimate in section 36.4 on the right order of magnitude?

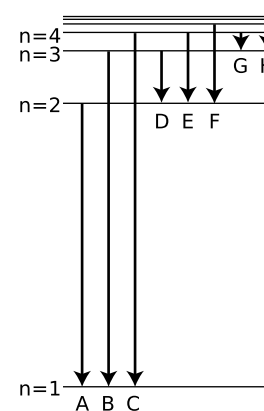
(b) Although we normally say the moon orbits the earth, actually they both orbit around their common center of mass, which is below the earth's surface but not at its center. The same is true of the hydrogen atom. Does the center of mass lie inside the proton or outside it?

**2** The figure shows eight of the possible ways in which an electron in a hydrogen atom could drop from a higher energy state to a state of lower energy, releasing the difference in energy as a photon. Of these eight transitions, only D, E, and F produce photons with wavelengths in the visible spectrum.

(a) Which of the visible transitions would be closest to the violet end of the spectrum, and which would be closest to the red end? Explain.

(b) In what part of the electromagnetic spectrum would the photons from transitions A, B, and C lie? What about G and H? Explain.

(c) Is there an upper limit to the wavelengths that could be emitted by a hydrogen atom going from one bound state to another bound state? Is there a lower limit? Explain.



Problem 2.

**3** Before the quantum theory, experimentalists noted that in many cases, they would find three lines in the spectrum of the same atom that satisfied the following mysterious rule:  $1/\lambda_1 = 1/\lambda_2 + 1/\lambda_3$ . Explain why this would occur. Do not use reasoning that only works for hydrogen — such combinations occur in the spectra of all elements. [Hint: Restate the equation in terms of the energies of photons.]

**4** Find an equation for the wavelength of the photon emitted when the electron in a hydrogen atom makes a transition from energy level  $n_1$  to level  $n_2$ . [You will need to have read optional section 36.4.] ✓

**5** Estimate the angular momentum of a spinning basketball, in units of  $\hbar$ . Explain how this result relates to the correspondence principle.

**6** Assume that the kinetic energy of an electron in the  $n = 1$  state of a hydrogen atom is on the same order of magnitude as the

absolute value of its total energy, and estimate a typical speed at which it would be moving. (It cannot really have a single, definite speed, because its kinetic and potential energy trade off at different distances from the proton, but this is just a rough estimate of a typical speed.) Based on this speed, were we justified in assuming that the electron could be described nonrelativistically?

**7** The wavefunction of the electron in the ground state of a hydrogen atom, shown in the top left of figure f on p. 959, is

$$\Psi = \pi^{-1/2} a^{-3/2} e^{-r/a} \quad ,$$

where  $r$  is the distance from the proton, and  $a = \hbar^2/kme^2 = 5.3 \times 10^{-11}$  m is a constant that sets the size of the wave. The figure doesn't show the proton; let's take the proton to be a sphere with a radius of  $b = 0.5$  fm.

- (a) Reproduce figure f in a rough sketch, and indicate, relative to the size of your sketch, some idea of how big  $a$  and  $b$  are.
- (b) Calculate symbolically, without plugging in numbers, the probability that at any moment, the electron is inside the proton. [Hint: Does it matter if you plug in  $r = 0$  or  $r = b$  in the equation for the wavefunction?] ✓
- (c) Calculate the probability numerically. ✓
- (d) Based on the equation for the wavefunction, is it valid to think of a hydrogen atom as having a finite size? Can  $a$  be interpreted as the size of the atom, beyond which there is nothing? Or is there any limit on how far the electron can be from the proton?

**8** Use physical reasoning to explain how the equation for the energy levels of hydrogen,

$$E_n = -\frac{mk^2e^4}{2\hbar^2} \cdot \frac{1}{n^2} \quad ,$$

should be generalized to the case of an atom with atomic number  $Z$  that has had all its electrons removed except for one. ★

**9** This question requires that you read optional section 36.4. A muon is a subatomic particle that acts exactly like an electron except that its mass is 207 times greater. Muons can be created by cosmic rays, and it can happen that one of an atom's electrons is displaced by a muon, forming a muonic atom. If this happens to a hydrogen atom, the resulting system consists simply of a proton plus a muon.

- (a) How would the size of a muonic hydrogen atom in its ground state compare with the size of the normal atom?
- (b) If you were searching for muonic atoms in the sun or in the earth's atmosphere by spectroscopy, in what part of the electromagnetic spectrum would you expect to find the absorption lines?

**10** Consider a classical model of the hydrogen atom in which the electron orbits the proton in a circle at constant speed. In this model, the electron and proton can have no intrinsic spin. Using the result of problem 14 in ch. 24, show that in this model, the atom's magnetic dipole moment  $D_m$  is related to its angular momentum by  $D_m = (-e/2m)L$ , regardless of the details of the orbital motion. Assume that the magnetic field is the same as would be produced by a circular current loop, even though there is really only a single charged particle. [Although the model is quantum-mechanically incorrect, the result turns out to give the correct quantum mechanical value for the contribution to the atom's dipole moment coming from the electron's orbital motion. There are other contributions, however, arising from the intrinsic spins of the electron and proton.]

**11** Hydrogen is the only element whose energy levels can be expressed exactly in an equation. Calculate the ratio  $\lambda_E/\lambda_F$  of the wavelengths of the transitions labeled E and F in problem 2 on p. 969. Express your answer as an exact fraction, not a decimal approximation. In an experiment in which atomic wavelengths are being measured, this ratio provides a natural, stringent check on the precision of the results. ✓

## Exercise 36: Quantum versus classical randomness

1. Imagine the classical version of the particle in a one-dimensional box. Suppose you insert the particle in the box and give it a known, predetermined energy, but a random initial position and a random direction of motion. You then pick a random later moment in time to see where it is. Sketch the resulting probability distribution by shading on top of a line segment. Does the probability distribution depend on energy?
2. Do similar sketches for the first few energy levels of the quantum mechanical particle in a box, and compare with 1.
3. Do the same thing as in 1, but for a classical hydrogen atom in two dimensions, which acts just like a miniature solar system. Assume you're always starting out with the same fixed values of energy and angular momentum, but a position and direction of motion that are otherwise random. Do this for  $L = 0$ , and compare with a real  $L = 0$  probability distribution for the hydrogen atom.
4. Repeat 3 for a nonzero value of  $L$ , say  $L = \hbar$ .
5. Summarize: Are the classical probability distributions accurate? What qualitative features are possessed by the classical diagrams but not by the quantum mechanical ones, or vice-versa?





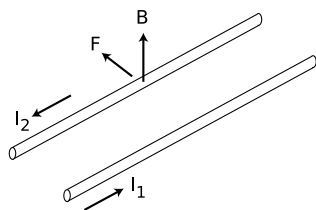
## Solutions for chapter 24

### Page 678, problem 11:

(a) Current means how much charge passes by a given point per unit time. During a time interval  $\Delta t$ , all the charge carriers in a certain region behind the point will pass by. This region has length  $v\Delta t$  and cross-sectional area  $A$ , so its volume is  $Av\Delta t$ , and the amount of charge in it is  $Avnq\Delta t$ . To find the current, we divide this amount of charge by  $\Delta t$ , giving  $I = Avnq$ . (b) A segment of the wire of length  $L$  has a force  $QvB$  acting on it, where  $Q = ALnq$  is the total charge of the moving charge carriers in that part of the wire. The force per unit length is  $ALnqvB/L = AnqvB$ . (c) Dividing the two results gives  $F/L = IB$ .

### Page 678, problem 12:

(a) The figure shows the case where the currents are in opposite directions.



The field vector shown is one made by wire 1, which causes an effect on wire 2. It points up because wire 1's field pattern is clockwise as view from along the direction of current  $I_1$ . For simplicity, let's assume that the current  $I_2$  is made by positively charged particles moving in the direction of the current. (You can check that the final result would be the same if they were negatively charged, as would actually be the case in a metal wire.) The force on one of these positively charged particles in wire 2 is supposed to have a direction such that when you sight along it, the  $B$  vector is clockwise from the  $v$  vector. This can only be accomplished if the force on the particle in wire 2 is in the direction shown. Wire 2 is repelled by wire 1.

To verify that wire 1 is also repelled by wire 2, we can either go through the same type of argument again, or we can simply apply Newton's third law.

Similiar arguments show that the force is attractive if the currents are in the same direction.

(b) The force on wire 2 is  $F/L = I_2B$ , where  $B = 2kI_1/c^2\pi r$  is the field made by wire 1 and  $r$  is the distance between the wires. The result is

$$F/L = 2kI_1I_2/c^2\pi r \quad .$$

### Page 679, problem 16:

(a) Based on our knowledge of the field pattern of a current-carrying loop, we know that the magnetic field must be either into or out of the page. This makes sense, since that would mean the field is always perpendicular to the plane of the electrons' motion; if it was in their plane of motion, then the angle between the  $v$  and  $B$  vectors would be changing all the time, but we see no evidence of such behavior. With the field turned on, the force vector is apparently toward the center of the circle. Let's analyze the force at the moment when the electrons have started moving, which is at the right side of the circle. The force is to the left. Since the electrons are negatively charged particles, we know that if we sight along the force vector, the  $B$  vector must be counterclockwise from the  $v$  vector. The magnetic field must be out of the page. (b) Looking at figure e on page 657, we can tell that the current in the coils must be counterclockwise as viewed from the perspective of the camera. (c) Electrons are negatively charged, so to produce a counterclockwise current, the electrons in the coils must be going clockwise, i.e., they are

counterrotating compared to the beam. (d) The current in the coils is keep the electrons in the beam from going straight, i.e. the force is a repulsion. This makes sense by comparison with figure w in section 23.2: like charges moving in opposite directions repel one another.

**Page 680, problem 19:**

The trick is to imagine putting together two identical solenoids to make one double-length solenoid. The field of the doubled solenoid is given by the vector sum of the two solenoids' individual fields. At points on the axis, symmetry guarantees that the individual fields lie along the axis, and similarly for the total field. At the center of one of the mouths, we thus have two parallel field vectors of equal strength, whose sum equals the interior field. But the interior field of the doubled solenoid is the same as that of the individual ones, since the equation for the field only depends on the number of turns per unit length. Therefore the field at the center of a solenoid's mouth equals exactly half the interior field.

**Page 680, problem 21:**

(a) Plugging in, we find

$$\sqrt{\frac{1-w}{1+w}} = \sqrt{\frac{1-u}{1+u}} \sqrt{\frac{1-v}{1+v}} \quad .$$

(b) First let's simplify by squaring both sides.

$$\frac{1-w}{1+w} = \frac{1-u}{1+u} \cdot \frac{1-v}{1+v} \quad .$$

For convenience, let's write  $A$  for the right-hand side of this equation. We then have

$$\begin{aligned} \frac{1-w}{1+w} &= A \\ 1-w &= A + Aw \quad . \end{aligned}$$

Solving for  $w$ ,

$$\begin{aligned} w &= \frac{1-A}{1+A} \\ &= \frac{(1+u)(1+v) - (1-u)(1-v)}{(1+u)(1+v) + (1-u)(1-v)} \\ &= \frac{2(u+v)}{2(1+uv)} \\ &= \frac{u+v}{1+uv} \end{aligned}$$

(c) This is all in units where  $c = 1$ . The correspondence principle says that we should get  $w \approx u + v$  when both  $u$  and  $v$  are small compared to 1. Under those circumstances,  $uv$  is the product of two very small numbers, which makes it very, very small. Neglecting this term in the denominator, we recover the nonrelativistic result.

**Solutions for chapter 26**

**Page 757, problem 2:**

(a) In the reaction  $p + e^- \rightarrow n + \nu$ , the charges on the left are  $e + (-e) = 0$ , and both charges on the right are zero. (b) The neutrino has negligible mass. The masses on the left add up to  $1.6736 \times 10^{-27}$  kg, which is less than the  $1.6750 \times 10^{-27}$  kg mass of the neutron on the right, so energy would be required from an external source in order to make this reaction happen.

**Page 758, problem 7:**

(a) The change in PE is  $e\Delta V$ , so the KE gained is  $(1/2)mv^2 = eV$ . Solving for  $v$ , we get  $v = \sqrt{2eV/m}$ . (b) Plugging in numbers, the velocity is  $5.9 \times 10^7$  m/s. This is about 20% of the speed of light. (Since it's not that close to the speed of light, we'll get a reasonably accurate answer without taking into account Einstein's theory of relativity.)

**Solutions for chapter 27****Page 778, problem 1:**

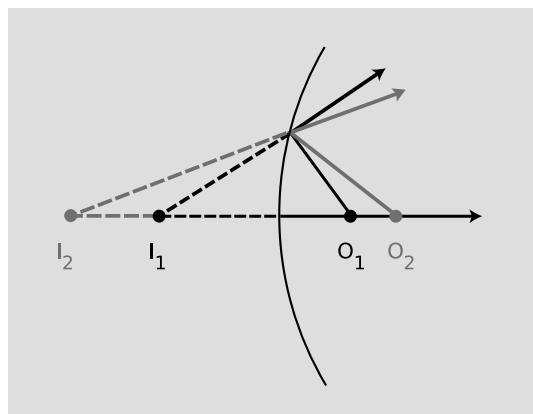
At the center of each of the large triangle's sides, the angles add up to  $180^\circ$  because they form a straight line. Therefore  $4s = S + 3 \times 180^\circ$ , so  $S - 180^\circ = 4(s - 180^\circ)$ .

**Page 778, problem 7:**

By the equivalence principle, we can adopt a frame tied to the tossed clock, B, and in this frame there is no gravitational field. We see a desk and clock A go by. The desk applies a force to clock A, decelerating it and then reaccelerating it so that it comes back. We've already established that the effect of motion is to slow down time, so clock A reads a smaller time interval.

**Solutions for chapter 30****Page 831, problem 2:**

See the ray diagram below. Decreasing  $\theta_o$  decreases  $\theta_i$ , so the equation  $\theta_f = \pm\theta_i + \pm\theta_o$  must have opposite signs on the right. Since  $\theta_o$  is bigger than  $\theta_i$ , the only way to get a positive  $\theta_f$  is if the signs are  $\theta_f = -\theta_i + \theta_o$ . This gives  $1/f = -1/d_i + 1/d_o$ .

**Page 832, problem 10:**

(a) The object distance is less than the focal length, so the image is virtual: because the object is so close, the cone of rays is diverging too strongly for the mirror to bring it back to a focus. (b) Now the object distance is greater than the focal length, so the image is real. (c),(d) A diverging mirror can only make virtual images.

**Solutions for chapter 31****Page 851, problem 13:**

Since  $d_o$  is much greater than  $d_i$ , the lens-film distance  $d_i$  is essentially the same as  $f$ . (a) Splitting the triangle inside the camera into two right triangles, straightforward trigonometry gives

$$\theta = 2 \tan^{-1} \frac{w}{2f}$$

for the field of view. This comes out to be  $39^\circ$  and  $64^\circ$  for the two lenses. (b) For small angles, the tangent is approximately the same as the angle itself, provided we measure everything in

radians. The equation above then simplifies to

$$\theta = \frac{w}{f}$$

The results for the two lenses are  $.70 \text{ rad} = 40^\circ$ , and  $1.25 \text{ rad} = 72^\circ$ . This is a decent approximation.

(c) With the 28-mm lens, which is closer to the film, the entire field of view we had with the 50-mm lens is now confined to a small part of the film. Using our small-angle approximation  $\theta = w/f$ , the amount of light contained within the same angular width  $\theta$  is now striking a piece of the film whose linear dimensions are smaller by the ratio  $28/50$ . Area depends on the square of the linear dimensions, so all other things being equal, the film would now be overexposed by a factor of  $(50/28)^2 = 3.2$ . To compensate, we need to shorten the exposure by a factor of 3.2.

**Page 854, problem 20:**

One surface is curved outward and one inward. Therefore the minus sign applies in the lens-maker's equation. Since the radii of curvature are equal, the quantity  $1/r_1 - 1/r_2$  equals zero, and the resulting focal length is infinite. A big focal length indicates a weak lens. An infinite focal length tells us that the lens is infinitely weak — it doesn't focus or defocus rays at all.

## Answers to self-checks for volume 2

### Answers to self-checks for chapter 21

**Page 545, self-check A:**

Either type can be involved in either an attraction or a repulsion. A positive charge could be involved in either an attraction (with a negative charge) or a repulsion (with another positive), and a negative could participate in either an attraction (with a positive) or a repulsion (with a negative).

**Page 546, self-check B:**

It wouldn't make any difference. The roles of the positive and negative charges in the paper would be reversed, but there would still be a net attraction.

**Page 561, self-check C:**

The large amount of power means a high rate of conversion of the battery's chemical energy into heat. The battery will quickly use up all its energy, i.e., "burn out."

### Answers to self-checks for chapter 22

**Page 605, self-check A:**

The reasoning is exactly analogous to that used in example 1 on page 603 to derive an equation for the gravitational field of the earth. The field is  $F/q_t = (kQq_t/r^2)/q_t = kQ/r^2$ .

**Page 615, self-check B:**

$$\begin{aligned} E_x &= -\frac{dV}{dx} \\ &= -\frac{d}{dx} \left( \frac{kQ}{r} \right) \\ &= \frac{kQ}{r^2} \end{aligned}$$

**Page 616, self-check C:**

(a) The voltage (height) increases as you move to the east or north. If we let the positive  $x$  direction be east, and choose positive  $y$  to be north, then  $dV/dx$  and  $dV/dy$  are both positive. This means that  $E_x$  and  $E_y$  are both negative, which makes sense, since the water is flowing in the negative  $x$  and  $y$  directions (south and west).

(b) The electric fields are all pointing away from the higher ground. If this was an electrical map, there would have to be a large concentration of charge all along the top of the ridge, and especially at the mountain peak near the south end.

**Answers to self-checks for chapter 23****Page 640, self-check A:**

At  $v = 0$ , we get  $\gamma = 1$ , so  $t = T$ . There is no time distortion unless the two frames of reference are in relative motion.

**Answers to self-checks for chapter 24****Page 664, self-check A:**

An induced electric field can only be created by a *changing* magnetic field. Nothing is changing if your car is just sitting there. A point on the coil won't experience a changing magnetic field unless the coil is already spinning, i.e., the engine has already turned over.

**Page 671, self-check B:**

Both the time axis and the position axis have been turned around. Flipping the time axis means that the roles of transmitter and receiver have been swapped, and it also means that Alice and Betty are approaching one another rather than receding. The time experienced by the receiving observer is now the longer one, so the Doppler-shift factor has been inverted: the receiver now measures a Doppler shift of  $1/2$  rather than  $2$  in frequency.

**Answers to self-checks for chapter 25****Page 687, self-check A:**

Yes. The mass has the same kinetic energy regardless of which direction it's moving. Friction converts mechanical energy into heat at the same rate whether the mass is sliding to the right or to the left. The spring has an equilibrium length, and energy can be stored in it either by compressing it ( $x < 0$ ) or stretching it ( $x > 0$ ).

**Page 687, self-check B:**

Velocity,  $v$ , is the rate of change of position,  $x$ , with respect to time. This is exactly analogous to  $I = \Delta q / \Delta t$ .

**Page 696, self-check C:**

The impedance depends on the frequency at which the capacitor is being driven. It isn't just a single value for a particular capacitor.

**Answers to self-checks for chapter 26****Page 706, self-check A:**

Yes. In U.S. currency, the quantum of money is the penny.

**Page 727, self-check B:**

Thomson was accelerating electrons, which are negatively charged. This apparatus is supposed to accelerate atoms with one electron stripped off, which have positive net charge. In both cases, a particle that is between the plates should be attracted by the forward plate and repelled by the plate behind it.

**Page 736, self-check C:**

The hydrogen-1 nucleus is simply a proton. The binding energy is the energy required to tear a nucleus apart, but for a nucleus this simple there is nothing to tear apart.

**Page 743, self-check D:**

The total momentum is zero before the collision. After the collision, the two momenta have reversed their directions, but they still cancel. Neither object has changed its kinetic energy, so the total energy before and after the collision is also the same.

**Page 750, self-check E:**

At  $v = 0$ , we have  $\gamma = 1$ , so the mass-energy is  $mc^2$  as claimed. As  $v$  approaches  $c$ ,  $\gamma$  approaches infinity, so the mass energy becomes infinite as well.

**Answers to self-checks for chapter 28****Page 792, self-check A:**

Only 1 is correct. If you draw the normal that bisects the solid ray, it also bisects the dashed ray.

**Answers to self-checks for chapter 29****Page 800, self-check A:**

You should have found from your ray diagram that an image is still formed, and it has simply moved down the same distance as the real face. However, this new image would only be visible from high up, and the person can no longer see his own image.

**Page 805, self-check B:**

Increasing the distance from the face to the mirror has decreased the distance from the image to the mirror. This is the opposite of what happened with the virtual image.

**Answers to self-checks for chapter 30****Page 822, self-check A:**

At the top of the graph,  $d_i$  approaches infinity when  $d_o$  approaches  $f$ . Interpretation: the rays just barely converge to the right of the mirror.

On the far right,  $d_i$  approaches  $f$  as  $d_o$  approaches infinity; this is the definition of the focal length.

At the bottom,  $d_i$  approaches negative infinity when  $d_o$  approaches  $f$  from the other side. Interpretation: the rays don't quite converge on the right side of the mirror, so they appear to have come from a virtual image point very far to the left of the mirror.

**Answers to self-checks for chapter 31****Page 839, self-check A:**

(1) If  $n_1$  and  $n_2$  are equal, Snell's law becomes  $\sin \theta_1 = \sin \theta_2$ , which implies  $\theta_1 = \theta_2$ , since both angles are between 0 and  $90^\circ$ . The graph would be a straight line along the diagonal of the graph. (2) The graph is farthest from the diagonal when the angles are large, i.e., when the ray strikes the interface at a grazing angle.

**Page 843, self-check B:**

(1) In 1, the rays cross the image, so it's real. In 2, the rays only appear to have come from the image point, so the image is virtual. (2) A ray is always closer to the normal in the medium with the higher index of refraction. The first left turn makes the ray closer to the normal, which is what should happen in glass. The second left turn makes the ray farther from the normal,

and that's what should happen in air. (3) Take the topmost ray as an example. It will still take two right turns, but since it's entering the lens at a steeper angle, it will also leave at a steeper angle. Tracing backward to the image, the steeper lines will meet closer to the lens.

### Answers to self-checks for chapter 32

#### Page 860, self-check A:

It would have to have a wavelength on the order of centimeters or meters, the same distance scale as that of your body. These would be microwaves or radio waves. (This effect can easily be noticed when a person affects a TV's reception by standing near the antenna.) None of this contradicts the correspondence principle, which only states that the wave model must agree with the ray model when the ray model is applicable. The ray model is not applicable here because  $\lambda/d$  is on the order of 1.

#### Page 863, self-check B:

At this point, both waves would have traveled nine and a half wavelengths. They would both be at a negative extreme, so there would be constructive interference.

#### Page 867, self-check C:

Judging by the distance from one bright wave crest to the next, the wavelength appears to be about  $2/3$  or  $3/4$  as great as the width of the slit.

#### Page 868, self-check D:

Since the wavelengths of radio waves are thousands of times longer, diffraction causes the resolution of a radio telescope to be thousands of times worse, all other things being equal. (To compensate for the wavelength, it's desirable to make the telescope very large, as in figure y on page 868.)

(1 rectangle =  $5 \text{ cm} \times 0.005 \text{ cm}^{-1} = 0.025$ ), but that would have been pointless, because we were just going to compare the two areas.

### Answers to self-checks for chapter 33

#### Page 891, self-check A:

(1) Most people would think they were positively correlated, but they could be independent. (2) These must be independent, since there is no possible physical mechanism that could make one have any effect on the other. (3) These cannot be independent, since dying today guarantees that you won't die tomorrow.

#### Page 893, self-check B:

The area under the curve from 130 to 135 cm is about  $3/4$  of a rectangle. The area from 135 to 140 cm is about 1.5 rectangles. The number of people in the second range is about twice as much. We could have converted these to actual probabilities

### Answers to self-checks for chapter 34

#### Page 915, self-check A:

The axes of the graph are frequency and photon energy, so its slope is Planck's constant. It doesn't matter if you graph  $e\Delta V$  rather than  $E_s + e\Delta V$ , because that only changes the y-intercept, not the slope.

### Answers to self-checks for chapter 35

#### Page 931, self-check A:

Wavelength is inversely proportional to momentum, so to produce a large wavelength we would

need to use electrons with very small momenta and energies. (In practical terms, this isn't very easy to do, since ripping an electron out of an object is a violent process, and it's not so easy to calm the electron down afterward.)

**Page 940, self-check B:**

Under the ordinary circumstances of life, the accuracy with which we can measure the position and momentum of an object doesn't result in a value of  $\Delta p \Delta x$  that is anywhere near the tiny order of magnitude of Planck's constant. We run up against the ordinary limitations on the accuracy of our measuring techniques long before the uncertainty principle becomes an issue.

**Page 940, self-check C:**

The electron wave will suffer single-slit diffraction, and spread out to the sides after passing through the slit. Neither  $\Delta p$  nor  $\Delta x$  is zero for the diffracted wave.

**Page 946, self-check D:**

No. The equation  $KE = p^2/2m$  is nonrelativistic, so it can't be applied to an electron moving at relativistic speeds. Photons always move at relativistic speeds, so it can't be applied to them, either.

**Page 947, self-check E:**

Dividing by Planck's constant, a small number, gives a large negative result inside the exponential, so the probability will be very small.

**Answers to self-checks for chapter 36**

**Page 955, self-check A:**

If you trace a circle going around the center, you run into a series of eight complete wavelengths. Its angular momentum is  $8\hbar$ .

**Page 958, self-check B:**

$n = 3, \ell = 0, \ell_z = 0$ : one state

$n = 3, \ell = 1, \ell_z = -1, 0, \text{ or } 1$ : three states

$n = 3, \ell = 2, \ell_z = -2, -1, 0, 1, \text{ or } 2$ : five states



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# Trig Table

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0.000	1.000	0.000	30°	0.500	0.866	0.577	60°	0.866	0.500	1.732
1°	0.017	1.000	0.017	31°	0.515	0.857	0.601	61°	0.875	0.485	1.804
2°	0.035	0.999	0.035	32°	0.530	0.848	0.625	62°	0.883	0.469	1.881
3°	0.052	0.999	0.052	33°	0.545	0.839	0.649	63°	0.891	0.454	1.963
4°	0.070	0.998	0.070	34°	0.559	0.829	0.675	64°	0.899	0.438	2.050
5°	0.087	0.996	0.087	35°	0.574	0.819	0.700	65°	0.906	0.423	2.145
6°	0.105	0.995	0.105	36°	0.588	0.809	0.727	66°	0.914	0.407	2.246
7°	0.122	0.993	0.123	37°	0.602	0.799	0.754	67°	0.921	0.391	2.356
8°	0.139	0.990	0.141	38°	0.616	0.788	0.781	68°	0.927	0.375	2.475
9°	0.156	0.988	0.158	39°	0.629	0.777	0.810	69°	0.934	0.358	2.605
10°	0.174	0.985	0.176	40°	0.643	0.766	0.839	70°	0.940	0.342	2.747
11°	0.191	0.982	0.194	41°	0.656	0.755	0.869	71°	0.946	0.326	2.904
12°	0.208	0.978	0.213	42°	0.669	0.743	0.900	72°	0.951	0.309	3.078
13°	0.225	0.974	0.231	43°	0.682	0.731	0.933	73°	0.956	0.292	3.271
14°	0.242	0.970	0.249	44°	0.695	0.719	0.966	74°	0.961	0.276	3.487
15°	0.259	0.966	0.268	45°	0.707	0.707	1.000	75°	0.966	0.259	3.732
16°	0.276	0.961	0.287	46°	0.719	0.695	1.036	76°	0.970	0.242	4.011
17°	0.292	0.956	0.306	47°	0.731	0.682	1.072	77°	0.974	0.225	4.331
18°	0.309	0.951	0.325	48°	0.743	0.669	1.111	78°	0.978	0.208	4.705
19°	0.326	0.946	0.344	49°	0.755	0.656	1.150	79°	0.982	0.191	5.145
20°	0.342	0.940	0.364	50°	0.766	0.643	1.192	80°	0.985	0.174	5.671
21°	0.358	0.934	0.384	51°	0.777	0.629	1.235	81°	0.988	0.156	6.314
22°	0.375	0.927	0.404	52°	0.788	0.616	1.280	82°	0.990	0.139	7.115
23°	0.391	0.921	0.424	53°	0.799	0.602	1.327	83°	0.993	0.122	8.144
24°	0.407	0.914	0.445	54°	0.809	0.588	1.376	84°	0.995	0.105	9.514
25°	0.423	0.906	0.466	55°	0.819	0.574	1.428	85°	0.996	0.087	11.430
26°	0.438	0.899	0.488	56°	0.829	0.559	1.483	86°	0.998	0.070	14.301
27°	0.454	0.891	0.510	57°	0.839	0.545	1.540	87°	0.999	0.052	19.081
28°	0.469	0.883	0.532	58°	0.848	0.530	1.600	88°	0.999	0.035	28.636
29°	0.485	0.875	0.554	59°	0.857	0.515	1.664	89°	1.000	0.017	57.290
								90°	1.000	0.000	$\infty$

# Mathematical Review

## Algebra

Quadratic equation:

The solutions of  $ax^2 + bx + c = 0$   
are  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

Logarithms and exponentials:

$$\ln(ab) = \ln a + \ln b$$

$$e^{a+b} = e^a e^b$$

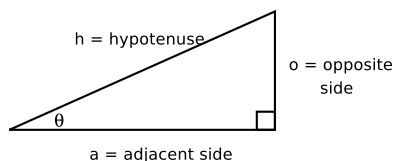
$$\ln e^x = e^{\ln x} = x$$

$$\ln(a^b) = b \ln a$$

## Geometry, area, and volume

area of a triangle of base $b$ and height $h$	$= \frac{1}{2}bh$
circumference of a circle of radius $r$	$= 2\pi r$
area of a circle of radius $r$	$= \pi r^2$
surface area of a sphere of radius $r$	$= 4\pi r^2$
volume of a sphere of radius $r$	$= \frac{4}{3}\pi r^3$

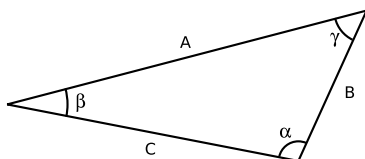
## Trigonometry with a right triangle



$$\sin \theta = o/h \quad \cos \theta = a/h \quad \tan \theta = o/a$$

Pythagorean theorem:  $h^2 = a^2 + o^2$

## Trigonometry with any triangle



Law of Sines:

$$\frac{\sin \alpha}{A} = \frac{\sin \beta}{B} = \frac{\sin \gamma}{C}$$

Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB \cos \gamma$$

## Properties of the derivative and integral (for students in calculus-based courses)

Let  $f$  and  $g$  be functions of  $x$ , and let  $c$  be a constant.

Linearity of the derivative:

$$\frac{d}{dx}(cf) = c \frac{df}{dx}$$

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

The chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Derivatives of products and quotients:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'}{g} - \frac{fg'}{g^2}$$

Some derivatives:

$$\begin{aligned} \frac{d}{dx}x^m &= mx^{m-1}, \text{ except for } m = 0 \\ \frac{d}{dx}\sin x &= \cos x & \frac{d}{dx}\cos x &= -\sin x \\ \frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x} \end{aligned}$$

The fundamental theorem of calculus:

$$\int \frac{df}{dx} dx = f$$

Linearity of the integral:

$$\int cf(x)dx = c \int f(x)dx$$

$$\int [f(x) + g(x)] = \int f(x)dx + \int g(x)dx$$

Integration by parts:

$$\int f dg = fg - \int g df$$

# Useful Data

## Metric Prefixes

M-	mega-	$10^6$
k-	kilo-	$10^3$
m-	milli-	$10^{-3}$
$\mu$ - (Greek mu)	micro-	$10^{-6}$
n-	nano-	$10^{-9}$
p-	pico-	$10^{-12}$
f-	femto-	$10^{-15}$

(Centi-,  $10^{-2}$ , is used only in the centimeter.)

## Notation and Units

quantity	unit	symbol
distance	meter, m	$x, \Delta x$
time	second, s	$t, \Delta t$
mass	kilogram, kg	$m$
density	$\text{kg}/\text{m}^3$	$\rho$
velocity	m/s	$\mathbf{v}$
acceleration	$\text{m}/\text{s}^2$	$\mathbf{a}$
force	$\text{N} = \text{kg} \cdot \text{m}/\text{s}^2$	$\mathbf{F}$
pressure	$\text{Pa} = 1 \text{ N}/\text{m}^2$	$P$
energy	$\text{J} = \text{kg} \cdot \text{m}^2/\text{s}^2$	$E$
power	$\text{W} = 1 \text{ J}/\text{s}$	$P$
momentum	$\text{kg} \cdot \text{m}/\text{s}$	$\mathbf{p}$
angular momentum	$\text{kg} \cdot \text{m}^2/\text{s}$ or $\text{J} \cdot \text{s}$	$\mathbf{L}$
period	s	$T$
wavelength	m	$\lambda$
frequency	$\text{s}^{-1}$ or Hz	$f$
gamma factor	unitless	$\gamma$
probability	unitless	$P$
prob. distribution	various	$D$
electron wavefunction	$\text{m}^{-3/2}$	$\Psi$

## The Greek Alphabet

$\alpha$	A	alpha	$\nu$	N	nu
$\beta$	B	beta	$\xi$	$\Xi$	xi
$\gamma$	$\Gamma$	gamma	$\omicron$	O	omicron
$\delta$	$\Delta$	delta	$\pi$	$\Pi$	pi
$\epsilon$	E	epsilon	$\rho$	P	rho
$\zeta$	Z	zeta	$\sigma$	$\Sigma$	sigma
$\eta$	H	eta	$\tau$	T	tau
$\theta$	$\Theta$	theta	$\upsilon$	Y	upsilon
$\iota$	I	iota	$\phi$	$\Phi$	phi
$\kappa$	K	kappa	$\chi$	X	chi
$\lambda$	$\Lambda$	lambda	$\psi$	$\Psi$	psi
$\mu$	M	mu	$\omega$	$\Omega$	omega

## Earth, Moon, and Sun

body	mass (kg)	radius (km)	radius of orbit (km)
earth	$5.97 \times 10^{24}$	$6.4 \times 10^3$	$1.49 \times 10^8$
moon	$7.35 \times 10^{22}$	$1.7 \times 10^3$	$3.84 \times 10^5$
sun	$1.99 \times 10^{30}$	$7.0 \times 10^5$	—

## Subatomic Particles

particle	mass (kg)	radius (fm)
electron	$9.109 \times 10^{-31}$	$\lesssim 0.01$
proton	$1.673 \times 10^{-27}$	$\sim 1.1$
neutron	$1.675 \times 10^{-27}$	$\sim 1.1$

The radii of protons and neutrons can only be given approximately, since they have fuzzy surfaces. For comparison, a typical atom is about a million fm in radius.

## Fundamental Constants

gravitational constant	$G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$
Coulomb constant	$k = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$
quantum of charge	$e = 1.60 \times 10^{-19} \text{ C}$
speed of light	$c = 3.00 \times 10^8 \text{ m/s}$
Planck's constant	$h = 6.63 \times 10^{-34} \text{ J} \cdot \text{s}$





